Introduction to Abstraction

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Summary

In this class, we will begin our study of abstraction. The essential idea behind abstraction is that, having built a compound object out of multiple components, we can throw away our knowledge of how the compound object was *constructed* without losing anything important. We will cover a couple examples of abstraction. Finally, we will look at examples of respecting abstraction barriers and of piercing them.

Intended Learning Outcomes

By the end of this session, students will be able to...

- define "abstraction"
- give at least one example of abstraction in mathematics, and justify why it's an example of abstraction
- give at least one example of abstraction in computer science, and justify why it's an example of abstraction
- give examples of operations that pierce abstraction barriers
- give examples of operations that respect abstraction barriers

Lesson Plan

Class Components	Description of Component	What will the teacher do?	What will the students do?	Justification or Rational[1]
1. Introduce class topics & communicate goals				
2. Introduce a specific idea / topic / piece of course content.	Definition of “abstraction”	First ask students to define abstraction themselves, then give definition	Attempt to recall, construct, or predict a definition; then listen to the given definition	Abstraction forms the basis for this entire unit. Asking students for the definition first engages prior knowledge and invites students’ attention and active engagement; having a definition sets them up to engage with examples
3. Demonstrate the idea / topic.	Example: defining the natural numbers in set theory	Recall the definition of sets. Describe the way we define natural numbers in formal set theory. Solicit questions.	Listen, think, ask questions	This example is a relatively simple one (perhaps still to complex though?), and giving the example sets the stage for the next questions which will require the students to make sense of this example in the context of abstraction.

Class Components	Description of Component	What will the teacher do?	What will the students do?	Justification or Rational[1]
4. Provide opportunities for students to apply and integrate new information	How is the definition of natural numbers an example of abstraction?	Ask students to think about what makes this an example of abstraction. Perhaps ask students to share thoughts. Share that in order to justify something as an example of abstraction, we need to point at (a) the compound object(s) we built; (b) the components we built it from; (c) what it means to throw away our knowledge of how the compound object was constructed; and (d) how to justify that we haven't lost anything important. Ask students to think again in light of this lens, and share what makes this an example of abstraction.	Think about how to integrate the definition of abstraction with this example. Communicate how to justify this example as an abstraction.	The first thing to learn, I think, after knowing the definition of abstraction, is knowing how to apply that definition to existing abstractions. This component asks students to engage with that application, and sets the stage for them to eventually think about which things are and are not abstractions. Finrally, this component will not fully answer (c) and (d), which I will ask students to keep in the back of their mind as we look at more examples.

Class Components	Description of Component	What will the teacher do?	What will the students do?	Justification or Rational[1]
3(b). Demonstrate the idea / topic.	Example: defining finite-domain functions as lists of pairs	Describe how to define finite-domain functions as lists of pairs. Solicit questions.	Listen, think, ask questions	This is another example similar to the natural number one, and I hope it will help solidify the knowledge of the students to see two different examples that use set theory to build compound objects. Furthermore, using lists of pairs rather than sets of pairs sets up the students to think about how some details of the construction can be irrelevant.
4(b). Provide opportunities for students to apply and integrate new information	How is the definition of functions an example of abstraction?	Ask students to recall what the parts of justifying an example of abstraction Ask students what makes this an example of abstraction.	Recall knowledge they were given recently. Think about integrating the definition of abstraction with this example, and respond with their thoughts.	Same as above.

Class Components	Description of Component	What will the teacher do?	What will the students do?	Justification or Rational[1]
3(c). Demonstrate the idea / topic.	Example: defining rational numbers as pairs of integers	Ask students how rational numbers can be defined as pairs of integers, and to specify in particular which subset of all pairs of integers we're considering, and how to define injection of the integers into the rationals, addition, and multiplication.	Recall prior mathematical knowledge about the components we're using. Come up with ideas for how to represent a compound object with these components.	This example provides the students with their first opportunity to attempt to construct the abstraction, given the target compound object and the components. I hope that it will be relatively obvious what the idea should be (numerator-denominator pairs), while still containing enough subtlety for the students to chew on (does the gcd need to be 1? if so, multiplication is non-trivial; if not, we get duplicate representation; note also we must forbid 0 in the denominator). This is a step towards being able to both recognize and construct abstractions in the wild. Additionally, it sets up an example of how you might define an abstraction in multiple ways (gcd = 1, and gcd not required to equal 1), and sets us up for seeing how this impacts what abstraction-breaking operations are available.

Class Components	Description of Component	What will the teacher do?	What will the students do?	Justification or Rational[1]
4(c). Provide opportunities for students to apply and integrate new information	How is this definition an example of abstraction?	Ask students to think about what makes this an example of abstraction, and then to share.	Think about how to integrate the definition of abstraction with this example. Communicate how to justify this example as an abstraction.	Same as above

Class Components	Description of Component	What will the teacher do?	What will the students do?	Justification or Rational[1]
3(d). Demonstrate the idea / topic.	Natural Number in Set Theory: Abstraction-Breaking, Abstraction-Respecting	Ask students to recall the definition of natural numbers using set theory. Point out the compound objects: 0, successor, and $\mathbb{N}$. Name this as our *API*, *interface*, or *abstraction barrier*. Ask how to define the function $+2$ in terms of successor. Ask students to think about how to define the function that is the identity on 0 and -1 on the other naturals. Describe induction and recursion and computation rule of recursion. Define predecessor. Solicit questions. Ask how to define addition, provide definition if students don't think of it. Ask how to define max. Point out two different definitions of max, one that uses only 0, successor, and recursion, and one that uses the definition-as-sets. Ask for two different definitions of min. Point out abstraction-respecting and abstraction-piercing.	Recall knowledge learnt in this class. Use knowledge of mathematics to define simple functions, subject to constraints about building blocks. Listen and engage with definition of induction and recursion. Ask questions about their confusions. Communicate their thoughts about how to define operations.	This example introduces the concept of interface/API/abstraction-barrier, which will hopefully be at least somewhat familiar to the students. I hope to guide students through seeing how to define operations in terms of the interface, and also to recognize when operations pierce the abstraction barrier. This is almost the culmination of the package of what an abstraction barrier is; it lets students see when given abstractions may not be sufficient.

Class Components	Description of Component	What will the teacher do?	What will the students do?	Justification or Rational[1]
4(d)(i). Provide opportunities for students to apply and integrate new information	Abstraction-Respecting and Abstraction-Piercing in the other examples	Ask students to recall the example of finite functions as lists of pairs Describe the interface: function creation, function calling, and equality of functions. Point out issues with list duplicates and missing elements, and function creation. Ask students to define how to glue two functions together, both in a way that pierces the abstraction barrier, and in a way that doesn't; use a concrete example for function gluing.	Recall recently given knowledge. Listen and ask questions. Construct an example of abstraction-barrier-piercing and abstraction-barrier-respecting.	This component gives students the opportunity to integrate and apply the knowledge from 3(d) to the example that we worked before. It still scaffolds their learning by providing the interface and drawing their attention to a particular example of an operation that can be easily defined in both an abstraction-breaking and an abstraction-preserving way.

Class Components	Description of Component	What will the teacher do?	What will the students do?	Justification or Rational[1]
4(d)(ii). Provide opportunities for students to apply and integrate new information	Abstraction-Respecting and Abstraction-Piercing in rational numbers as pairs of integers	Ask students to recall the definition(s) of the compound object and the example. Write down both definitions (gcd = 1, and gcd allowed to be anything). Ask students to think about the interface, then pair up (or perhaps groups of three) and discuss what interface they think should apply to each abstraction—tell students to attend to whether or not the interfaces are the same. Ask for volunteers to share, and share sufficient interfaces (division of integers into rationals; reduced numerator & denominator; axiom that gcd = 1 and deominator is positive $\implies$ that we get the same numerator and denominator out that we put in; extensional equality). Ask students to come up with an operation that has both an abstraction-piercing way of defining it and an abstraction-respecting one, and to write down both.	Same as above, but this time with a bit less scaffolding, on an example that is in some ways easier and in some ways more nuanced.	Same as above, but with a bit less scaffolding.

Class Components	Description of Component	What will the teacher do?	What will the students do?	Justification or Rational[1]
5. Summarize topics/material; Look ahead to next class				Note that we haven't covered what makes an abstraction barrier sufficient, and we'll be looking at that in the upcoming classes.

[1]

- Why have you decided to include this component?
- How does this component sequence or scaffold learning?