Universal Properties, Abstraction Barriers, Isomorphism Invariance, and Homotopy Type Theory

Jason Gross

Course Information
Course Number: TBD
Credit Hours: 3-0-9
Meeting Times: TBD
Classroom Location: TBD
Website URL: TBD

Instructor Information
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Course Description

Mathematicians interact with many mathematical objects over their courses of study. We learn in grade school about natural numbers, integers, rational numbers, and later real numbers; we talk about functions in high school. We perform geometric constructions. In undergrad mathematics, we might interact with matrices, linear transformations, vectors, vector spaces, sets, topological spaces, groups, rings, fields, modules, and more. Finally, we read and write proofs, which, depending on whom you ask, might or might not be mathematical objects in and of themselves.

Through osmosis, you may have picked up a vague notion of what a mathematical object is allowed to be, and what it means to have one. Some basic questions, though, have likely been swept under the rug. Is a red "3" the same as a blue "3"? If you take the formal set-theoretic definition of natural numbers where $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, …, is this 0 "the same as" the 0 you'd get if you started off with $0 = \{\emptyset\}$, $1 = \{0\}$, $2 = \{0, 1\}$, …?

In this class, I will aim to convey a particular mathematical aesthetic, a lens through which you can look at not just the world of math, but the world at large. The underlying, unifying thread of this class will be the question "What is a mathematical object?", or, said another way, "What is the structure of the rules by which we play this game of math?"

We'll be covering some basic category theory: a universal property is a way of pinning down a mathematical object in a way that is unique up to isomorphism (and the isomorphism itself is in fact unique). We'll look at the relationship between abstraction barriers and APIs in computer science and mathematical objects. Finally, we'll dive into some basic Homotopy Type Theory, an exciting new field of math which provides an alternative to set theory as mathematical foundations, and allows stating, proving, and using the Univalence Axiom, that isomorphic objects can be considered equal.

Prerequisites/Corequisites

Intended Learning Outcomes

By the end of this course, students will be able to. . .

- see the world through the lens of objects being defined only and uniquely by their relationships with other objects; (Note: I have no idea how to measure/assess this one)
- know and understand the basic category-theoretic definitions—that is, students should feel comfortable using fluidly definitions including those of category, functor, natural transformation, isomorphism, initial and terminal objects, and adjoint functors, and will be able to justify why each component of the definition is useful or necessary, and why no additional properties are needed;
- understand isomorphisms, universal properties, and what it means to identify something uniquely up to unique isomorphism;
- approach problems from a morphism-centric or relation-centric view rather than an object-centric one;
- identify when two theorems, conjectures, proofs, or problems in different areas of math are in fact the same;
- describe their frames, lenses, or aesthetics for approaching problems, as well as discover others' lenses;
- feel comfortable with category-theoretic language and framing;
- turn standard mathematics definitions into definitions phrased in terms of universal properties