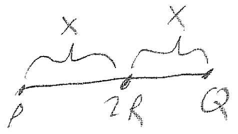
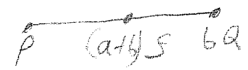
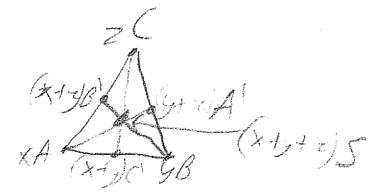
VM&S





$\frac{PS}{QS} = \frac{b}{a}$



DEF: Affine combination of points P & Q is: $aP+bQ$, $a+b=1$

THM: Ceva's Theorem: In $\triangle ABC$ (top), $\exists x,y,z$ s.t. $\frac{AC'}{C'B}\cdot\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}=1$, $x+y+z=1$

Pf: $xA+yB=(x+y)C'$ | $yB+zC=(y+z)A'$ | $xA+zC=(x+z)B'$ | $\frac{AC'}{C'B}=\frac{y}{x}$ | $\frac{BA'}{A'C}=\frac{z}{y}$ | $\frac{AB'}{B'C}=\frac{z}{x}$ | $\frac{AC'}{C'B}\cdot\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}=\frac{y\cdot z\cdot x}{x\cdot y\cdot z}$

For $AB \parallel CD$, if $\exists$ 4 pts xA, yB, zC, wD s.t. $xA+zC=yB+wD$, $AB \parallel CD$

DEF: Polar vector is a "directed line segment" with a magnitude: tail $\longrightarrow$ head

2 polar vectors $\vec{PQ}$ & $\vec{RS}$ are same: $Q+S=P+R \Leftrightarrow \vec{PQ}=\vec{SR}$

DEF: Covector is a "directed pair of lines":

DEF: axial vector is an axis of rotation, a magnitude, and a "sense of rotation"

DEF: a basis for a vector space, V, is a set $\{\bar{e}_i\}$ s.t. $\forall \bar{v}\in V$, $\bar{v}=\sum a^i\bar{e}_i$

Polar/covector bases: $\bar{e}_1$, $\bar{e}_2$, $\bar{\omega}^1$, $\bar{\omega}^2$

DEF: twisted scalar is a point with a rotation: (in 3D it has a handed mass)

DEF: Bivector is an area w/ rotation:

DEF: Twisted vector is a magnitude and handedness: $\bar{v}=$

DEF: A vector space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R},\mathbb{C}$) is a set of V things, called vectors such that: $\forall \bar{v},\bar{w}\in V$, $\bar{v}+\bar{w}\in V$, $\forall a\in\mathbb{K}$, $a\bar{v}\in V$, and: $\forall \bar{v},\bar{w},\bar{u}\in V$, $a,b\in\mathbb{K}$,

assoc. 1) $\bar{v}+(\bar{w}+\bar{u})=(\bar{v}+\bar{w})+\bar{u}$ | 2) comm $\bar{v}+\bar{w}=\bar{w}+\bar{v}$ | 3) $\exists\,\bar{0}\in V$ s.t. $\bar{0}+\bar{v}=\bar{v}$ | 4) $\exists(-\bar{v})$ s.t. $\bar{v}+(-\bar{v})=\bar{0}$

5) $a(b\bar{v})=(ab)\bar{v}$ | 6) $1\cdot\bar{v}=\bar{v}$ | 7) $a(\bar{v}+\bar{w})=a\bar{v}+a\bar{w}$ | 8) $(a+b)\bar{v}=a\bar{v}+b\bar{v}$

DEF: A linear combination in V is an expression $\sum a_i\bar{v}_i$

Given a set $S\subset V$ its span is the set $\langle S\rangle$ of all linear combinations of elements in V

$\langle\bar{v},a\bar{v}\rangle=\langle S\rangle$

DEF: a set S is degenerate ~~if~~ • some $\bar{v}\in S$ is a linear combination of vectors $\bar{s}\neq\bar{v}$, but in S • $\exists$ a linear combination equal to $\bar{0}$, w/ at least one $a_i\neq 0$

degenerate = dependent

DEF: S is a basis of V if S is independent and $\langle S\rangle=V$

DEF: A vector space V is finite dimensional if $V=\langle S\rangle$ for a finite S

THM: V is F-D iff all bases of V have the same cardinality

Lemma: If $V=\langle S\rangle$, and no subset of S spans V, S is independent

Pf: BWOC, $V=\langle S\rangle$ & $\nexists P\subset S$ spans V, and S is dependent. Then, $S-\{\bar{v}\}$ spans V ⇒⇐

Cor: Any finite spanning set contains a basis

THM: If $|S|=n$, $T\subset\langle S\rangle$, with $|T|>n$, S is dependent

Cor: All bases of an F-D space have the same size covector · polar vector = scalar $\bar{\bar{w}}\cdot\vec{PQ} = PQ/PR$

polar vector + polar vector: $\vec{PQ}+\vec{QS}=\vec{PS}$

covector + covector: $\bar{\bar{w}}_1+\bar{\bar{w}}_2=\bar{\bar{w}}_3$

polar · polar = axial; $\vec{PQ}\cdot\vec{PR}=\vec{PS}$, $\vec{PS}\perp\vec{PQ},\vec{PR}$, length $\vec{PS}$ = area PQSR

point − point = polar;   P−Q

polar vector + point = point; $P+\vec{PQ}=S$ $\bar{\bar{w}}_1+\bar{\bar{w}}_2=\bar{\bar{w}}_3$

twisted vector + twisted vector; $\vec{AC}+\vec{CB}=\vec{AB}$

twisted vector · twisted scalar = polar;

twisted scalar · twisted scalar = scalar

wedge product   polar ∧ polar = bivector: $\vec{x}\wedge\vec{y}=-\vec{y}\wedge\vec{x}$

polar · polar = polar · covector; $QR=\frac{1}{QP}$, $PQ\cdot QS=\frac{PS}{QT}$

DEF: if X is a set, let $FX = \{a_1x_1 + \dots + a_nx_n \mid a_i \in \mathbb{K}, x_i \in X\}$ be the set of formal linear combinations

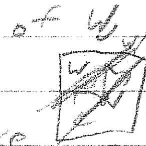
DEF: a function $f: V \to \mathbb{K}$ is a linear functional on V if it is a linear map

DEF: V^* = set of all linear functionals on V

DEF: $V \times W = \{\underbrace{(\bar{v}, \bar{w})}_{\bar{v} \times \bar{w}} \mid \bar{v} \in V, \bar{w} \in W\}$

V^*	covector $\bar{w}$
V	polar vector $\bar{v}$
$\tilde{V}$	axial (twisted) vector
$\Lambda^2 V$	bivector
$\tilde{\mathbb{K}}$	twisted scalar
$\mathbb{K}$	scalar
$\Lambda^3 V$	trivector
$\widetilde{\Lambda^2 V}$	
$\widetilde{\Lambda^3 V}$	
$\widetilde{V^*}$	
$\Lambda^2 V^*$	
$\widetilde{\Lambda^2 V^*}$	

Quotients: if V is a subspace of W, W/V = parts of vectors not in V:



DEF: $V \subset W$ means V is a subspace of W

DEF: $\bar{v} \in V$, $W \subset V$, $\bar{v} + W = \{\bar{v} + \bar{w} \mid \bar{w} \in W\}$

DEF: $V/W = \{\bar{v} + W \mid \bar{v} \in V\}$: $\bar{v}_1 + W = \bar{v}_2 + W$ iff $\bar{v}_1 - \bar{v}_2 \in W$

If V is FD, $\dim V/W = \dim V - \dim W$

DEF: If V, W are vector spaces, a linear map is a function $T: V \to W$ s.t. $T(\bar{v}_1 + \bar{v}_2) = T(\bar{v}_1) + T(\bar{v}_2)$ and $T(a\bar{v}) = aT(\bar{v})$

DEF: A (linear) isomorphism is a linear map which is a bijection

If there is an isomorphism from V to W, $V \cong W$ (V and W are isomorphic)

"Formal Products": $V \otimes W = \{\overbrace{(\bar{v}, \bar{w})}^{\bar{v} \otimes \bar{w}} \mid \bar{v} \in V, \bar{w} \in W\}$

Let $X = \{f: V \times W \to \mathbb{K} \mid f$ is a linear isomorphism (?) following rules ... and the ...

$(f+g)(\bar{v}, \bar{w}) = f(\bar{v}, \bar{w}) + g(\bar{v}, \bar{w}) \mid (af)(\bar{v}, \bar{w}) = af(\bar{v}, \bar{w})$

$f(\bar{v}) = a$ — $g(a\bar{v}) = 1$, $f(\text{else}) = 0$ — $g(\text{else}) = 0$

$$\text{Let } S = \left\{ \begin{array}{l|l} a(\bar{v} \otimes \bar{w}) - a\bar{v} \otimes \bar{w} & \bar{v}_i, \bar{v} \in V \\ a(\bar{v} \otimes \bar{w}) - \bar{v} \otimes a\bar{w} & \bar{w}_i, \bar{w} \in W \\ (\bar{v}_1 + \bar{v}_2) \otimes \bar{w} - \bar{v}_1 \otimes \bar{w} - \bar{v}_2 \otimes \bar{w} & a \in \mathbb{K} \\ \bar{v} \otimes (\bar{w}_1 + \bar{w}_2) - \bar{v} \otimes \bar{w}_1 - \bar{v} \otimes \bar{w}_2 & \end{array} \right\}$$

DEF: $V \otimes W = F(V \times W)/\langle S \rangle$

THM: if $\{\bar{e}_i\}$ is a basis for V, $\{\bar{f}_j\}$ is a basis for W, then $\{\bar{e}_i \otimes \bar{f}_j\}$ is a basis for $V \otimes W$

RE-DEF: $V \otimes W$ = formal linear combinations of $\{\bar{e}_i \otimes \bar{f}_j\}$

COR: $\dim(V \otimes W) = (\dim V)(\dim W)$

DEF: an $\binom{r}{s}$-tensor is an element of $\overbrace{V \otimes \dots \otimes V}^{r} \otimes \overbrace{V^* \otimes \dots \otimes V^*}^{s}$

eg: $\binom{1}{0}$-tensor $\in V$, $\binom{0}{1}$-tensor $\in V^*$, $\binom{0}{0}$-tensor $\in \mathbb{K}$

DEF: an orientation of V is a direction in $\Lambda^n V$, $\dim V = n$

DEF: $\tilde{\mathbb{K}} = \{(\Omega, a) \mid \Omega$ is an orientation of V, $a \in \mathbb{K}$, $a > 0\} \cup \{0\}$

$\dim(\tilde{\mathbb{K}}) = 1$

DEF: $\tilde{V} = \tilde{\mathbb{K}} \otimes V$

Question: $\tilde{V}$, $\Lambda^n V$ related?

DEF: $\Lambda^2 V = V \otimes V/\langle S \rangle$, $\dim(\Lambda^2 V) = \binom{n}{2}$, $\bar{v} \wedge \bar{w} \in \Lambda^2 V$

DEF: $\Lambda^3 V = V \otimes V \otimes V/\langle S \rangle$, $S = \left\{ \begin{array}{l|l} \bar{v} \otimes \bar{w} \otimes \bar{u} + \bar{v} \otimes \bar{u} \otimes \bar{w} & \bar{v}, \bar{w}, \bar{u} \in V \\ \bar{v} \otimes \bar{w} \otimes \bar{u} + \bar{w} \otimes \bar{v} \otimes \bar{u} & \end{array} \right\}$

$\dim \Lambda^n V = \binom{n}{?}$

Rotations:

How do rotations act on vectors?

- Preserve length, addition, scalar multiplication, angles, dimension of subspace
- don't kill stuff
- preserves orientation

How can we define length & angle in abstract vector space

- pick a basis

$\bar{v} = v^1\bar{e}_1 + v^2\bar{e}_2$

some $f(v^1, v^2) =$ "length of $\bar{v}$"

$\sqrt{(v^1)^2 + (v^2)^2}$???

Define $\|\bar{v}\| \doteq \sqrt{\theta(\bar{v})\cdot\bar{v}}$, where θ is an isomorphism from V to V^*

Questions:

- Is $\bar{v}_1 \cdot \bar{v}_2 = \bar{v}_2 \cdot \bar{v}_1$ ~~not~~ No
- Is $\|\bar{v}\| \geq 0$ No, but we can make it

$\boxed{K = \mathbb{R}}$ We want to consider θ s.t. $\theta(\bar{v})(\bar{v}) \geq 0$, and if $\bar{v} \neq \bar{0}$, then $\theta(\bar{v})(\bar{v})$

Metric Structure

- bases s.t. $\bar{e}_i \cdot \bar{e}_j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$ $\longrightarrow$ $\theta: V \cong V^*$ s.t. $\theta(\bar{v})(\bar{v}) \geq 0$, $\theta(\bar{v})(\bar{v}) > 0$ if $\bar{v} \neq \bar{0}$ and $\theta(\bar{v}_1)(\bar{v}_2) = \theta(\bar{v}_2)(\bar{v}_1)$

DEF: Such a basis is orthonormal

DEF: an inner product / dot product:

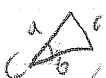
Bilinear map $\cdot: V \times V \to K$ s.t.

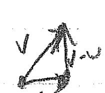
- nondegenerate: if $\bar{v}_1 \cdot \bar{v}_2 = 0 \ \forall \bar{v}_2$, then $\bar{v}_1 = 0$
- $\bar{v}\cdot\bar{v} \geq 0$, $\bar{v}\cdot\bar{v} \neq 0$ if $v \neq 0$
- $\bar{v}_1 \cdot \bar{v}_2 \doteq \bar{v}_2 \cdot \bar{v}_1$

DEF: If V is a vector space equipped with an inner product, it is called a Euclidean space

DEF: $\|\bar{v}\| = \sqrt{\bar{v}\cdot\bar{v}}$ (length of $\bar{v}$)

Angles: law of cos: $c^2 = a^2 + b^2 - 2ab\cos C$





$\|v-w\|^2 = \|v\|^2 + \|w\|^2 - \dots$

DEF: $\cos\varphi = -\left(\frac{\|v-w\|^2 - \|v\|^2 - \|w\|^2}{2\|v\|\|w\|}\right)$ $\quad \|v-w\|^2 = (v-w)\cdot(v-w)$ $\quad 2\|v\|\|w\|\cos\varphi$

$= v\cdot v - v\cdot w - w\cdot v + w\cdot w = \|v\|^2 + \|w\|^2 - 2\, v\cdot w$

$\cos\varphi = \frac{v\cdot w}{\|v\|\|w\|}$

In particular: $v \cdot w = 0 \Leftrightarrow v = 0$, $w = 0$, or v and w are "perpendicular"

DEF: a linear map or linear transformation a map that preserves + and scalar multiplication.

dot product preserves length & angles

DEF: a lin. map $L: V \to V$ is called orthogonal if it preserves the dot product $(L\bar{v} \cdot L\bar{w}) = (\bar{v} \cdot \bar{w})$

DEF a rotation transformation is an orthogonal linear map that preserves orentation

DEF: L preserves orientation if $\Lambda^n L$ is "not a flip"

In 2D

If V has a basis $\{\bar{e}_1, \bar{e}_2\}$, $\Lambda^2 V$ has basis $\{\bar{e}_1 \wedge \bar{e}_2\}$ so $L\bar{e}_1 \wedge L\bar{e}_2 = a(\bar{e}_1 \wedge \bar{e}_2)$ L preserves orentation $\Leftrightarrow a > 0$

FACT: (Cauchy-Schwartz inequality) $\langle v, w \rangle^2 \leq \langle v, v \rangle \cdot \langle w, w \rangle \Leftrightarrow |\langle v, w \rangle| \leq \|v\| \cdot \|w\|$

$\boxed{\text{Matrices}}$

How do we represent a linear map $L: V \to W$ in coordinates?

choose bases $\{d_i\}$, $\{e_j\}$ for V and W respectively

$\bar{v} = \sum v^i \bar{d}_i$, $\bar{w} = \sum w^j \bar{e}_j$

Write $L(\bar{d}_i) = \sum l_i^j \bar{e}_j$ Then $L(\bar{v}) = L(\sum v^i d_i) = \sum_{i=1}^{n} v^i L(d_i) = \sum_{i=1}^{n} v^i \sum_{j=1}^{m} l_i^j \bar{e}_j$

We write

$$\begin{pmatrix} l_1^1 & l_2^1 & \cdots & l_n^1 \\ l_1^2 & l_2^2 & \cdots & l_n^2 \\ \vdots & \vdots & & \vdots \\ l_1^m & l_2^m & \cdots & l_n^m \end{pmatrix}$$

this is an $m \times n$ matrix

If $A: V \to W$, $B: W \to U$, A has matrix $\begin{pmatrix} a_i^j & \cdots \\ \vdots & \end{pmatrix}$ B has $\begin{pmatrix} b_j^k & \cdots \\ \vdots & \end{pmatrix}$ in some bases $\{c_i\}$ $\{d_j\}$ $\{e_k\}$ Then $B \circ A$ has matrix $\begin{pmatrix} g_i^k \end{pmatrix}$ Then

$B(A(\bar{c}_i)) = B(\sum_j a_i^j d_j) = \sum a_i^j B(d_j) = \sum_j a_i^j \sum b_j^k \bar{e}_k$ $\quad B \circ A(c_i) = \sum_k g_i^k \bar{e}_k$

$= \sum_k (\sum_j a_i^j b_j^k) \bar{e}_k$ so $g_i^k = \sum_j a_i^j b_j^k$

$$\begin{pmatrix} b_j^k & \\ \vdots & \end{pmatrix} \begin{pmatrix} a_i^j & \\ \vdots & \end{pmatrix} = \begin{pmatrix} g_i^k \\ \end{pmatrix}$$

inner products of multiplication of row and column vectors

V is n-dim $\bar{v} \in V$ choose a basis $\{\bar{c}_i\}$

$\bar{v} = \sum v^i \bar{c}_i$, $M_{\{c_i\}}(\bar{v}) = M(\bar{v}) = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$

W is m-dim
$A: V \to W$ chose basis $\{\bar{c}_i\}$ & $\{\bar{d}_j\}$ $A\bar{c}_i$

$A\bar{c}_i = \sum a_i^j \bar{d}_j$ $M(A) = \begin{pmatrix} a_1^1 & \cdots & \\ \vdots & \ddots & \\ \vdots & & a_n^m \end{pmatrix}$

U is p-dim $B: W \to U$, $\{c_i\}$, $\{d_j\}$, $\{\bar{e}_k\}$

$M(B) = \begin{pmatrix} b_1^1 & \cdots & \\ \vdots & \ddots & \\ \vdots & & b_m^p \end{pmatrix}$ $M(B \circ A) = M(B)M(A)$

DEF if $M = \begin{pmatrix} a_1^1 & & \\ & \ddots & \\ & & a_n^m \end{pmatrix}$ $N = \begin{pmatrix} b_1^1 & & \\ & \ddots & \\ & & b_m^p \end{pmatrix}$

$NM = \begin{pmatrix} g_1^1 & & \\ & \ddots & \\ & & g_n^p \end{pmatrix}$ $g_i^k = \sum_{j=1}^{m} a_i^j b_j^k$

THM Matrix multiplication is associative
Pf: Composition of functions is associative

$\bar{w} \in V^*$ choose dual basis $\{\bar{f}^i\}$ $\bar{w} = \sum w_i \bar{f}^i$ $M(\bar{w}) = (w_1 \cdots w_n)$

$\bar{w}(\bar{v}) = w_1 v^1 + \dots + w_n v^n = M(\bar{w})M(\bar{v}) = (\bar{w}_1 \cdots \bar{w}_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$A: V \to W$
$M(A\bar{v}) = M(A)M(\bar{v}) = \begin{pmatrix} a_1^1 & \cdots & a_n^m \end{pmatrix}\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$

V^* = lin. maps $V \to \mathbb{K}$, use $\{1\}$ for $\mathbb{K}$, special case is $(\cdots)$

Identity Matrix $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} = M(Id)$, where $Id(\bar{v}) = \bar{v}$

Given $\bar{v} \in V$, define $A_v : \mathbb{K} \to V$, $A_v(a) = a\bar{v}$, $M(A_v) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = M(\bar{v})$

$\mathbb{K} \xrightarrow{A_v} V \xrightarrow{A} W$

$1 \mapsto \bar{v} \mapsto A\bar{v}$ $\quad M(A \circ A_v) = M(A)M(A_v) = M(A)M(\bar{v})$

$\mathbb{K} \xrightarrow{A_{\bar{v}}} V \xrightarrow{\bar{w}} \mathbb{K}$

$1 \mapsto \bar{v} \longmapsto w(\bar{v})$

$M(\bar{w} \circ A_{\bar{v}}) = M(\bar{w})M(\bar{v})$

If we have an inner product, we should choose orthogonal bases.

$\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \begin{pmatrix} w^1 \\ \vdots \\ w^n \end{pmatrix}$

$\downarrow$

$(v^1 \cdots v^n) \begin{pmatrix} w^1 \\ \vdots \\ w^n \end{pmatrix} = v^1 w^1 + \ldots + v^n w^n$

$\theta(\bar{v}) \in V^*$

~~DEF~~. The *transpose* of $M = \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \\ a_1^m & & a_n^m \end{pmatrix}$ is $M^T = \begin{pmatrix} a_1^1 & \cdots & a_1^m \\ \vdots & \ddots & \\ a_n^1 & & a_n^m \end{pmatrix}$

$\bar{v} \cdot \bar{w} \longrightarrow M(\bar{v})^T M(\bar{w})$

rotations $\longrightarrow$ $R : V \to V$ is orthogonal if $(R\bar{v}) \cdot (R\bar{w}) = \bar{v} \cdot \bar{w}$

· Try on basis vectors. Let $\{\bar{e}_i\}$ be orthonormal $\begin{pmatrix} r_1^1 & & \\ & \ddots & \\ & & r_n^n \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} =$

$\begin{pmatrix} r_i^1 \\ \vdots \\ r_i^n \end{pmatrix}$ The cols of $M(R)$ are coordinates of image of basis vectors $\quad \underline{M(R)} \quad M(\bar{e}_i)$

$M(R)^T M(R) = I$ dentity matrix

$(MN)^T = N^T M^T$

~~Ex~~ R is orthogonal iff $M(R)^T M(R) = I$

orientation preserving!

Ex: $L : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \bar{e}_1 \longmapsto \bar{e}_1 + \bar{e}_2$ $\quad$ Not ~~pres~~ ~~ove~~ ~~ora~~

$\bar{e}_2 \longmapsto \bar{e}_1 - \bar{e}_2$

Ex: $L: \mathbb{R}^3 \to \mathbb{R}^3$

$\bar e_1 \mapsto 0\bar e_1 + 2\bar e_2 - 5\bar e_3$

$\bar e_2 \mapsto 0\bar e_1 - \bar e_2 + 3\bar e_3$

$\bar e_3 \mapsto -\bar e_1 + \bar e_2 + \bar e_3$

DEF. A linear map $L: V \to V$ perserves orentation iff $v \wedge w = k(Lv \wedge Lw)$ for some $k > 0$

if $\bar e_1, \dots, \bar e_n$ is a basis of V,

$\Lambda^m V$ = vectorspace with basis $\{\bar e_i \wedge \dots \wedge \bar e_i\}$ s.t. $\bar e_i \wedge \bar e_j = -\bar e_j \wedge \bar e_i$ and $\bar e_i \wedge \bar e_i = \bar 0$

$\dim V = n$, basis for $\Lambda^n V$ is $\bar e_1 \wedge \bar e_2 \wedge \dots \wedge \bar e_n$

If $L: V \to V$ is linear map, then $\Lambda^n L: \Lambda^n V \to \Lambda^n V$ s

$\Lambda^n L(\bar e_1 \wedge \bar e_2 \wedge \dots \wedge \bar e_n) = L\bar e_1 \wedge \dots \wedge L\bar e_n$

$\Lambda^n L(\bar e_1 \wedge \dots \wedge \bar e_n) = L\bar e_1 \wedge \dots \wedge L\bar e_n = (\det L)\, \bar e_1 \wedge \dots \wedge \bar e_n$

DEF. The determinant of L is the number that fits

ex: $L: \mathbb{R}^2 \to \mathbb{R}^2$

$L(\bar e_1) = a\bar e_1 + b\bar e_2$, $\quad L(\bar e_2) = c\bar e_1 + d\bar e_2$

$L(\bar e_1 \wedge \bar e_2) = L\bar e_1 \wedge L\bar e_2 = (a\bar e_1 + b\bar e_2) \wedge (c\bar e_1 + d\bar e_2) =$

$ad\,\bar e_1 \wedge \bar e_2 + bc\,\bar e_2 \wedge \bar e_1 = (ad - bc)\,\bar e_1 \wedge \bar e_2$

determinant of L

DEF. Determinant of a matrix is the determinant of the linear map it represents

DEF. an atomic rotation is a rotation about a line ℓ with angle θ

Rotations in 2D can be represented by complex numbers

DEF. A **quaternion** is an expression $a + bi + cj + dk$

• add in obvious way

multiply: $i^2 = j^2 = k^2 = -1$

consider $e = bi + cj + dk$, say $\|e\| = 1$, so $\sqrt{b^2+c^2+d^2} = 1$

$e^2 = (bi+cj+dk)^2 = b^2i^2 + c^2j^2 + d^2k^2 + bc\,ij + cb\,ji +$

$bd\,ik + cd\,jk + db\,ki + dc\,kj = -(b^2+c^2+d^2) + bc(ij+ji) +$

$bd(ik+ki) + cd(jk+kj) = -1 + bc(ij+ji) + bd(ik+ki) +$

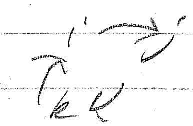
$cd(jk+kj)$

we want $ik = -ki$, $jk = -kj$, and $ij = -ji$

$ij = -ji = k$

$ik = -ki = j$

$jk = -kj = i$



$z = a+bi$, $\bar z = a-bi$, $|z|^2 = z\bar z = a^2+b^2$

$q = a+bi+cj+dk$

$\bar q = a-bi-cj-dk$ $\quad |q|^2 = a^2+b^2+c^2+d^2 = q\bar q$

$z^{-1} = \frac{1}{z} = \frac{\bar z}{|z|^2}$ $\quad q^{-1} = \frac{1}{q} = \frac{\bar q}{|q|^2}$

if $|z| = 1$, $\exists\theta$ s.t. $z = \cos\theta + i\sin\theta$

if $|q| = 1$, let θ be s.t. $a = \cos\theta$, $q = \cos\theta + \sin\theta\, e$

with $e = b'i + c'j + d'k$ with $|e| = 1$

$\overline{q_1 q_2} \overset{?}{=} \cancel{\bar q_1 \bar q_2} \overset{?}{=} \bar q_2 \bar q_1$ $\quad$ let $q_1 = i$, $q_2 = j$; $\overline{ij} = \bar k = -k = (-j)(-i)$

$\bar q = -i$, $\bar j = -j$

$|q_1 q_2| = |q_1| \cdot |q_2|$

$\cos\theta + i\sin\theta \sim \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$

$a+bi \sim \begin{pmatrix} a & -b \\ b & a\end{pmatrix}$ $\qquad a \sim \begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix} = aI$

$\sqrt{-1} \sim \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \sim \begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix}$

$a+bi+cj+dk \sim$ 2x2 complex matrices or 4x4 real matrices

$q = a + bi + cj + dk$

$i \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$j \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$k \sim \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$q \sim \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$

if $z = a+bi$ $\quad z \sim \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $\quad \bar{z} = a - bi$

$\bar{z} \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^T \sim z^T$

$\bar{q} = a - bi - cj - dk = \begin{pmatrix} a-bi & -c-di \\ c-di & -a+bi \end{pmatrix}$ = conjugate transpose $= q^+$

$\cos\theta + e\sin\theta$

$e = bi + cj + dk$

<u>DEF</u> a <u>vector quarternion</u> is a quaternion with no real part

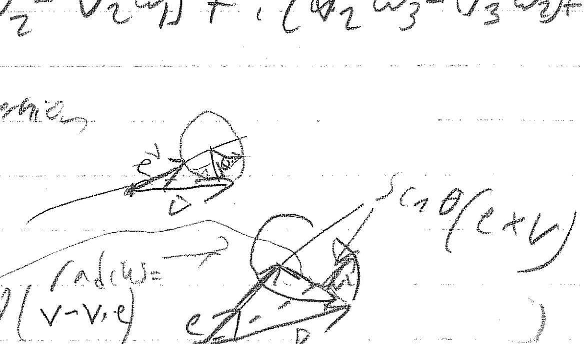
e is a vector quaternion with $|e| = 1$

$v = v_1 i + v_2 j + v_3 k \qquad w = w_1 i + w_2 j + w_3 k$

$vw = (-v_1w_1 - v_2w_2 - v_3w_3) + v_1w_2 k - v_2w_1 k + v_2w_3 i - v_3w_2 i + v_3w_1 j - v_1w_3 j = -(v_1w_1 + v_2w_2 + v_3w_3) + k(v_1w_2 - v_2w_1) + i(v_2w_3 - v_3w_2) + j(v_3w_1 - v_1w_3) = \underbrace{-(v\cdot w)}_{\text{real}} + \underbrace{(v\times w)}_{\text{vector quarternion}}$

What does an atomic rotation do

$|e| = 1$



$v' = (e\cdot v)e + \cos\theta(v - (e\cdot v)e) + \sin\theta(e\times v)$

$q = \cos\theta + e\sin\theta$

$qv\bar{q} = (\cos\theta + e\sin\theta)\, v\, (\cos\theta - e\sin\theta) =$

$\left((\cos\theta)v - (e\cdot v)\sin\theta + \sin\theta(e\times v)\right)(\cos\theta - \sin\theta) =$

$\cos^2\theta v - \cancel{\sin\theta\cos\theta(e\cdot v)} + \sin\theta\cos\theta(e\times v) + \cancel{\sin\theta\cos\theta(e\cdot v)}\sin\theta$
$\cos\theta(v\times e) + \sin^2\theta(e\cdot v)e + \cancel{\sin^2\theta((e\times v)\cdot e)} - \sin^2\theta((e\times v)\times e) =$

$\cos^2\theta v + 2\sin\theta\cos\theta(e\times v) + (1-\cos^2\theta)(e\cdot v)e - \sin^2((e\times v)\times e) =$
$(e\cdot v)e + \cos^2\theta(v-(e\cdot v)e) + 2\sin\theta\cos\theta(e\times v) - \sin^2\theta(v-(e\cdot v)e) =$

$(e\times v)\times e = v-(e\cdot v)e$
$(e\cdot v)e + \cos 2\theta(v-(e\cdot v)e) + \sin 2\theta(e\times v)$

THM: If v a vector quaternion ($v^1 i + v^2 j + v^3 k$) and q is a unit quaternion ($|q|=1$), then $qv\bar q$ = vector quaternion representing the rotation of v around the axis determined by e at an angle 2θ ($q=\cos\theta + e\sin\theta$) (this is an atomic rotation)

Cor: Composite of atomic rotations are atomic

Pf: Let q_1, q_2 represent atomic rotations
$v \to q_1 v \bar q_1 \to (q_2 q_1) v (\bar q_1 \bar q_2) = (q_2 q_1) v \overline{(q_2 q_1)}$
if $q_3 = q_2 q_1$, q_3 represents the composite

Thm: All (3d) rotations are atomic

Pf: Suppose $\{e_1, e_2, e_3\}$ orthonormal basis, R is rotation.
Then $\{Re_1, Re_2, Re_3\}$ is a [illegible] orthonormal
$\exists$ atomic rotation R_1 takes e_1 to Re_1
Re_2 is $\perp$ Re_1, and $R_1 e_2 \perp R_1 e_1 = Re_1$, so $\exists R_2$ (atomic rotation)
such that $R_2 Re_1 = Re_1$ and $R_2 R_1 e_2 = Re_2$
Because R preserves orientation, $R_2 R_1 e_3 = Re_3$

How many q's represent same rotation? $q = \cos\theta + e\sin\theta$
$q' = \cos(\theta+\pi) + e\sin(\theta+\pi) = -\cos\theta - e\sin\theta$
$qv\bar q = (-q)v(\overline{-q})$
2 choices for axis, each gives a unique angle, so 2 values of θ
going [illegible] to the other case [illegible] same
[illegible]

we have a map $\{\text{unit quaternions}\} \xrightarrow{\Phi} \{\text{rotations of 3 space}\}$

and inverse image of φ : any rotation has 2 points

Let V be a VS $\{e_i\}$ a basis and let $\{e_i'\}$ be another w/ same origin

$\vec{v}$ in first basis is $\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$

Define $R_{e_i} = e_i'$ because $e_i' = a_i e_1 + a_2 e_2 + \dots + a_n e_n$

$\vec{v} = v^1 e_1 + \dots v^n e_n = v^1(b_{11} e_1' + \dots + b_{1n} e_n') + \dots$

DEF: A spinor is a pair $\binom{\mu}{\nu}$ of complex #'s s.t. when we change basis by rotation $q = a + bi + cj + dk$ it changes to

$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}$

Why is inertia a tensor?

Linear motion: $\vec{F} = m\vec{a}$

$\vec{p} = m\vec{v}$ $\quad K = \frac{1}{2} m |\vec{v}|^2$

rotational motion:

$\vec{\omega}$ $\quad \vec{\tau} = I\vec{\alpha}$

$\vec{L} = I\vec{\omega}$ $\quad K = \frac{1}{2} I \omega^2$

$a = r\alpha$

$\vec{a} = \vec{\alpha} \times \vec{r}$ $\quad \vec{\tau} = \vec{r} \times \vec{F} = r \times (m\vec{a}) = r \times (m(\alpha \times r)) = m(r \times (\alpha \times r))$

$r(r\alpha) = \dots \Rightarrow r \times (r \times \alpha) = (r \cdot \alpha) r - |r|^2 \alpha$

so $\tau = r \times F = \dots = m|r|^2 \alpha - m(r \cdot \alpha) \vec{r}$

I is an operator

$\alpha \longmapsto m|r|^2 \alpha - m(r \cdot \alpha) r$, in general, total I is

$\alpha \longmapsto \underset{\text{bits of object}}{\Sigma} (m|r|^2 \alpha - m(r \cdot \alpha) r)$

This is a linear map $V \to V$

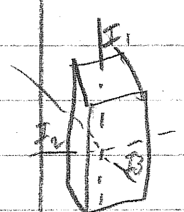
$End(V) \cong V \otimes V^*$

so I is a $\binom{1}{1}$ tensor $\in V \otimes V^*$

DEF: If L is a linear map $L: V \to V$, an eigenvector is a $v \in V$, $v \neq 0$, s.t. $L(v) = \lambda v$, for some $\lambda \in \mathbb{K}$, λ is the eigenvalue

DEF: Principal axes of an object are lines spanned by eigenvectors

axes of symmetry are principal directions



$I_1 < I_2 < I_3$

unstable

$L = I\omega = (I_i \omega_i)$ $K = \frac{1}{2}(I\omega)\cdot\omega$

$2KI_1 - |L|^2 = (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)I_1 - I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2) = (I_1 - I_2)I_2\omega_2^2 + (I_1 - I_3)I_3\omega_3^2$ conserved

$\omega_1 \approx$ all

$\omega_2 \approx 0$

$\omega_3 \approx 0$

so whole stays about 0, constant

$\omega_1 \approx 0$

$\omega_2 \approx 0$

$\omega_3 \approx$ all, ~~whole stays~~ stays around 0 $(I_3 - I_2) I_2\omega_2^2 + (I_3 - I_1)I_1\omega_1^2$

$\omega_1 \approx 0$

$\omega_2 \approx$ all $(I_2 - I_1) I_1\omega_1^2 + (I_2 - I_3) I_3\omega_3^2$

$\omega_3 \approx 0$

Xtreme division by 0 in probability

Statements, combine with logical symbols

$P(A) \in \mathbb{R}$, and P satisfies the following

~~P~~ Kolmogorov's axioms

1. $\forall A,\ P(A) \geq 0$
2. if A is a tautology, $P(A) = 1$
3. if $A \wedge B$ is a contradiction, $P(A) + P(B) = P(A \vee B)$

Subjective probability: For some rational agent, $P(A)$ = degree of belief this person has in A

The amount the agent would be willing to pay for a bet that pays $1 if A is true, and nothing otherwise

Dutch book theorem:

THM: If an agent's prices for bets are given by P, then $P(x)$ satisfies Kolmogorov's axioms ~~or~~ there is a set of bets, each of which the agent is guaranteed to accept, which will ~~allow~~ cause loss

If $A_i \wedge A_j$ is a contradiction $\forall i,j\ (i \neq j)$, then $P(\text{each of } A_i\text{'s}) = \sum_{i=0}^{n} P(A_i)$

Conditional Probability:

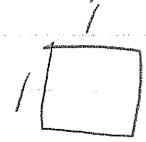
$P(A)$, $P(B)$, $P(A|B)$ is $P(A$ given $B)$ is how strongly you would believe A if you knew B. Same as a conditional bet: I'll bet this on A ~~anyway~~, only if ~~A~~ if not B, 0)

$P(A)$ — A: $1-P(A)$; $\neg A$: $-P(A)$

$$\boxed{P(A|B) = \frac{P(A \wedge B)}{P(B)}}$$ Undefined for $P(B)=0$

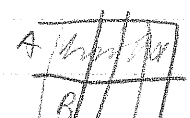
$P(A|B)$ — $\neg B$: 0; $B \wedge A$: $1-P(A|B)$; $B \wedge \neg A$: $-P(A|B)$

Dart Board



$P(\text{hit in area } R) = \text{Area of } R$

$$\boxed{P(B) \cdot P(A|B) = P(A \wedge B)}$$



$P(A|B) = \frac{1}{2}$

If A is fixed, If B is limit of B_i's then $P(A|B) = \lim_{i \to \infty} P(A|B_i)$

Look at As and Bs, (???) doesn't work

~~one way~~
Look at it as not a well defined event. one way Look at it as defined differently for each limit.

Use 6 axioms:

Define $P(A|B)$ on pairs A, B

1. $P(A|A) = P(B|B)$
2. if $P(A|C) = P(B|C)$ $\forall C$ then $P(D|A) = P(D|B)$ $\forall D$
3. $P(A \& A|C) = P(A|C)$
4. $P(A \& B|C) \leq P(A|C)$
5. $P(A|B \& C)P(B|C) = P(A \& B|C)$
6. $P(A|C) + P(\neg A|C) = P(C|C)$

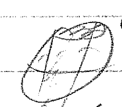
This assumes for any pair A, B, $\exists! P(A|B)$.

Dartboard: $P(A|l_i) = \frac{1}{2}$

Conglomerability: ~~If~~ e_i's form a partition (exclusive probability), and probability of e_i is in interval $x \leq e_i \leq y$, then the total probability is in same interval is in same range

Look at sphere, surface area 1

Pick a point at random



60°

E_i - line of longitude of $i°$

E_i is partition, dropping poles

$P(A|E_i) = P(A|E_j)$

$P(A) \approx \frac{2-\sqrt{3}}{4} \approx \frac{1}{16}$

$P(A|E_i) = \frac{2\sqrt{3}}{4}$

$P(A|B)$ depends on how you arrive at probabilities

$P(A|B) = \int_B f(x)\, d\mu$

N-category theory

Mericans! Nothing exists in a vacuume,
Everything should be studied taking into account its relations to other things.

DEF: A category consists of things called objects and the relationships

Ex:

	Objects	relationships
1)	people	relationships
2)	cities	possible trips
3)	real numbers	$a \leq b$
4)	Let X be a set. Let objects be $\subseteq X$	$A \subseteq B$
4a)	objects: open sets in X (topological space)	$A \subseteq B$
5)	points in a space	distance
6)	sets	functions $f: X \to Y$

DEF: A category is enriched in ____ if $\forall x, y$ the collection of morphisms from x to y is a ____.

DEF: An n-category is a category enriched over n-1 categories

A 0-category is a category w/ only identity morphisms

1-category: a category w/ morphisms between objects

2-category: a category with 0-cells (objects), 1-cells, et of a category that is a relation between 0-cells and 2-cells (a morphism between 1-cells)

Bayesian Statistics

Ex: Suppose 20 people get assigned jobs and worst job requires 6 people and all 4 minority workers get assigned to this job.

Statistician says:

H_0 = by chance H_1 = not by chance

compute $P(\text{data}|H_0)$ If sufficiently small, we reject H_0

$P(\text{data}|H_0) = \binom{16}{2} / \binom{20}{6} = \frac{1}{323}$ small ∴ reject H_0

Look at other groups of 4 (exi last name ends in n) ∴

difference is prior knowledge

Pascal invented probability in mid-1600's

What is probability?

① long term frequency of repeatable events

② degree of belief (how much are you willing to bet)

Scientist threw coin 12 times, 9H, 3T, is coin biased

H_0 = not biased

H_1 = biased

if $P(\text{data}|H_0) <$ cut off, then publish

(1%, 5%, 10%)

\$ coin thrown n times, k heads,

random variable (assumes certain values w/ certain probabilities)

Define probability function $P(k)$ is

$P(\text{prob of event})$

if coin comes up heads w/ prob f,

$P(k) = f^k(1-f)^{n-k}\binom{n}{k}$

in our case, compute probability of data or worse is

$\left(\frac{1}{2}\right)^{n}\left(\binom{12}{9}+\binom{12}{10}+\binom{12}{11}+\binom{12}{12}\right) \approx .07, 7\%$

$P(n)$ = prob it takes n throws to get 3 tails

$P(n) = \binom{n-1}{2}\frac{1}{2^{n-1}} \cdot \frac{1}{2} = \binom{n-1}{2} \cdot \frac{1}{2^n}$

$P(n \geq 12 | H_0) = \sum_{n=12}^{\infty}\binom{n-1}{2}\frac{1}{2^n} \approx .03, 3\%$

Type 1 error: when you reject H_0 when H_0 is true
Type 2 error: when you don't reject H_0 when H_1 is true
For a given test you want probability of Type 1 error and type 2 error
$P(\text{type 1}) = \text{cutoff}$
$P(\text{type 2})$ - uncomputable by this type of experiment

Bayes' Rule (derivation):

$$P(A \text{ and } B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$$

$$\boxed{P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}}$$

H = hypothesis, D = data

$$\boxed{P(H|D) = \frac{P(D|H)\,P(H)}{P(D)}}$$

P(H) ← Prior probability of H
P(D|H) ← likelihood of data given H
P(D): evidence
P(H|D): posterior probability

$$P(H|D) \propto P(D|H)P(H)$$

eg say X is a rare disease w/ frequency is 1%
we have a test for X w/ error rate 5% false positives. Probability where you test negative when you have it = 0%
You take a screening test, test positive

$$P(\text{sick} \mid \text{positive}) = \frac{P(+|\text{sick})\,P(\text{sick})}{P(+)}$$

$$P(\text{not sick} \mid +) = \frac{P(+|\text{not sick})\,P(\text{not sick})}{P(+)}$$

odds: likelihood ratio

$$\frac{P(+|\text{sick})\,P(\text{sick})}{P(+|\text{not sick})\,P(\text{not sick})}$$

× ratio of prior =

$$\frac{1 \cdot .01}{.05 \cdot .99} = 20 \cdot \frac{1}{99} = \frac{20}{99} \approx \frac{1}{5} \quad P(\text{sick}) = \frac{1}{6}$$

suppose freq. blood type O is 60%, blood type AB is 10%.
2 blood samples are found from criminals
1 is O, 1 is AB

Mr Smith is a suspect w/ bt O. How much does the data incriminate him
We want $\frac{P(\text{Guilty} \mid \text{Data})}{P(\text{guilty})}$

$$\frac{P(g \mid d)}{P(d)} = \frac{\frac{P(D \mid g) \cdot P(g)}{P(d)}}{P(g)} = \frac{P(d \mid g)}{P(d)} = \frac{.1}{2 \cdot .6 \cdot .1} = \frac{.1}{.12}$$

$$\frac{1}{1.2} = \frac{5}{6}$$

Coin thrown n times, k heads
what is P(biased)?
What is P(~~[illegible]~~ bias for heads is f) : $P(H) = f$
$P(T) = 1 - f$

What is P(heads) if thrown again, it will be heads

Fisher, uniform prior on f.

P(f) is a density function for continuous random vars

$$P(A \le f \le B) = \int_A^B P(f)\,df$$

$$P(f \mid k) = \frac{P(k \mid f)\,P(f)}{P(k)} \propto P(k \mid f)\,P(f)$$

$$\binom{n}{k} f^k (1-f)^{n-k}$$

$$P(k) = \int_0^1 P(k \mid f)\,P(f) = \boxed{\binom{n}{k} \int_0^1 f^k (1-f)^{n-k}\,df}$$

$$\int_0^1 f^k (1-f)^{n-k}\,df = \frac{k!\,(n-k)!}{(n+1)!}$$

$Z(a,b) = \frac{a!\,b!}{(a+b+1)!}$

$\int_0^1 f^a(1-f)^b\,df = \frac{a!\,b!}{(a+b+1)!}$

So before is $\binom{n}{k}\frac{k!\,(n-k)!}{(n+1)!} = \frac{1}{n+1}$

$P(f|k) = (n+1)\binom{n}{k} f^k (1-f)^{n-k} = \frac{1}{Z(k,n-k)} f^k (1-f)^{n-k}$

$\frac{k}{n-k} = \frac{f}{1-f}$, so $f = \frac{k}{n}$

We don't want to maximize, but to ~~average~~ get expectation or mean

$\int_0^1 f\, p(f|k)\,df = \frac{1}{Z(k,n-k)}\int f^{k+1}(1-f)^{n-k}\,df =$

$\frac{Z(k+1,n-k)}{Z(k,n-k)} = \frac{(k+1)!\,(n-k)!}{(n+2)!}\cdot\frac{(n+1)!}{k!\,(n-k)!}$, so

$\boxed{\frac{k+1}{n+2}}$

Throw a coin n times, get k heads

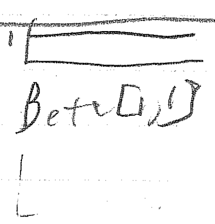
Hypotheses: f; $p(H) = f$

$P(f) = 1$ (uniform distribution on $[0,1]$)

$P(f|D) = \frac{\binom{n}{k} f^k (1-f)^{n-k}}{\binom{n}{k} Z(k,n-k)}$ ← Beta

DEF: $\boxed{\mathrm{Beta}[a,b](f) = \frac{f^{a-1}(1-f)^{b-1}}{Z(a-1,b-1)}}$

Expected value of $\mathrm{Beta}[a,b]$ is $\frac{a}{a+b}$



uniform

$\mathrm{Beta}[1,1]$

prior

What should prior $p(f)$ be?
It's reasonable to take prior to be $Beta[a,b]$ for some a, b
- if prior is this, and you see k H in n tosses, posterior is $Beta[a+k, b+(n-k)]$
- prior should look like what you've seen before
conjugate prior β is conjugate to binomial $\binom{n}{k} f^k (1-f)^{n-k}$

What if I got k heads, n tosses

$$P(f|D) = \frac{f^k (1-f)^{n-k}}{Z(k, n-k)}$$

Now throw again, get k' heads, n' tosses

Now prior is $Beta[k+1, n-k+1](f) = \frac{f^k (1-f)^{n-k}}{Z(k, n-k)}$

$$P(D'|f) = \binom{n'}{k'} f^{k'} (1-f)^{n'-k'}$$

$$P(D') = \int_0^1 P(D'|f) P(f) df = \int_0^1 \binom{n'}{k'} f^{k'} (1-f)^{n'-k'} \frac{f^k (1-f)^{n-k}}{Z(k, n-k)}$$

$$P(f|D') = \frac{f^{k+k'} (1-f)^{(n+n')-(k+k')}}{Z(k+k', (n+n')-(k+k'))} = \frac{f^K (1-f)^{N-K}}{Z(K, N-K)}$$ if $N = n+n'$, $K = k+k'$

$\Gamma(n) = (n-1)!$

Compare H_0 = unbiased and H_1 = biased

$$\frac{P(H_1|D)}{P(H_0|D)} = \frac{P(D|H_1) \cancel{P(H_1)}}{P(D|H_0) \cancel{P(H_0)}} = .48$$ let $P(H_1) = P(H_0)$

$$P(D|H_0) = \left(\frac{1}{2}\right)^{250} \binom{250}{140}$$

$$P(D|H_1) = \int_0^1 P(D|H_1, f) P(f) df = \frac{1}{n+1} = \frac{1}{251}$$

Say f dist is $\beta[a, a]$

you get $P(D|H_a) = \frac{Z(k+a-1,\ n-k+a-1)}{Z(a-1,\ a-1)}$

$P(D|H_0) = \left(\frac{1}{2}\right)^{250} \binom{250}{140}$

To ~~ma~~ ~~the~~ $\frac{P(D|H_a)}{P(D|H_0)}$ max,

$a \approx 55$, ratio is 1.9

Big Numbers

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Given $\{1, 2, 3, \ldots, N\}$ if we color them with c colors, can we always find an arithmetic progression of length l which is monochromatic

Strategy: There is a number $W(l, c)$ s.t. any coloring of $1, \ldots, W(l,c)$ in c colors has a mono AP of length l

We'll find lower bound and upper bound

$W(3,2) = 9$ $\quad W(4,2) = 35$

$W(3,3) = 27$

$W(3,4) = 76$

Lower bound: $w(l,c)$

Color $\{1, \ldots, N\}$ randomly w/ c colors

Take l numbers in AP, $P(\text{all same color}) = \frac{1}{c^{l-1}}$

if $\{1 \ldots n\}$ has k AP of length l, then expected # of monochromatic ones is $\frac{k}{c^{l-1}} \approx \frac{N^2}{2lc^{l-1}}$

Gap 1: $1, 2, \ldots, l, \ldots, N-l+1, \ldots, N \quad N-(l-1)$

Gap 2: $\quad N-2(l-1)$

$\vdots$ $\quad \frac{N}{l}$

Total $\approx \frac{N(\frac{N}{l})}{2} \approx \frac{N^2}{2l}$ A.P.'s of length l

If $\frac{N^2}{2lc^{l-1}} < 1$, can't guarantee a monochromatic AP

$N^2 < 2l\, c^{l-1}$

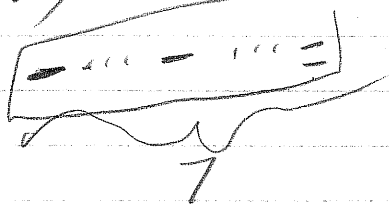
$N < \sqrt{2l}\, c^{\frac{l-1}{2}}$

$W(l, c) \geq \sqrt{2l}\, c^{\frac{l-1}{2}}$

$W(3,c)$

$W(3,2) \le 2^{6\cdot 5}+5+1$

$W(3,3) \le 2\times\left(3^{2\times(3^7+1)\times 7}+1\right)\times 2\times(3^7+1)\times 7$



$2\cdot 3^7+1$ blocks

7,115 !

$W(3,4)$

for $c=4$

$\cancel{2\left(c^{2\cdot\left(c^{2\times(c^{2c+1}+1)\times(2c+1)}+1\right)2(c^{2c+1}+1)(2c+1)}+1\right)}$

$W(4,2) =$

B B B G

$2\cdot W(3,2)$

$W(4,2) \le 2\times W\left(3, 2^{2\cdot W(3,2)}+1\right)\times 2W(3,2)$

Math Until You Die

Point set Topology

Topological space: A set X with a collection of subsets T called open: $\emptyset \in T$, $X \in T$, $X, Y \in T \Rightarrow X \cap Y \in T$, $\{X_i\} \subset T \Rightarrow \bigcup X_i \in T$

[illegible]

[illegible]

Haskell
www.haskell.org/tutorial
GHC (compiler)
Types:

5 :: Integer
'a' :: Char
inc :: Integer -> Integer (inc is a function
[1,2,3] :: [Integer]
('b',4) :: (Char, Integer)
defining functions:
inc n = n+1

inc (inc 3)

[char]
[a]

[1,2,3] = 1:(2:(3:[])) = 1:2:3:[]

length :: [a] -> Integer

length [] = 0
length (x:xs) = 1 + length xs   order matches, goes to top first

head :: [a] -> a
head (x:xs) = x
tail :: [a] -> a
tail x:xs = xs

[y+1 | y <- [1,2,3], y<2]

```
quicksort :: [a] -> [a]
quicksort [] = []
quicksort (x:xs) = quicksort [y | y <- xs, y < x] ++ [x]
                   ++ quicksort [y | y <- xs, y >= x]


data Bool = False | True
data Point a = Pt a a
      OR
data Point a = Point a a

data Tree a = Leaf a | Branch (Tree a) (Tree a)

Leaf :: a -> Tree a
Branch :: Tree a -> Tree a -> Tree a

Foliage : Tree a -> [a]
Foliage (Leaf x) = [x]
foliage (Branch l r) = foliage l ++ foliage r

type String = [Char]
type Assoc a b = [(a, b)]

add :: Integer -> Integer -> Integer
add x y = x + y

inc = add 1

map :: (a -> b) -> [a] -> [b]
map f [] = []
map f (x:xs) = (f x) : (map f xs)
```

λ is lambda, written as \

\x -> x+1

(++) :: [a] -> [a] -> [a]

[] ++ ys = ys
(x:xs) ++ ys = x:(xs ++ ys)

$f \circ g$
f.g

(.) :: (b -> c) -> (a -> b) -> (a -> c)
f.g = \x -> f (g x)

(x+) = \y -> x+y

x + y ~~is~~ x \`add\` y

bot = bot $\perp$ (bottom)

f is strict if f bot is bot

const1 x = 1
x = 1/0 means define x to " 1/0

ones = 1 : ones

take 5 ones is [1,1,1,1,1]

```
numsfrom n = n : numsfrom (n+1)
squares = map (^2) (nums from 0)

zip (x:xs) (y:ys) = (x,y) : zip xs ys
zip xs ys = []

fib = 1 : 1 : [a+b | (a,b) <- zip fib (tail fib)]
x `elem` [] = False
x `elem` (y:ys) = x == y || (x `elem` ys)
let y = a*b
let f x = (x+y)/y
in f c + f d
```

Type Classes

```
class Eq a where
  (==) :: a -> a -> Bool
instance Eq Integer where
  x == y = x `IntegerEq` y

instance Eq a => (Tree a)
  Leaf x == Leaf y = x == y
```

/= is not equal

IO:

~~get Char :: Char~~

```
get :: IO Char
put :: Char -> IO Char
```

```
ready :: IO Bool
ready = do c <- getChar
           putChar c
           return (c == 'y')

Class Monad m where
  (>>=) :: m a -> (a -> m b) -> m b

  return :: a -> m a
```

Circuits over Sets of Natural Numbers & PCP

2 char Lang, 2 char/expression

$\begin{bmatrix} 10 \\ 10 \end{bmatrix} = 3^{9}3^{0}$ $\begin{bmatrix} ab \\ cd \end{bmatrix} = abcd_5$

$Cards = \{abcd \mid [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \text{ is a valid card}\}$

required: $\forall a_1b_1c_1d_1, a_2b_2c_2d_2 \in Cards$, $a_1b_1c_1d_1 + a_2b_2c_2d_2$ is not a card

~~Invalid~~ $Val_{1,2} = \{n \mid$ all digits of n in base 3 are either 0 or 1$\}$

For k-sum of cards, $\left(\sum_{i=1}^{k} Cards\right) \times Powers_5 \cap \sum_{i=1}^{k}(Powers_5 \times Cards)$ is legal combos

for good to length $\frac{k}{2}$, and $\frac{k}{2} \leq$ length expression

if $(num \times 5) \bmod 5^k + (num \bmod 5) = num \bmod 5^k$

$a \bmod b = \sum_{i=0}^{b-1}(multiples(b)+i), a, 0)$

Congruent to $a \bmod b$ in S: $(\mathbb{N} \times \{b\} + \{a\}) \cap S$

$S \bmod a = \bigcup_{i=0}^{a-1}((\text{Congruent to } i \bmod a \text{ in } S) \times \{0\} + \mathbb{N}) \times \{0\} + \{i\}$

Fibonacci & phyllotaxis - leaf arrangements

n	0	1	2	3	4	5	6	7
f_n	0	1	1	2	3	5	8	13
l_n	2	1	3	4	7	11	18	29

if $x^2 = 1+x$, then $1, x, x^2, x^3, x^4, \ldots$ solves $a_{n+2} = a_n + a_{n+1}$

τ: $x = \frac{1+\sqrt{5}}{2}$ $\quad$ σ: $x = \frac{1-\sqrt{5}}{2}$

$$\begin{array}{cccccc} & 1 & 1 & \tau & \tau^2 & \tau^3 \\ - & 1 & 1 & \sigma & \sigma^2 & \sigma^3 \\ \frac{1}{\sqrt{5}} & 0 & 1 & 1 & \ldots & \end{array}$$

so $\frac{\tau^n - \sigma^n}{\tau - \sigma} = \boxed{\frac{\tau^n - \sigma^n}{\sqrt{5}}}$

$$\begin{array}{cccccc} & 1 & 1 & \tau & \tau^2 & \tau^3 \\ + & 1 & 1 & \sigma & \sigma^2 & \sigma^3 \\ & 2 & 1 & 3 & 4 & \ldots \end{array}$$

$\tau^n - \sigma^n = \sqrt{5} f_n$

$\tau^n + \sigma^n = l_n$

if $\tau = e^{i\theta}$ $\sigma = e^{-i\theta}$, $\tau^n - \sigma^n = 2i \sin n\theta$

$\tau^n + \sigma^n = 2\cos\theta$

f_n, l_n satisfy trig-like identities

$\sin^2\theta + \cos^2\theta = 1$ $\qquad$ $l_n^2 - 5f_n^2 = 2(-1)^n$

$2\sin\theta\cos\theta = \sin 2\theta$ $\qquad$ $\sqrt{5} f_n l_n = \frac{\tau^{2n} - \sigma^{2n}}{\sqrt{5}}$

$f_n l_n = f_{2n}$

(mod p)

$f_{p-1} \equiv 0$ if $p \equiv \pm 1 \ (5)$

$f_{p+1} \equiv 0$ if $p \equiv \pm 2 \ (5)$

$f_p \equiv 0$ if $p \equiv 0 \ (5)$

~~1~~ 1
~~1 1~~ 1 2
~~1 2 1~~ 1 2 3
~~1 3 3 1~~ 3 5
1 4 6 4 1 8

extra - fib series

fib	0	1	1	2	3	5	8	13 ...	0th extra fib series			
luc	2	1	3	4	7	11	18		1st	"	"	"
	2	2	4	6	10	16	26					
	3	5		9	15...							
	4	8										
	5	9										
	6	11										
	7	12										
	8	14										

Angel & Devil ~~[illegible]~~ Game
played on infinite checkerboard between
an angel of power n and ~~t~~ the devil

Complex Numbers

$\mathbb{R}$ are a "field" = we have $+$, $\cdot$, $-$, $\div$, and they "behave well"
There are things we cannot do:
Solve $x^2+1=0$
Make up a new number: i
declare $\boxed{i^2=-1}$
Pretend four operations still behave well
Must accept all expressions of form $a+bi$, $(a,b\in\mathbb{R})$
This set is the set of complex numbers $\mathbb{C}$
$\bar{z}=a-bi$ if $z=a+bi$

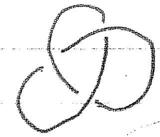
Thm: <u>Fundamental Thm of Algebra</u>:
Every polynomial has solutions in $\mathbb{C}$

$f(x)=e^{ax}$ is the solution to $f'(x)=af(x)$ for some const a
and $f(0)=1$

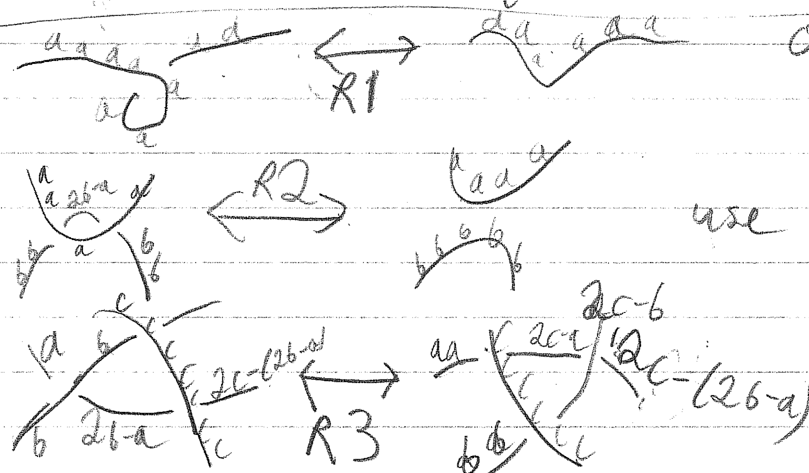
$g(x)=\cos x+i\sin x$ $\quad g'(x)=-\sin x+i\cos x=ig(x)$
$g(0)=1$

(Tangles, Bangles, &) Knots

A Knot is ~~a~~ map $f:[0,1] \to \mathbb{R}^3$ s.t. $f(a)=f(b)$ iff $a \equiv b \pmod 1$

 un knot  trefoil

one way to fix: Make knot equivalent to torus

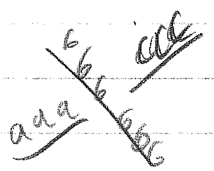


OR

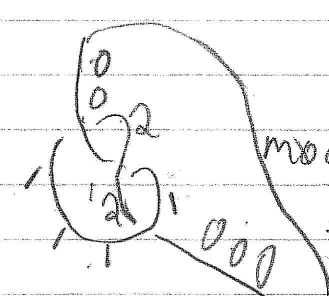
use only R_1, R_2, R_3

Knot (k)numbering

where you can see string cannot change



at each crossing, a, b, c are an arithmetic progression
i.e. $b-a=c-b \iff c=2b-a$



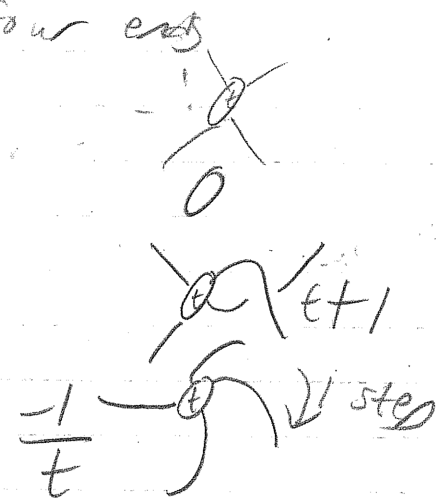
modulo 3

this is numbering (mod 3)

Thm: the number of (k)numberings (with any system of numbers) never changes when you do a Reidemeister(sp?) move

A tangle is a "knot" with four ends

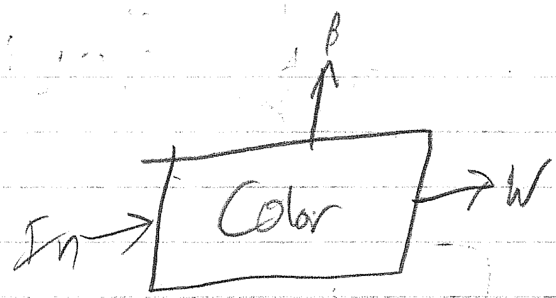
$0 \to 1 \to 2 \to 3 \to$

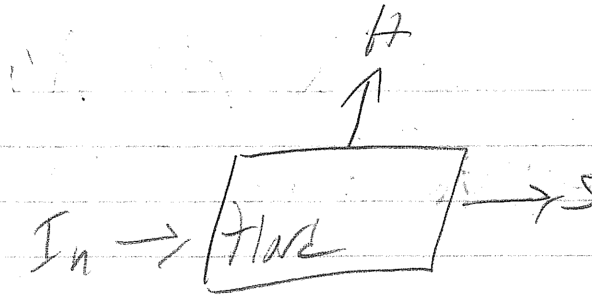


-0

(Quantum Mechanics)

Pad Thai with Electrons

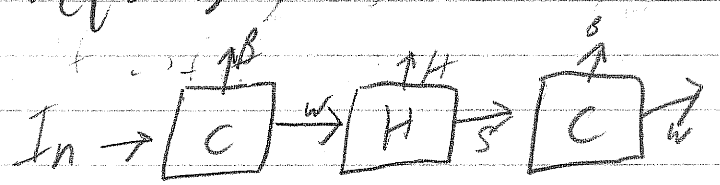




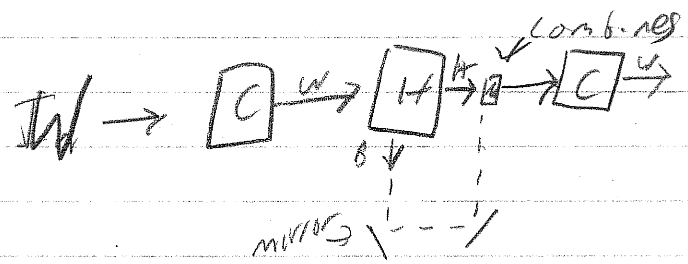
electrons are either Black or white, soft or hard

repeatable: if same property is measured multiple times

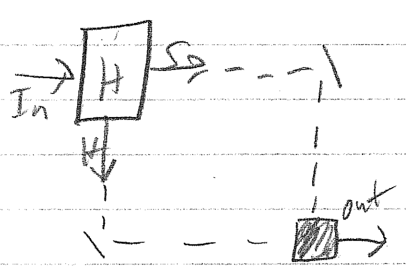
equally distributed: ~~[illegible]~~ independent



50/50 W/B



100% W/B



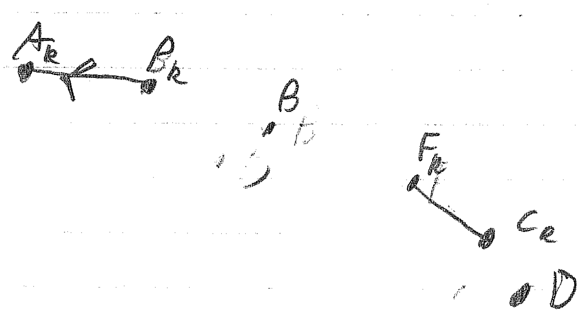
$A_k : \{n\} \mapsto \{a \mid a+k=n\}$ $n-k$ $P_k : S \mapsto \{n \mid n+k \in S\}$ $S-k$ $B : S \mapsto \{a \mid \forall b \in S,\ a \geq b,\ a \in S\}$ max

$F_k : \{n\} \mapsto \{\lfloor \frac{n}{k} \rfloor\}$ $C_k : \{n\} \mapsto \{\lceil \frac{n}{k} \rceil\}$ $D : S \mapsto \{n \mid \exists a, b \in S \text{ s.t. } a \leq n \leq b\}$ Fill

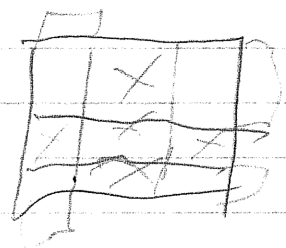
$E : S \mapsto \{n \mid \exists a_0, a_1, \ldots, a_{n-1} \in S \text{ s.t. } \forall 0 < i < n,\ a_{i-1} < a_i\}$ size $G : \max E$

$P_k \Longrightarrow A_k$ $F_k \Longleftrightarrow C_k$ $D, A_1 \Longrightarrow B$ $E, B \Longrightarrow G$ $G \Longrightarrow E$

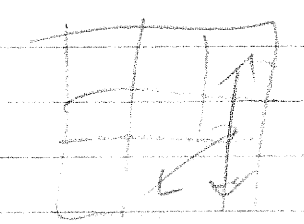
Fill $\Longleftrightarrow$ Max



$5 \times 5 \times 5$



LU'R'UL'URUR'UR



$(r^{-1}ur^{-1})u^{-1}r^{-1}u^{-1}(r^{-1}ur')(ur2)$

$\text{RemoveKSmallest}(S,k) = S \setminus \text{K-Min}(S,k)$

$\text{K-Fill}(S) =$

$\text{K-Min}(S,k) = S \setminus (S + GTEQ(\{k\}))$

Want: # of elements

given: $\{n\} \mapsto \{n-1\}$

$LTEQ(\{n\}) + \{k\} = LTEQ(\{n\}) \setminus LT(\{k\})$

$LTEQ(\{k\})$

$\{a \mid a \equiv 0 \pmod{4}, a \leq n\}$ never:

$\{n^4\}$:

$\{0, 2, 4, 8, \ldots, 32\}$?

$\{0, 2, 4, 8, \ldots, 64\}$

$\{34, 36, \ldots, 66\}$

$\{0, 4, 8, \ldots, 64\}$

$\{n\} \mapsto \{\lfloor \frac{n}{2} \rfloor\}$

$\{n\} \mapsto \{\lceil \frac{n}{2} \rceil\}$

$S \mapsto \max S$

$\to \{n \mid \exists a \in S \text{ s.t. } n = a-1\} \to \{n\} \mapsto \{n-1\}$

$\overline{\{p\} \times \mathbb{N}} \cup \{1\} \times \mathbb{N}$

Primes $\cap \overline{\{p\}}$

fly: a   pay: b

$\frac{a}{b}$ $\frac{2a + a}{2b}$ $\frac{\frac{9}{2}a + a}{3b}$ $\frac{\frac{22}{3}a + a}{4b}$

Jason

$S \bmod n = \bigcup_{i=0}^{n-1} \left(\left[(\{n\} \times \mathbb{N} + \{i\}) \cap S \right] \times \{0\} + \{i\} \right)$

$= \{ n \mid \exists a, b, c \in S \text{ s.t. } a \le b \le c \text{ and } a \le n \le c \text{ and } |n - b| \le k \}$

$k\text{-fill}(S, \{k\}) \neq S \cap (S + LTEQ(\{k\}))$

$\{n\} \cap$

$\{2n - k\} = $ $(LT(\{n\}) \cap EVENS) + (LT(\{n\}))$

$LTEQ(S + \{k\})$

$(S + \{k\}) + \mathbb{N}$

$S+$

~~Regular~~ Expressions

$\cup$ $\cap$ $*$ Kleene Star

$S + \{0, 1\}$ $\quad LTEQ(S)$

$(01)^* = \{\varepsilon, 01, 0101, \ldots\}$

n even

$n \mapsto \{0, 2, 4, \ldots, n-2\}$

$\{1, 3, 5, 7, n-3, n-1, \}$

$\{0, 1, 2, \ldots, n-1\}$

$\{1, 3, \ldots, n-1\}$

~~$\{1, 3, 5, \ldots, 2n-3\}$~~

$+ \{0, 1, 2, \ldots, 2n-3\} \rightarrow \{2n-2\}$

8	1	2
7	0	3
6	5	4

$2n-1$

$2n-2$ (n even or odd) ✓

$2n-k$, any fixed $k < n$
or if k is related to n
(know where n is mod k)

$\mathrm{Pred}(\{n\}) = \begin{cases} \text{1-Fill}(LT(\{n\}) \cap \mathrm{EVENS}) \\ \end{cases}$

n even

$\{0, 1, 2, \ldots, n-1\}$

$\{0, 2, 4, \ldots, n-2\}$

$\{0, 1, 2, 3, 4, \ldots, n-2\}$

$\{n-1, n, n+1, \ldots\}$

$\{n-1\}$

Symmetry of Flexible Molecules

Def: chemically achiral means can change onto mirror image
Chemically chiral not change onto mirror image

useful to know if a molecule will be chiral before being manufactured

The Mysterious Numbers of Professor Hansel

$312 = 3\cdot 10^2 + 1\cdot 10 + 2\cdot 1$

$X = 10q_0 + r_0 \quad 10 > r_i \geq 0,\ r_i \in \mathbb{Z}$
$q_0 = 10q_1 + r_1$
$\vdots$

$-1 = X = \quad \ldots 9999$
$X = 10\cdot q_0 + r_0$

$-15 = \ldots\ldots 9985$

$\frac{1}{3} = \quad \ldots 6666667$

$\frac{1}{3} = 10q_0 + r_0 \quad \frac{1}{3} - r_0 = 10q_0$

$\frac{2}{3}$

$\frac{1-3r_0}{3} = 10q_0$

$\frac{1}{2} = 10q_0 + r_0$
$\frac{1}{2} = \frac{1}{10}\cdot 5 = 0.5$

$\frac{1}{6} = \frac{1}{10}\left(\frac{3}{5}\right) = \frac{1}{10}(\ldots 3335) = \ldots 3333.5$

$\mathbb{Q}_{10} \leftarrow$ 10-adics
all things that can be written like this

$\sqrt[3]{11} = \quad \ldots 2771$
$11 = (10q_0 + r_0)^3$

$1^3 = 1$
$71^3 = 357911$
$771^3 = \ldots 011$
$2771^3 = \ldots 0011$

$x^2 = x$
$5, 6 \quad 5^2 = 25$
$25^2 = 625$
$625^2 = 390625$

$e = \quad 0625$

$\left(\left(\left(5^2\right)^2\right)^2\right)^{2\cdots} = \left(5^{2^\infty}\right)^2 = 5^{2^\infty}$

$e \neq 0,1 \quad e^2 = e \quad (1-e)^2 = 1-e = f \quad ef = 0 \quad e+f = 1$

$e+f=1$ $\qquad x - y = 10^n q$ x and y are "close"

$xe + xf = x$

$e^n + f^n = 1^n$

$\mathbb{R} \supset \mathbb{Q} \subset \mathbb{Q}_p$

What is A Tensor?

Quick, dirty, and wrong definition

A rank k (≥ 0) tensor in n dimensions is an $\underbrace{n \times n \times \cdots \times n}_{k}$ array of #s

$k=1 \qquad k=2$

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \qquad \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \quad (w_1, \ldots, w_n)$

Re-def #1: An $\binom{r}{s}$-tensor in n-dim is an $\underbrace{\overbrace{n \times \cdots \times n}^{r}}_{\text{vertical}} \underbrace{\overbrace{n \times \cdots \times n}^{s}}_{\text{horizontal}}$ array of #s

$\binom{0}{1} \quad \binom{1}{0} \quad \binom{2}{0} \quad \binom{1}{1} \quad \binom{0}{2}$

addition is point-wise

$\binom{r}{s}$-tensors form an n^{r+s}-dim vectorspace

Let V be an n-dim vectorspace

if $\vec{e}_1, \ldots, \vec{e}_n$ basis, $\vec{v} = \sum_{a=1}^{n} v^a \vec{e}_a \rightsquigarrow \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$ represents $\vec{v}$ in basis $\{\vec{e}_a\}$

can think of vector as a collection of $\{v^a\}$ of components for each basis

s.t. $\vec{e}_a \to f_\mu$

$v^a \to \sum_\mu A^\mu_a v^a$

could have $\vec{f}_\mu = \sum_a B^a_\mu \vec{e}_a$

~~so~~ $\{w_a\} \rightsquigarrow \{\sum_a B^a_\mu w_a\}$

this called covector or covariant vector

$$R^{ab} \rightsquigarrow \sum_{a,b} \overset{\text{number}}{A^{\mu}_{a}} A^{\nu}_{b} R^{ab} \quad \binom{2}{0}$$

$$S^{a}_{i} \rightsquigarrow \sum_{a,b} A^{\mu}_{a} B^{b}_{\nu} S^{a}_{i} \quad \binom{1}{1}$$

Physicist's defi
A tensor is a collation of components which transform tensorially

$$T_{ab} \rightsquigarrow \sum_{a,b} B^{a}_{\mu} B^{b}_{\nu} T_{ab}$$

~~eg~~ e.g.: (1) $L: V \to V$

$$L(\vec{e}_a) = \sum_b L^b_a \vec{e}_b$$

form a $\binom{1}{1}$-tensor

(2) inner product $\langle -, - \rangle$

$g_{ab} = \langle \vec{e}_a, \vec{e}_b \rangle$ metric tensor

(g_{ab}) is a $\binom{0}{2}$-tensor which determines inner product

Inertia: $\binom{1}{1}$-tensor

$L = I\omega \quad p = mv$

$\tau = I\alpha \quad F = ma$

gravitational field $G_{ab} = T_{ab}$ $\binom{0}{2}$

Electromagnetic field φ_{ab} $\binom{0}{2}$ tensor

Summation convention: write $\varphi^a w_a$ to mean $\sum_a \varphi^a w_a$

any covector w gives map

$\hat{w}: V \to \mathbb{R}$ as $v^a \mapsto w_a v^a$

$\{\text{covectors}\} \rightleftarrows \mathrm{Hom}(V, \mathbb{R}) = V^* =$ dual space of V

Defi a $\binom{0}{1}$-tensor is an element of V^*

$g_{ab} v^a v^b$

takes $(\vec{v}, \vec{w}) \mapsto g_{ab} v^a w^b$ bilinear

$\{\binom{0}{2}\text{-tensors}\} \xrightarrow{\cong} \mathrm{Bilin}(V,V;\mathbb{R})$

$g_{ab} = g(\vec{e}_a, \vec{e}_b)$

Redef: A $\binom{0}{s}$-tensor is a s-multilinear map $\underbrace{V \times V \times \cdots \times V}_{s} \to \mathbb{R}$

An $\binom{r}{s}$-tensor is an $(r+s)$-multilinear map $\underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s}$

$V \times V \to \binom{2}{0}$-tensors

$(\vec{v}, \vec{w}) \mapsto \{v^a w^b\}$

universal property

$V \times V \xrightarrow{\beta} \{\binom{0}{2}\text{-tensors}\} = V \otimes V$

bilinear, linear, $\exists!$ linear map s.t. triangle commutes

R-module is vectorspace, but replace $\mathbb{R}$ with comm ring R

A tensor product $M \otimes_R N$ is an R-module w/ that universal property

1

How to Use Matematical Models to Study the Mind

events are unique; to learn, the mind must generalize

similarity: $S(a, b)$

dif 1   sim 20

15   19

16   -1 -4 -5

19

5

3   6

3   ABC

Mira's other Favorite Mathematical Magic trick

Hall's Marriage Thm:

n guys, n girls each girl has a list of guys liked,

Necessary conditions: ~~all~~ guy will ~~take~~ marry one girl

each girl likes at least 1 guy

any pair of girls like at least 2 guys

any k girls like between them at least k guys

→ This is sufficient

Cor: If each girl likes exactly k guys, and each guy is liked by exactly k girls → match exists

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Encoder instructions:

~~52~~ 24 cards total $(c_0, c_1, c_2, c_3, c_4)$

given 5 cards, hide 1, order rest $c_0 < c_1 < c_2 < c_3 < c_4$

1) Add $c_0 + c_1 + c_2 + c_3 + c_4 \bmod 5 = i$ Hide c_i

2) re number cards 0 1 2 3 4 5 6 7 8

encode $\underbrace{c_i}_{\text{from } 0 \text{ to } 119} = h$ $| | | c_0 | | c_1 | | c_2 | c_3$

say hidden

3) write h as $5q + r$ q from 0 to 23

Decoder's instructions

3') figure out q ($q = 5$ in ex)

2') $h = 5q + \text{rem}$ ← sum of revealed mod 5

1') find i, get $i^{th} = c_i$

3, 8, 17, 21 1, 2, 25, 26

1 2 1 2 2

3 0 2 white 1 6 black

25 23

$(v_1 + r_1)t_1 = 15$ 1, 2, 26

$(v_1 - r_1)(t_1 + 1) = 15$

$(\frac{v_1}{2} + r)t_2 = 15$ 1 8

6

t_2 3, 17, 21

$(\frac{v_1}{2} - r)(t_2 + 5) = 15$

1 $v_1 = kr$

$v_1 t_1 + r_1 t_1 = 15$ $(k+1)r t_1 = 15$

$v_1 t_1 - r_1 t_1 = 15$ $(k-1)r(t+1) = 15$

$+ v_1 + r$ $2v_1 t_1 + v + r$ $\frac{6}{r}$

$2v_1 - 2r = v_1 + r$ $t_1 = \frac{15 \cdot 2}{}$

$v_1 - 3r = 0$ $\frac{3}{2}r + r$

$v_1 = 3r$

$\frac{5}{2}r t_1 = 15$

$(\frac{3}{2}r - r)(t_1 + 5) = 15$

$\frac{1}{2}r(\frac{6}{r}) = 15$