Dominating sets in graphs with no long induced paths

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Introduction

k-coloring

- k-**coloring**: $c : V(G) \to [k]$ such that $c(u) \neq c(v)$ for all edges (u, v)
- NP-complete problems:
  - Graph coloring problem: Determining the smallest k such that G admits a k-coloring [1]
  - k-coloring problem: Determining whether G admits a k-coloring for fixed $k \geq 3$ [2]

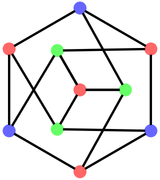

Figure: A 3-coloring of a graph.

[1] Karp. 1972.

[2] Stockmeyer. 1973.

k-coloring H-free graphs

- **Forbidden subgraphs**: G is H-**free** if H is not an induced subgraph of G

Theorem (Lozin and Kamiński [3])

For any $k, g \geq 3$, the k-coloring problem on graphs with no cycles of length $\leq g$ is NP-complete.

Theorem (Holyer, Leven and Galil [4])

For any $k \geq 3$ and any forest H with a vertex of degree ≥ 3, the k-coloring problem on H-free graphs is NP-complete.

[3] Lozin and Kamiński. 2007.

[4] Holyer. 1981; Leven and Galil. 1983.

k-coloring P_t-free graphs

k \ t	5	6	7	8	9
3		P [6]	P [7]		$\cdots$
4	P [5]	P [8]			$\cdots$
5			NP [9]		$\cdots$
6	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

[5] Hoàng et al. 2010.
[6] Randerath and Schiermeyer. 2004.
[7] Bonomo et al. 2017.
[8] Chudnovsky, Spirkl, and Zhong. 2018; Chudnovsky, Spirkl, and Zhong. 2018.
[9] Huang. 2016.

Approach to 3-coloring

- **List k-coloring** of (G, L) where $L : V(G) \to \mathcal{P}(\mathbb{Z}^+)$:
  $c : V(G) \to \cup_{v \in V(G)} L(v)$ such that
  1. $c(u) \neq c(v)$ for all edges (u, v),
  2. $c(v) \in L(v)$ for all vertices v, and
  3. $|L(v)| \leq k$ for all vertices v
- List k-coloring problem: Determining whether G admits a list k-coloring for fixed k

Theorem (Edwards [10])

The list 2-coloring problem is polynomial.

[10] Edwards. 1986.

Approach to 3-coloring

- **Dominating set**: $S \subseteq V(G)$ such that every vertex in $V(G) \setminus S$ has a neighbor in S
- $\mathcal{G}$ has constant bounded dominating sets if every $G \in \mathcal{G}$ has a dominating set S such that $|S| \leq K_{\mathcal{G}}$, for constant $K_{\mathcal{G}}$
- Approach to 3-coloring $G \in \mathcal{G}$:
  1. Find dominating set S such that $|S| \leq K_{\mathcal{G}}$
  2. Consider all possibilities of 3-coloring S
  3. Solve the remaining list 2-coloring problem on each possibility

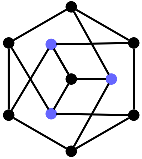

Figure: A dominating set.

Main problem

Do P_t-free graphs admit constant bounded dominating sets?

P_5-free graphs

Theorem (Bacsó and Tuza [11])

Every connected P_5-free graph has a dominating clique or a dominating P_3.

- In the context of 3-coloring ...
  - Any graph with a clique of size ≥ 4 is clearly not 3-colorable
  - Checking for cliques of size ≥ 4 takes polynomial time
- $\therefore$ it suffices that P_5-free graphs without cliques of size ≥ 4 have constant bounded dominating sets

[11] Bacsó and Tuza. 1990.

Our results

- Excepting **reducible configurations**, ...
  - $\{P_6, \text{triangle}\}$-free graphs have constant bounded dominating sets
  - $\{P_7, \text{triangle}\}$-free graphs have constant bounded dominating sets

Preliminaries

Reducible configurations

- Dominating vertices:
    - **Dominating vertex**: v such that there exists u where $N(u) \subseteq N(v)$
    - **Twin**: v such that there exists u where $N(u) = N(v)$
- Vertices with degree < 3

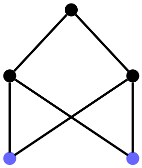

Figure: A pair of twins.

[12] Chudnovsky. 2014.

Reducible configurations

- Nontrivial homogeneous pairs of stable sets:
  - **Homogeneous pair**: disjoint, non-empty $U, V \subseteq V(G)$ such that every vertex not in $U \cup V$ is complete or anticomplete to U, and similarly with V
  - (U, V) is **nontrivial** if there exists an edge b/w U and V, and $|U| + |V| \geq 3$
  - (U, V) is **stable** if U and V are stable
  - Relax to bipartite graphs in $\{P_7, \text{triangle}\}$-free graphs case

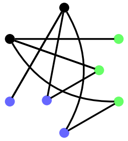

Figure: A nontrivial homogeneous pair of stable sets.

[13] Chudnovsky. 2014.

$\{P_6, \text{triangle}\}$-free graphs

Setup

- **Clebsch graph**: five-dimensional cube graph, identifying all pairs of opposite vertices
  - H is **Clebschian** if it is contained within the Clebsch graph
- **Climbable graph**:
  - Construct H_n by taking $K_{n,n} = (\{v_i\}_{i \in [n]}, \{u_i\}_{i \in [n]})$ + subdividing each edge (v_i, u_i) (add w_i)
  - H is **climbable** if it is contained within H_n for some n

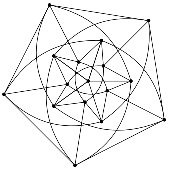

Figure: The Clebsch graph.

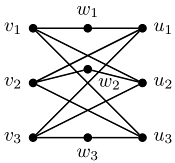

Figure: A climbable graph.

Setup

- V_8 **graph**: $C_8 = \{v_i\}_{i \in [8]}$ with an edge b/w all pairs of opposite vertices
  - Bipartite $H = (U, V)$ is an **antisubmatching** if every vertex in U has $\leq$ one non-neighbor in V, and vice versa
  - V_8 **expansion**:
    - Let $H_{1,5} = (V_1, V_5)$, $H_{3,7} = (V_3, V_7)$ be antisubmatchings
    - Take the V_8 graph
    - Replace (v_1, v_5) with (V_1, V_5), and (v_3, v_7) with (V_3, V_7)
    - Delete some vertices in $\{v_2, v_4, v_6, v_8\}$ (or none)

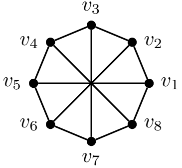

Figure: The V_8 graph.

Setup

- **Simplicial**: homogeneous pair of stable sets (U, V) such that every vertex not in $U \cup V$ with a neighbor in U is adjacent to every vertex not in $U \cup V$ with a neighbor in V

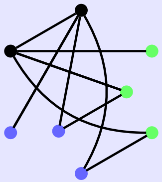

Figure: A nontrivial simplicial homogeneous pair of stable sets.

Setup

Theorem (Chudnovsky *et al.* [14])

If G is a connected $\{P_6, triangle\}$-free graph with no twins, then either

1. *G is Clebschian, climbable, or a V_8 expansion, or*
2. *G admits a nontrivial (simplicial) homogeneous pair of stable sets.*

[14] Chudnovsky et al. 2018.

Proof

- Clebschian: # of vertices in G is bounded by 16
- Climbable:
  - Recall H_n: subdivide (v_i, u_i) in $K_{n,n}$, adding w_i
  - $\deg(w_i) \leq 2$ for all $w_i \in V(G)$ (reducible configuration)
  - $\therefore$ G is an induced subgraph of $K_{n,n}$ (dominating set ≤ 2)
- V_8 expansion:
  - For odd i, let $D_i = V_i$ if $|V_i| = 1$, else $\{x_i, y_i\}$ for any $x_i, y_i \in V_i$
  - For even i, let $D_i = \{v_i\}$ if $v_i \in V(G)$, else $\emptyset$
  - $D = \cup_{i \in [8]} D_i$ is a dominating set
- Nontrivial (simplicial) homogeneous pair of stable sets: reducible configuration

$\{P_7, \text{triangle}\}$-free graphs

Setup

- G is not bipartite $\rightarrow$ G contains C_5 or C_7
- If G is C_5-free:
  - $V(G) = V_1 \cup \ldots \cup V_7$
  - V_i is nonempty + stable
  - V_i is complete to V_{i+1}
  - Take $v_i \in V_i$ for each i $\rightarrow$ dominating set
- $\therefore$ G contains a C_5, $C = \{c_1, \ldots, c_5\}$ (**base graph**)

[15] Bonomo, Schaudt, and Stein. 2014.

Setup

- **Anchors**: neighborhood of C:
  - **Clones**: vertices adj to c_{i-1} and c_{i+1} (index i)
  - **Leaves**: vertices adj to c_i (index i)
- **Linkers**: $E = V(G) \setminus (C \cup N(C))$
  - Components are singletons or edges
  - Edge components are anticomplete to leaves

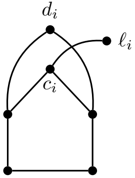

Figure: A clone d_i and a leaf ℓ_i, both on index i.

[16] Bonomo, Schaudt, and Stein. 2014.

Proof

- Assume G has no constant bounded dominating set
- $\therefore$ non-constant stable set of linkers E' with pairwise disjoint neighborhoods
  - **k-independent set**: $U \subseteq V(G)$ such that distance b/w any pair of vertices in U is $> k$
  - $\gamma(G) \leq 11\alpha_2(G) - 5$, where α_2 is the 2-independence number and γ is the domination number [17]
- Each linker $e^r \in E'$ is adj to at least 2 anchors ($\deg(e^r) \geq 3$)
- Use PHP so each e^r is adj to the same 2 types + indices of anchors a^r, b^r

[17] Du and Wan. 2013.

a^r, b^r have different types or indices

- Main idea:
  - e^r must be adjacent to either another linker or a third anchor
  - Consider all possible edges
  - Either there exists a P_7 or a triangle, or e^r is adj to 2 anchors of the same type and index

Code opt: a^r, b^r have different types or indices

- `check_base_anchors`:
  - Consider all possibilities of a^r, b^r
  - Consider all possible edges between a^r, b^s for $r \neq s$
  - Return the edges that do not create a triangle or a P_7 in a dict, and the graphs corresponding to these cases
- `check_add_rep`:
  - Add another e^r to E', with the requisite adj anchors
  - Consider all possible edges between the new anchors and the previous anchors, based on a dict of allowable edges
  - Update the dict of allowable edges, and return the graphs corresponding to these cases

Code opt: a^r, b^r have different types or indices

- `check_add_anchor`:
  - Similar to `check_add_rep`, except add a new anchor adj to each e^r in E'
- `check_add_linkers`:
  - Similar to `check_add_rep`, except add a new linker adj to each e^r in E'

Overall: Add repetitions (3), anchors, and linkers, and show that in every scenario, e^r is adj to 2 anchors of the same type and index

a^r, b^r have the same type and index

- Main idea:
  - b^r dominates a^r (and vice versa) $\rightarrow$ there exists d_a^r adj to a^r but not b^r (similarly for d_b^r)
    - $d_a^r, d_b^r \neq d_a^s, d_b^s$: otherwise, if WLOG $d_a^r = d_a^s$, $b^s, e^s, a^s, d_a^r, a^r, e^r, b^r$ is a P_7
  - Prove that d_a^r and d_b^r are non-adj to e^s for all $r \neq s$
  - c_i dominates e^r $\rightarrow$ there exists d_e^r adj to e^r but not c_i
    - $d_e^r \neq d_a^s, d_b^s$ by the previous point
  - Consider all other possible edges + show there exists a P_7 or a triangle

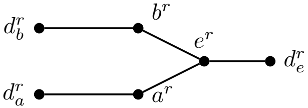

Figure: An induced subgraph of G, based on vertices introduced thus far.

Pseudocode: a^r, b^r have the same type and index

```
function IS_ALL_CONTRA(g, nonedges_set)
    isg ← triangle or P7 as an induced subgraph of g, if one exists
    if isg is None then
        return False
    is_contra ← True
    for all nonedge in (non_edges of isg) do
        if nonedge not in nonedges_set then
            nonedges_set.update(nonedge)
            g_new ← g.copy()
            g_new.add_edge(*nonedge)
            is_contra ← is_contra and is_all_contra(g_new, nonedges_set)
            if not is_contra then
                return is_contra
    return is_contra
```

d_a^r, d_b^r are non-adjacent to e^s

- Each d_a^r is adj to at most one vertex in $E' \rightarrow$ there are $\leq |E'|$ edges between $\{d_a^r\}_r$ and E'
- Construct H where $V(H) = \{h_r\}_r$, $E(H) = \{(h_r, h_s) \mid (d_a^r, e^s) \in E(G)\}$
- There exists a stable set in H of size $\geq \sum_{h_r}(1 + \deg(h_r))^{-1}$ [18]
- $\sum_{h_r}(1 + \deg(h_r))^{-1} \geq |E(H)|/3 \geq |E'|/3$ (by AM-HM)
- Remove all e^r such that h_r is not in this stable set
- E' remains nonconstant and all vertices satisfy the proposition

[18] Wei. 1981.

Conclusion

Conclusion

- Showed that excepting reducible configurations,
  - $\{P_6, \text{triangle}\}$-free graphs have constant bounded dominating sets
  - $\{P_7, \text{triangle}\}$-free graphs have constant bounded dominating sets (with a semi-automatic proof)
- Future work:
  - Extend work to P_6-free and P_7-free graphs
  - Potentially finding constant bounded dominating sets in P_8-free graphs

Thank you!