

## 8.311 Spring 2013: Recitation Problems 2

E.I. Sfakianakis\*

*Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

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1. *Two identical capacitors are connected via a switch. One of them initially holds a charge  $Q$ . Describe what happens to the energy of the system once the switch closes.*

Due to charge conservation, both capacitors have a charge of  $Q/2$ . The energy of each capacitor is  $\frac{1}{2} \frac{Q^2}{4C}$ , so the total energy of the system is  $2 \frac{1}{2} \frac{Q^2}{4C} = \frac{1}{4} \frac{Q^2}{C}$ . The energy of the system initially was  $\frac{1}{2} \frac{Q^2}{C}$ . This means that half the initial energy is wasted (heat and/or radiation)!

2. *A capacitor connected in series with a voltage source  $V$  and a resistor  $R$ . Once the switch is closed, the capacitor is being charge. Calculate the charge on the capacitor as a function of time and the total energy dissipated on the resistor.*

$$V_{IN} = V_R + V_C = I(t) \cdot R + \frac{1}{C} \int_0^t I(t) dt + V_C(0)$$

We use the Laplace transform to solve this equation.

$$\frac{V_{IN}C}{s} = I(s)RC + \frac{1}{s}I(s) + \frac{V_C(0)C}{s} \Rightarrow I(s) = \frac{RC}{RCs + 1} \frac{V_{IN} - V_C(0)}{R}$$

The inverse Laplace transform gives

$$I(t) = \frac{V_{IN} - V_C(0)}{R} e^{-t/RC}$$

the known exponential behavior with the time constant  $RC$ .

We take  $V_C(0) = 0$ . The energy lost in the resistance is

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\*Electronic address: <http://web.mit.edu/esfaki/www/8.311.html>

$$W_R = \int_0^\infty \frac{V^2}{R} dt = \int_0^\infty \frac{V_{IN}^2}{R} e^{-2t/RC} dt = \frac{V_{IN}^2 C}{2}$$

The energy given by the source is

$$W_s = \int_0^\infty V I dt = \frac{V_{IN}^2}{R} \int_0^\infty e^{-t/RC} dt = V_{IN}^2 C$$

We see that half the energy provided by the battery is lost, regardless of the value of the resistance.

3. ***A dielectric plate is inserted in a parallel plate capacitor. Calculate the force on the dielectric if the capacitor has a constant charge  $Q$  or is held at a constant voltage  $V$ .***

Let's start with the case of an isolated capacitor with constant  $Q$ . The capacitor has length  $L$ , width  $w$  and height  $d$ . If the dielectric is inserted a distance  $x$  the stored energy is

$$W = \int \frac{1}{2} \epsilon(x) |E|^2 dV = \frac{Q^2 d}{2w} \frac{1}{\epsilon x + \epsilon_0(l - x)}$$

$$F = -\frac{dW}{dx} = \frac{Q^2 d}{2w} \frac{\epsilon_1 - \epsilon_0}{(\epsilon_1 x + \epsilon_0(l - x))^2}$$

In the opposite case of a constant potential

$$W_c = \int \frac{1}{2} \epsilon(x) |E|^2 dV = \frac{wd}{2} \frac{V^2}{d^2} (\epsilon x + \epsilon_0(l - x)) = CV^2/2$$

When we move the dielectric by  $\delta x$   $W$  increases! Does this mean that the dielectric gets expelled from the capacitor? No. To prove it let's look at what the source does when the dielectric moves by  $\delta x$ . In order to keep a constant voltage, it sends a charge of  $\delta Q$ , so the energy of the battery DECREASES by  $W_b = -\delta QV = -V^2 dC$ .

So the total change in energy is

$$dW = dW_c + dW_b = \frac{dC}{2} V^2 - V^2 dC = -\frac{dC}{2} V^2 = -dW_c$$

This means that in order to calculate the force, we will either calculate

$$F = -\frac{dW_{total}}{dx}$$

or equivalently

$$F = +\frac{dW_{capacitor}}{dx}$$

The take-home message of this problem is that nature always minimizes the energy of a system by expelling as much energy as possible into heat. However we must be really careful when defining our system and calculating its energy. When a capacitor is connected to a battery, the capacitor AND the battery form the system. Even when the energy of the capacitor seems to increase, the stored (chemical) energy in the battery decreases more to compensate and lower the total energy of the system!