Cube versus Torus Models and the Euclidean Minimum Spanning Tree Constant*

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Abstract

We show that the length of the minimum spanning tree through points drawn uniformly from the d-dimensional torus is almost surely asymptotically equivalent to the length of the minimum spanning tree through points drawn uniformly from the d-cube. This result implies that the analytical expression recently obtained by Avram and Bertsimas for the MST constant in the d-torus model is in fact valid for the traditional d-cube model. We also show that the number of vertices of degree k for the MST in both models are asymptotically equivalent with probability one. Finally we show how our results can be extended to other combinatorial problems such as the traveling salesman problem.

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1 Introduction

In Beardwood et al. [2], the authors prove that for any bounded i.i.d. random variables $\{X_i : 1 \leq i < \infty\}$ with values in $\mathbf{R}^d$, $d \geq 2$, the length of the shortest tour through $\{X_1, \ldots, X_n\}$ is asymptotic to a constant times $n^{(d-1)/d}$ with probability one (the same being true in expectation). This theoretical result has been the inspiration for a growing interest in the area of probabilistic analysis of combinatorial optimization problems. An important contribution is contained in Steele [6] in which the author uses the theory of independent subadditive processes to obtain strong limit laws for a class of problems in geometrical probability which exhibit non-linear growth. Examples include the traveling salesman problem (TSP), the Steiner tree problem, and the minimum weight matching problem. Among other problems, not in this class, but with a similar asymptotical behavior, is the minimum spanning tree problem (MST) and some weighted versions of it (see Steele [7]).

For most of these problems, the results concern the almost sure convergence of a sequence of normalized random variables to a constant. One of the persistently important open problems in this area is the determination of the exact value of the constant for any interesting functional. In fact progress has been made by Avram and Bertsimas [1] who have recently obtained an exact expression (as a series expansion) for the MST constant when the points are drawn uniformly from the d-dimensional torus. The authors used the d-torus in order to avoid boundary effects and obtain tractable derivations. They also conjectured that their resulting constant was in fact the same than for the traditional d-cube model.

In this paper, we prove this conjecture by showing that the length of the optimal spanning trees in the d-torus and d-cube models are almost surely asymptotically equivalent. Note that, for comparison with related results on the d-torus versus d-cube model, it has been shown, in Steele and Tierney [9], that, when $d \geq 3$, the limiting distribution for the largest of the nearest-neighbor links is different in the two models. The paper is structured as follows. In the next section, after presenting the d-torus and d-cube models, we characterize the asymptotic growth of a largest edge in an optimal tree. In Section 3 we then use this result to prove the almost sure equivalence of the length of the optimal trees in both models. Then, in Section 4, we show that the number of vertices of degree k in both optimal trees are asymptotically equivalent with probability one. Finally in Section 5 we consider other combinatorial problems such as the traveling salesman problem.

2 Notation and preliminary results

The minimum spanning tree problem consists of finding a spanning tree of minimum total length in an undirected weighted graph . We will consider two special models for this undirected graph:

The d-cube model:

Let $\{x_i : 1 \leq i < \infty\}$ be an arbitrary infinite sequence of points in $[0,1]^d$ (the unit cube in $\mathbf{R}^d$, $d \geq 2$, the d-dimentional space of real numbers, with the Euclidean metric and the Lebesgue measure), and let $x^{(n)} = \{x_1, x_2, \ldots, x_n\}$ denote its first n points. For each finite n, $x^{(n)}$ will be the vertex set, and $K_n(x) = \{\{x_i, x_j\} : 1 \leq i < j \leq n\}$ the edge set of our graph. The weight of an edge $\{x_i, x_j\}$ will be the Euclidean distance $||x_i - x_j||$ between x_i and x_j.

The d-torus model:

In order to eliminate the boundary effects of the previous model, consider the previous sequence $x_1, x_2, \ldots, x_n, \ldots$ modulo 1 in each component. Alternatively, one can imagine a sequence on the d-torus $T^d = ([0,1] \bmod 1)^d$ (the metric space with its Lebesgue measure and Euclidean d-torus metric). Note that the weight of an edge $\{x_i, x_j\}$ is now taken to be $||\{x_i - x_j\}(\bmod 1)^d||$. We recall that for $y \in [-1,1]$,

$y(\mathrm{mod}1)$ is the minimum of $|y|$ and $1-|y|$.

Other notations:

We will write $L^{(c)}_{mst}(x^{(n)})$ for the length of an optimal MST (described by its set of edges $\mathcal{K}^{(c)}_{mst}(x^{(n)})$) for the problem in the d-cube. We will use $L^{(t)}_{mst}(x^{(n)})$ and $\mathcal{K}^{(t)}_{mst}(x^{(n)})$ for the corresponding quantity in the d-torus. Also $|\{.\}|$ will stand for the cardinality of the set $\{.\}$.

Finally let $\ell^{(t)}_{mst}(x^{(n)}) = \max\left\{||\{x_i - x_j\}(\mathrm{mod}1)^d|| : \{x_i, x_j\} \in \mathcal{K}^{(t)}_{mst}(x^{(n)})\right\}$ be the length of the largest edge in an optimal tree in the d-torus. The main result of this section is concerned with the asymptotic growth of this largest edge, and is expressed in the following proposition:

Proposition 1 *Let $\{X_i : 1 \le i < \infty\}$ be a sequence of points independently and uniformly distributed over $[0,1]^d$. Then for the corresponding MST in the d-torus, we have, for n sufficiently large,*

$$\mathbf{P}\left(\ell^{(t)}_{mst}(X^{(n)}) > \lambda_d \left(\frac{\log n}{n}\right)^{1/d}\right) \le \frac{1}{24n^2 \log n}, \tag{2.1}$$

where $\lambda_d = 13^{1/d}\sqrt{d+3}$.

In order to prove this proposition we need two intermediate lemmas.

Lemma 1 *Let m be a positive integer, and $(Q_j)_{1 \le j \le m^d}$ be a partition of the d-cube $[0,1]^d$ into cubes with edges parallel to the axes and of length $1/m$. If for a sequence of points $\{x_i : 1 \le i < \infty\}$, $x^{(n)} \cap Q_j$ is not empty for all j, then the MST in the d-torus is such that*

$$\ell^{(t)}_{mst}(x^{(n)}) \le \frac{\sqrt{d+3}}{m}. \tag{2.2}$$

Proof:

This proof is a generalization of an argument used in [4] for the MST in the square. It goes as follows. Let e be an edge of $\mathcal{K}^{(t)}_{mst}(x^{(n)})$ so that its weight is $\ell^{(t)}_{mst}(x^{(n)})$. By definition of an optimal MST, we then have the following property: "If we discard e, we end up with a forest with two components, with point sets, say V_e and W_e, such that for all $x_i \in V_e$ and all $x_j \in W_e$ we have $||\{x_i - x_j\}(\text{mod}1)^d|| \geq \ell_{mst}^{(t)}(x^{(n)})$".

We will now prove the lemma by contradiction. Let us assume that $\ell_{mst}^{(t)}(x^{(n)}) > \sqrt{d+3}/m$. Then $\ell_{mst}^{(t)}(x^{(n)}) > \sqrt{d}/m$ and thus each Q_j either contains points from V_e or from W_e but not from both. So the partition of the points into V_e and W_e leads to a partition of the cubes into two sets, I and J such that for all $i \in I$ we have $x^{(n)} \cap Q_i \subset V_e$, and for all $j \in J$ we have $x^{(n)} \cap Q_j \subset W_e$. Now, since all cubes are non-empty, we can always find a pair of adjacent (i.e, sharing a facet) cubes Q_i and Q_j with $i \in I$ and $j \in J$. But now, the largest possible edge connecting these two squares is bounded from above by $\sqrt{d+3}/m$ and thus, using our working hypothesis, by $\ell_{mst}^{(t)}(x^{(n)})$. This crearly contradicts the property above. Note that the same arguments hold for the problem in the d-cube.

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Lemma 2 *Let $\{X_i : 1 \leq i < \infty\}$ be a sequence of points independently and uniformly distributed over $[0,1]^d$, let m be a positive integer, and let $(Q_i)_{1 \leq i \leq m^d}$ be a partition of the d-cube $[0,1]^d$ into cubes with edges parallel to the axes and of length $1/m$. If N_j denotes the cardinality of $X^{(n)} \cap Q_j$, then, with we have, for $h \geq 12$ and $n \geq 3$,*

$$\mathbf{P}\left(\forall j, N_j > n/m^d - \sqrt{hn \log n/m^d}\right) \geq 1 - \frac{m^d}{2n^{h/4}}. \tag{2.3}$$

Proof:

Let $p = 1/m^d$, and for all j let $\mathcal{B}_{n,j}$ be the event $\left\{N_j \leq np - \sqrt{hnp \log n}\right\}$. We obviously have

$$\mathbf{P}\left(\exists j : \mathcal{B}_{n,j}\right) \leq \sum_{j=1}^{m^d} \mathbf{P}\left(\mathcal{B}_{n,j}\right) = m^d \mathbf{P}\left(\mathcal{B}_{n,1}\right) = \mathbf{P}\left(\mathcal{B}_{n,1}\right)/p. \tag{2.4}$$

Now N_1 is a binomial random variable with n trials and parameter p. Using classical bounds on the tail of a binomial distribution (see [3, Corollary 4, p.11]) we have, with $q = 1 - p$,

$$\mathbf{P}\left(N_1 \le np - \sqrt{hnp\log n}\right) \le \frac{1}{2}\exp\left\{-h\log n/3q + 1/q\right\}. \tag{2.5}$$

For $h \ge 12$ and $n \ge 3$, we have $-h\log n/3q + 1/q \le -h\log n/4q \le -h\log n/4$ which together with (2.5) gives

$$\mathbf{P}\left(N_1 \le np - \sqrt{hnp\log n}\right) \le \frac{1}{2n^{h/4}}. \tag{2.6}$$

Now the lemma follows from (2.4) and (2.6).

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We are now in position to prove Proposition 1.

Proof of Proposition 1:

For (2.3) to be of interest we need $n/m^d - \sqrt{hn\log n/m^d}$ to be non-negative and thus $m^d \le n/(h\log n)$. Let us choose $m^d = \lfloor n/(h\log n)\rfloor$ (we will suppose n large enough so that $m \ge 1$ for a given $h \ge 12$). We then have from Lemma 2

$$\mathbf{P}\left(\forall j, N_j > 0\right) \ge 1 - \frac{\lfloor n/(h\log n)\rfloor}{2n^{h/4}} \ge 1 - \frac{1}{2hn^{h/4-1}\log n}. \tag{2.7}$$

Also, from Lemma 1 we have

$$\mathbf{P}\left(\ell^{(t)}_{mst}(X^{(n)}) \le \sqrt{d+3}/\left(\lfloor n/(h\log n)\rfloor\right)^{1/d}\right) \ge \mathbf{P}\left(\forall j, N_j > 0\right). \tag{2.8}$$

The proposition follows from (2.7) and (2.8) by taking $h = 12$ and by realizing that $\lfloor n/(12\log n)\rfloor > n/(13\log n)$ for n sufficiently large (here $n \ge 1092$). Note again that the same arguments hold for the problem in the d-cube.

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We finally need a last result before stating our main theorem.

Lemma 3 *Among $\mathcal{K}^{(t)}_{mst}(x^{(n)})$, let k be the number of edges $\{x_i, x_j\}$ such that $||\{x_i - x_j\}(\text{mod}1)^d|| < ||x_i - x_j||$. Then there exists a feasible solution to the d-cube problem, of weight bounded by $L^{(t)}_{mst}(x^{(n)}) + \gamma_d k^{(d-1)/d}$, where $\gamma_d \ge 0$.*

Proof:

From $\mathcal{K}_{mst}^{(t)}(x^{(n)})$, delete the edges $\{x_i, x_j\}$ such that $||x_i - x_j(\text{mod}\,1)^d|| < ||x_i - x_j||$. If k is the number of such edges, we end up with a forest of $k+1$ components. Pick one representative from each component, and construct the MST (in the d-cube) through these $k+1$ points. From well-known results (see for example [7]), the length of such a tree is $O(k^{(d-1)/d})$. Now the forest together with this tree form a spanning tree of $x^{(n)}$ in the d-cube, and this shows the validity of our lemma.

■

3 Almost sure equivalence

Theorem 1 *Let $\{X_i : 1 \leq i < \infty\}$ be a sequence of points independently and uniformly distributed over $[0,1]^d$. Then for the MST we have*

$$\lim_{n\to\infty} \frac{L_{mst}^{(t)}(X^{(n)})}{n^{(d-1)/d}} = \lim_{n\to\infty} \frac{L_{mst}^{(c)}(X^{(n)})}{n^{(d-1)/d}} = \beta_{mst}(d) \;\; (a.s.). \tag{3.9}$$

Proof:

First let us consider an arbitrary sequence $(x_i)_i$. For all edges $\{x_i, x_j\}$ of $\mathcal{K}_{mst}^{(c)}(x^{(n)})$, replace $||x_i - x_j||$ by $||\{x_i - x_j\}(\text{mod}\,1)^d||$. We then obtain a feasible solution to the d-torus model of length less than $L_{mst}^{(c)}(x^{(n)})$. Hence we have

$$L_{mst}^{(t)}(x^{(n)}) \leq L_{mst}^{(c)}(x^{(n)}). \tag{3.10}$$

Now let $\mathcal{F}(x^{(n)}) = \Big\{\{x_i, x_j\} \in \mathcal{K}_{mst}^{(t)}(x^{(n)}) : ||x_i - x_j(\text{mod}\,1)^d|| < ||x_i - x_j||\Big\}$. Also, for $r < 1/2$, let $Q(r) = [0,1]^d \setminus [r, 1-r]^d$ be a layer of width r on the inside of the d-cube. Partition $\mathcal{F}(x^{(n)})$ into two sets $\mathcal{F}_1^{(r)}(x^{(n)}) = \{\{x_i, x_j\} \in \mathcal{F}(x^{(n)}) : x_i \in [r, 1-r]^d, x_j \in [r, 1-r]^d\}$, and $\mathcal{F}_2^{(r)}(x^{(n)}) = \mathcal{F}(x^{(n)}) \setminus \mathcal{F}_1^{(r)}(x^{(n)})$. Call their respective cardinalities $k_1(r,n)$ and $k_2(r,n)$. From Lemma 3 we then have

$$L_{mst}^{(c)}(x^{(n)}) \leq L_{mst}^{(t)}(x^{(n)}) + \gamma_d \left(k_1(r,n) + k_2(r,n)\right)^{(d-1)/d}. \tag{3.11}$$

Now, it is well-known (see [7]) that for any $x^{(n)} = \{x_1, x_2, \ldots, x_n\}$ the degree of the points in $\mathcal{K}_{mst}^{(t)}(x^{(n)})$ is bounded by a constant D_d. Hence it is easy to see that

$$k_2(r, n) \leq D_d |\{x_i \in Q(r)\}|. \tag{3.12}$$

Let us now consider a sequence $(X_i)_i$ of points independently and uniformly distributed over $[0, 1]^d$. We then have

$$L_{mst}^{(t)}(X^{(n)}) \leq L_{mst}^{(c)}(X^{(n)}) \leq L_{mst}^{(t)}(X^{(n)}) + \gamma_d \left(K_1(r, n) + D_d |\{X_i \in Q(r)\}|\right)^{(d-1)/d}, \tag{3.13}$$

Now let $r_n \stackrel{\text{def}}{=} \lambda_d (\log n/n)^{1/d}$. For all $\varepsilon > 0$, there exists n_1 such that, for all $n \geq n_1$, we have $r_n < \varepsilon$. Hence, for all $0 < \varepsilon < 1/2$, we have

$$\limsup_{n \to \infty} \frac{|\{X_i \in Q(r_n)\}|}{n} \leq \lim_{n \to \infty} \frac{|\{X_i \in Q(\varepsilon)\}|}{n} = 1 - (1 - 2\varepsilon)^d \leq 2d\varepsilon \text{ (a.s.)}. \tag{3.14}$$

Also, from Proposition 1, we have, for a sufficiently large constant n_2

$$\sum_{n=1}^{\infty} \mathbf{P}(K_1(r_n, n) > 0) \leq \sum_{n=1}^{\infty} \mathbf{P}(\ell_{mst}^{(t)}(X^{(n)}) > r_n) \leq n_2 + \sum_{n=n_2+1}^{\infty} \frac{1}{24 n^2 \log n} < \infty. \tag{3.15}$$

From the Borel-Cantelli lemma this implies that

$$\lim_{n \to \infty} K_1(r_n, n) = 0 \text{ (a.s.)}. \tag{3.16}$$

The result then follows from (3.13) with $r = r_n$, (3.14), (3.16), and from the almost sure convergence of $L_{mst}^{(c)}(X^{(n)})/n^{(d-1)/d}$ to $\beta_{mst}(d)$ (as obtained in [7]).

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Consequence: As a corollary to Theorem 1, the series expansion recently obtained for the MST constant in the d-torus (see below) is also valid for the classical Euclidean model of the MST. This is then one rare example, among this class of problems, for which we have been able to characterize the limiting constant analytically.

For completeness, let us recall the result obtained in [1]:

$$\beta_{mst}(d) = \frac{1}{d(b_d)^{1/d}} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\infty} \frac{f_k(y)}{y^{(d-1)/d}} dy,$$

where $b_d = \pi^{d/2}/\Gamma(1 + d/2)$ is the volume of the ball of unit radius in dimension d, $f_1(y) = e^{-y}$, and for $k \geq 2$,

$$f_k(y) = \frac{y^{k-1}}{b_d^{k-1}(k-1)!} \int_{\Omega_k} e^{\frac{-y}{b_d} g_k(u_0,\ldots,u_{k-1})} du_1 \ldots du_{k-1},$$

where the integration is performed on the set Ω_k of all points $\{u_0, \ldots, u_{k-1}\}$ of the d-torus (u_0 being the 'center' of the torus) such that the spheres $S(u_j, 1/2)$, $0 \leq j \leq k-1$, form a connected set and $g_k(u_0, \ldots, u_{k-1})$ is the volume of $\cup_j S(u_j, 1)$.

Before concluding this section, let us mention that in [7], the author studies the asymptotics of generalizations of the minimum spanning tree problem in which the distance between points are some fixed power of the Euclidean distance. It is quite clear that Theorem 1 can be readily extended to cover this case as well.

4 Node degree equivalence for the MST

Proposition 2 *For a given k, let $V_k^{(c)}(x^{(n)})$ and $V_k^{(t)}(x^{(n)})$ be the number of vertices of degree k in the MST in the d-cube and d-torus, respectively. Let $\{X_i : 1 \leq i < \infty\}$ be a sequence of points independently and uniformly distributed over $[0,1]^d$. Then there are positive constants $\alpha_{k,d}$ such that*

$$\lim_{n\to\infty} n^{-1} V_k^{(t)}(X^{(n)}) = \lim_{n\to\infty} n^{-1} V_k^{(c)}(X^{(n)}) = \alpha_{k,d} \text{ (a.s.)}. \tag{4.17}$$

Proof:

The existence of the constants verifying the second equality was proved in Steele et al. [10]. Also, if $\mathcal{F}(X^{(n)})$ denotes the set of edges of $\mathcal{K}_{mst}^{(t)}(X^{(n)})$ that 'crosses' the boundary of the d-cube (see the proof of Theorem 1), it is easy to see that, with probability one, we have

$$\mathcal{K}_{mst}^{(t)}(X^{(n)}) \setminus \mathcal{F}(X^{(n)}) \subset \mathcal{K}_{mst}^{(c)}(X^{(n)}). \tag{4.18}$$

Now consider any spanning tree T on an arbitrary connected graph $G = (V, E)$, and any pair of edges $e \in T$ and $e' \in E \setminus T$ such that $T' = T \setminus \{e\} \cup \{e'\}$ is still a spanning tree. For any given k, let N_k and N'_k be the number of vertices of degree k in T and T', respectively. Then it is easy to see that

$$|N_k - N'_k| \leq 4. \tag{4.19}$$

(In fact, for leaves, this can be improved to $|N_1 - N'_1| \leq 2$). From (4.18) and (4.19) we then have

$$\left|V_k^{(c)}(X^{(n)}) - V_k^{(t)}(X^{(n)})\right| \leq 4\left|\mathcal{F}(X^{(n)})\right|. \tag{4.20}$$

The rest of the proof parallels the one for Theorem 1.

5 Generalizations to other problems

5.1 Sufficient conditions

The result of Section 3 can be extended to other combinatorial optimization problems. Let us consider a problem (generically labeled "$*$"), defined on an undirected graph $G = (V, E)$ with positive weighted edges, which requires finding, among all feasible subsets of edges, a subset of minimum weight (the weight of a subset of edges being the sum of the weight of the edges belonging to this subset). Suppose we are interested in comparing $L_*^{(c)}(x^{(n)})/n^{c_d}$ and $L_*^{(t)}(x^{(n)})/n^{c_d}$ for a given positive constant c_d.

The following properties are sufficient for showing that, for a sequence of points independently and uniformly distributed over $[0, 1]^d$, these quantities are almost surely asymptotically equivalent (refer to Section 2 for the definition of terms, replacing "mst" by "$*$").

1. (Bounded degree). For any $x^{(n)} = \{x_1, x_2, \ldots, x_n\}$ the degree of the points in $\mathcal{K}_*^{(t)}(x^{(n)})$ is bounded by a constant D_d.

2. (Bounded passage from torus to cube). Among $\mathcal{K}_*^{(t)}(x^{(n)})$, let k be the number of edges (x_i, x_j) such that $||x_i - x_j(\text{mod} 1)^d|| < ||x_i - x_j||$. Then there exists a feasible solution to the d-cube problem, of weight bounded from above by $L_*^{(t)}(x^{(n)}) + O(k^{c_d})$.

3. (Probabilistically small largest edge). For $\{X_i : 1 \leq i < \infty\}$ a sequence of points independently and uniformly distributed over $[0,1]^d$, the largest edge is such that, for all $\varepsilon > 0$, $\sum_{n=1}^{\infty} \mathbf{P}(\ell_*^{(t)}(X^{(n)}) > \varepsilon) < \infty$.

Properties 1 and 2 insure that an inequality similar to (3.13) still holds. Such an inequality, in conjonction with Property 3, is then sufficient for showing that $L_*^{(c)}(X^{(n)})/n^{c_d}$ and $L_*^{(t)}(X^{(n)})/n^{c_d}$ are almost surely asymptotically equivalent. As a consequence, if $L_*^{(c)}(X^{(n)})/n^{c_d}$ converges almost surely to a constant $\beta_*(d)$, it will be the same for $L_*^{(t)}(X^{(n)})/n^{c_d}$, and vice-versa. As an application, let us consider the case of the traveling salesman problem with $c_d = (d-1)/d$.

5.2 The Traveling Salesman Problem

The traveling salesman problem consists of finding an hamiltonian tour through a given set of points of minimum total length. For this problem, Property 1 is obvious. The easiest way to see that the TSP verifies Property 2 is as follows (see [8] for details): Delete the k edges (x_i, x_j) such that $||x_i - x_j(\text{mod} 1)^d|| < ||x_i - x_j||$, and then greedily connect the resulting (possibly degenerate) paths. For k paths (maximum $2k$ path endpoints), it is well-known that there exist two endpoints at a maximum distance of $O(k^{-1/d})$. Connecting these endpoints and repeating the process until a tour results costs $O(k^{(d-1)/d})$.

It remains to show that Property 3 is valid for the TSP. In order to do so, we will consider a slightly modified (but, for our purpose, asymptotically equivalent) probabilistic model in which we use a Poisson point process. More precisely, let π_n denote a Poisson point process in $[0,1]^d$ with intensity equal to n times the Lebesgue measure ν. For any bounded Borel set $A \subset [0,1]^d$, let $\pi_n(A)$ denote the random set of points in A (almost surely a finite set of points) and $N_n(A)$ the cardinality of $\pi_n(A)$ (a Poisson random variable with parameter $n\nu(A)$). When A is $[0,1]^d$, we simply write π_n and N_n. Now we have the following result from which Property 3 obviously holds.

Proposition 3 *Let π_n be a Poisson point process in $[0,1]^d$ with intensity equal to n times the Lebesgue measure. Then, for the corresponding TSP in the d-torus, we have*

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\ell_{tsp}^{(t)}(\pi_n) > \frac{\log n}{n^{1/d}}\right) < \infty. \tag{5.21}$$

In order to prove this proposition we need two intermediate results. The first lemma is a special application of a geometrical property of TSP tours.

Lemma 4 *Let $\{x_i : 1 \leq i < \infty\}$ be an arbitrary infinite sequence of points in $[0,1]^d$. For any $r < 1/8$ let $B^{(t)}(0,r) = \{y \in [0,1]^d : ||y(\mathrm{mod} 1)^d|| \leq r\}$ be the (d-torus) ball with radius r and center 0. If $|B^{(t)}(0,r) \cap x^{(n)}| \geq 12$, then , for any optimal solution to the TSP in the d-torus, there exists a point among $B^{(t)}(0,r) \cap x^{(n)}$ such that both its adjacent points along the tour belong to $B^{(t)}(0,4r)$.*

Proof:

By definition of an optimal TSP tour, a 2-exchange step (replacing any two edges of a tour by two other edges) cannot lead to a better tour. Let us then consider an optimal TSP tour through $x^{(n)}$ together with an arbitrary orientation. For each point x_i, let

$x_{b(i)}$ be its successor along the tour. For any two points x_i and x_j, let $\Delta_{ij}L(x^{(n)})$ be the change in the length of the tour after replacing $\{x_i, x_{b(i)}\}$ and $\{x_j, x_{b(j)}\}$ by $\{x_i, x_j\}$ and $\{x_{b(i)}, x_{b(j)}\}$. Finally let $\Delta^* L(x^{(n)}) = \min_{j \neq b(i), i \neq b(j)} \Delta_{ij}L(x^{(n)})$ (note that, for a TSP tour, $\Delta^* L(x^{(n)}) \geq 0$).

Now, if we have k points in $B^{(t)}(0,r)$, each of them having its successor outside of $B^{(t)}(0,4r)$, then $k \leq 5$. Indeed, otherwise, $\Delta^* L(x^{(n)}) < -r$, a contradiction. To see that, suppose that $k = 6$. From the "pigeonhole principle", $\sup_{x^{(n)}} \Delta^* L(x^{(n)})$ will correspond to 6 points evenly spread on the boundary of $B^{(t)}(0,r)$, with each successor $x_{b(i)}$ directly across from it, on the boundary of $B^{(t)}(0, 4r + \varepsilon)$ for an arbitrarily small ε. But in this case we have $\Delta^* L(x^{(n)}) = r + (4r + \varepsilon) - (3r + \varepsilon) - (3r + \varepsilon) = -r - \varepsilon$.

The same argument hold if we reverse the orientation of the tour. Hence, we can be sure that, among 12 points, there is at least one point for which *both* its adjacent points along the tour belong to $B^{(t)}(0,4r)$.

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The second lemma is a probabilistic statement based on the previous result. For any points x, y and z in $[0,1]^d$, let $S(x,y,z) = ||x - y(\text{mod}1)^d|| + ||y - z(\text{mod}1)^d|| - ||x - z(\text{mod}1)^d||$.

Lemma 5 *Let π_n be a Poisson point process in $[0,1]^d$ with intensity equal to n times the Lebesgue measure. Consider the following event: $\mathcal{H}(r,\mu) \stackrel{\text{def}}{=}$ "there exists a point Y among $\pi_n(B^{(t)}(0, r/n^{1/d}))$ such that both its adjacent points along the TSP tour, X, Z, belong to $\pi_n(B^{(t)}(0, 4r/n^{1/d}))$, and such that $S(X,Y,Z) \geq \mu/n^{1/d}$". Then, for any $\varepsilon > 0$, there exists two positive constants r and μ so that*

$$\mathbf{P}\left(\mathcal{H}(r,\mu)\right) \geq 1 - \varepsilon. \tag{5.22}$$

Proof:

Let $N(r) = N_n(B^{(t)}(0, r/n^{1/d}))$. Suppose first that $N(r) \geq 12$. Then, from Lemma

4, there exists a point Y in $B^{(t)}(0, r/n^{1/d})$, such that both its adjacent points along the TSP tour, X and Z, are in $B^{(t)}(0, 4r/n^{1/d})$, and thus in $B^{(t)}(Y, 5r/n^{1/d})$. Suppose that, in addition, there exists a constant M such that $N(6r) \leq M$, and thus $N_n(B^{(t)}(Y, 5r/n^{1/d})) \leq M$. Then, for X_i and Z_i among $\pi_n(B^{(t)}(Y, 5r/n^{1/d}))$, there exists a constant μ such that $\mathbf{P}\left(\inf S(X_i, Y, Z_i) \geq \mu/n^{1/d}\right) \geq 1 - \varepsilon/2$. Indeed, note that in the d-torus, everything is unchanged through translation, and thus that it suffices to show that for two independent points U and W uniformly distributed on $B^{(t)}(0, 5r/n^{1/d})$, we have $\lim_{\eta\to 0} \mathbf{P}(S(U, 0, W) \leq \eta/n^{1/d}) = 0$, where the limit is uniform in n (see [5, Lemma 3] for a similar argument). But this is obvious since the probability in the limiting expression is independent of n. It remains to evaluate the probability of "$N(r) \geq 12$ and $N(6r) \leq M$". $N(r)$ is a Poisson random variable with a parameter independent of n. Hence, for any $\varepsilon > 0$, one can always choose a large r and a large constant M, so that: $\mathbf{P}(N(r) \geq 12 \text{ and } N(6r) \leq M) \geq 1 - \varepsilon/2$.

∎

We are now ready to complete the proof of Proposition 3.

Proof of Proposition 3:

First note that Lemma 5 remains valid for any d-torus ball translated from the origin. Also if, for such a ball, the event $\mathcal{H}(r, \mu)$ is true, say with Y, then we get savings of at least $\mu/n^{1/d}$ by skipping point Y from the tour.

Let us look at the probability that a given edge {U,W} of the tour has a length D (in the d-torus) greater than or equal to $\log n/n^{1/d}$. Divide the edge into three equal segments, and further divide the middle segment into $m-1$ equal segments. Let $(z_j)_{1\leq j\leq m}$ be the m endpoints defining the small segments and, for a given r, consider m adjacent d-balls of same radius $4r/n^{1/d}$ centered at these points. We then have $m = 1 + \log n/24r = \Theta(\log n)$. Now suppose that for at least one of the balls, say $B^{(t)}(z_j, r/n^{1/d})$ the event $\mathcal{H}(r, \mu)$ is true with a given point, say Y. Then one can transform the current solution by connecting Y to U and W instead of its previous adjacent points. Since $Y \in B^{(t)}(z_j, r/n^{1/d})$ and $\min\{||z_j - U(\text{mod } 1)^d||, ||z_j - W(\text{mod}1)^d||\} \geq \log n/3n^{1/d}$ the extra cost of going from U to W via Y will be at most a constant times $1/n^{1/d}\log n$, and will be less than the savings in the ball, i.e. $\mu/n^{1/d}$, for n large enough. In conclusion, from Lemma 5, we have for large n

$$\mathbf{P}(D > \log n/n^{1/d}) \leq \varepsilon^{1+\log n/24r} = \varepsilon n^{\log\varepsilon/24r} \tag{5.23}$$

Now, since N_n is a Poisson random variable with parameter n, one can always find, for any $c < 1$, a constant K such that $\mathbf{P}(N_n > Kn) = O(c^n)$. Hence we finally have

$$\mathbf{P}\left(\ell_{tsp}^{(t)}(\pi_n) > \frac{\log n}{n^{1/d}}\right) \leq mn(\varepsilon n^{\log\varepsilon/24r}) + O(c^n). \tag{5.24}$$

Now we can always choose ε so that the proposition is true.

∎

6 Concluding remarks

In the course of proving the main theorem of this paper we have obtained several results of independent interests. For example, in proposition 1, we have proved that for n points i.i.d. uniform on $[0,1]^d$, the length of the largest edge of the optimal MST solutions (in the d-cube or d-torus) is almost surely asymptotically bounded from above by $\lambda_d(\log n/n)^{1/d}$. In fact, it is not difficult to show (see for example [4]) that, for a Poisson point process π_n with intensity n times the Lebesgue measure on $[0,1]^d$, the growth of the largest edge is $\Theta((\log n/n)^{1/d})$ almost surely.

Also, in Section 4, we have noted that in [10], the authors prove that for any independent and uniform random variables $\{X_i : 1 \leq i < \infty\}$ in $[0,1]^d$, $d \geq 2$, the number of vertices of degree k in the MST through $\{X_1, \ldots, X_n\}$ is asymptotic to a constant $\alpha_{k,d}$ times n with probability one. In the case $k = 1$ and $d = 2$ (i.e., for the number of leaves of the MST in the square), the authors have shown that the constant $\alpha = \alpha_{1,2}$ is positive and that Monte Carlo simulation results suggest that $\alpha = 2/9$ is a reasonable approximation. If one attempts to get any more information on this constant, one rapidly finds that the boundary effects of the square are a serious limitation on any analytical approach. Now from Theorem 2 any attempts on characterizing these constants could be made within the torus model, with no boundary problems. For example, it is clear, from the symmetry induced by the d-torus model, that $\alpha_{k,d}$ is equal to $\lim_{n\to\infty} \mathbf{P}(H_d^{(n)} = k)$, where $H_d^{(n)}$ is the degree of any point, say X_1, in a minimal spanning tree through $\{X_1, \ldots, X_n\}$ in the d-torus.

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