# Relativistic Dynamics


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I present the energy-momentum-force relations of Newtonian and relativistic dynamics. I investigate the goodness of fit of classical and relativistic models for energy, momentum, and charge-to-mass ratio for electrons traveling at 60%–80% the speed of light. I find that the relativistic models do much better than the classical ones ($\chi^2_\nu$ ratios range from about 5 to about about 100). I find the electron charge-to-mass ratio to be $(1.54 \pm 0.03) \cdot 10^{11}\ \mathrm{C/kg}$, as compared with a book value of $1.758\,820\,088(39) \cdot 10^{11}\ \mathrm{C/kg}$. I find the charge of the electron to be $(1.5 \pm 0.1) \cdot 10^{-19}\,\mathrm{C}$ as compared with a book value of $1.602\,176\,565(35) \cdot 10^{-19}\,\mathrm{C}$. I find the mass of the electron to be variously $(9 \pm 2) \cdot 10^{-31}\,\mathrm{kg}$ and $(1.13 \pm 0.04) \cdot 10^{-30}\,\mathrm{kg}$, as compared with a book value of $9.109\,382\,91(40) \cdot 10^{-31}\,\mathrm{kg}$.


## I. NEWTONIAN AND RELATIVISTIC DYNAMICS

Newtonian dynamics begins with the assumption that space and time are flat and absolute. Relativistic dynamics begins with the assumption that space and time are flat, and that the speed of light is the same in all inertial reference frames. From these assumptions, the Minkowski metric can be derived, as well as length contraction and time dilation and the relativistic formulae.

See Appendix A for the conventions and symbolic definitions I use in this paper.

Newtonian and relativistic dynamics are governed by the following relations, among others:

| Newtonian | Relativistic |
|---|---|
| $\vec{p} = m\vec{v}$ | $\vec{p} = \gamma m\vec{v}$ |
| $E = K + U$ | $E^2 = p^2c^2 + m^2c^4$ |
| $K = p^2/2m$ | $K = (\gamma - 1)mc^2$ |
| | $K = mc^2\left(\sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1\right)$ |
| $\vec{F} = d\vec{p}/dt$ | $\vec{F} = d\vec{p}/dt$ |

Lastly, the electromagnetic force on a charged particle is $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$. This formula is valid both for Newtonian dynamics and relativistic dynamics.

We set out to test relativity, and to determine the electron charge and electron mass.

## II. METHODOLOGY AND EXPERIMENTAL SETUP

We used high-energy electrons to test relativistic dynamics, using the setup in Figure 1. High-energy electrons are emitted from a radiation source and travel in a circular path through the vacuum chamber due to the uniform magnetic field. At the end of a semi-circular path, electrons are absorbed by a PIN diode which measures their energy.

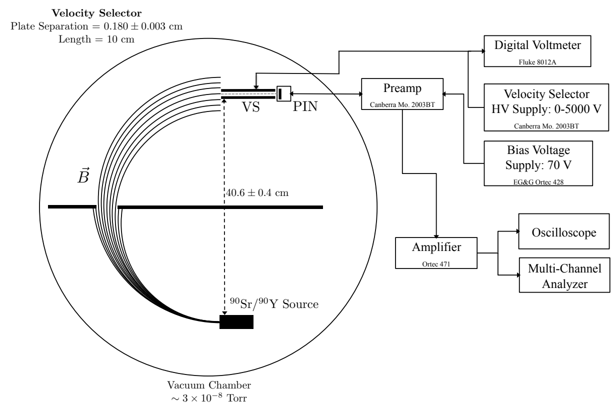


FIG. 1. Electrons are emitted from the $^{90}\mathrm{Sr}/^{90}\mathrm{Y}$ source, travel through the vacuum chamber, enter the velocity selector (VS) and hit the PIN diode, which measures their energy. Figure taken from the lab guide.[1]

A baffle in the middle of the semi-circle selects for electrons with a small range of momenta, corresponding to the small range of radii which pass through a slit in the baffle.

A capacitive velocity selector further restricts the velocities; anything going too slow will be accelerated into one side of the selector by the electric field, and anything going too fast will not be significantly affected by the electric field, and will hit the other wall due to the magnetic field. At approximately the center of this range of momenta, the forces due to the magnetic and electric fields are equal and opposite. It is at this momentum that the PIN diode registers a peak in its energy spectrum.

## III. EXPERIMENTAL THEORY

We have three relevant parameters for each stream of electrons: the energy, from the PIN diode and MCA, the magnetic field strength, and the peak electric field strength (the value of $E$ for which count rate is maximized).


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Each of these parameters is related to the momentum of the electron.

When the electron feels only a magnetic field, its motion is circular; it's governed by the equation $evB = |d\vec{p}/dt| = \omega p = vp/\rho$ which becomes $p = eB\rho$.

The peak in count rate occurs at the momentum at which electrons in the velocity selector feel no force:

$$\vec{F} = e\vec{E} + e\vec{v} \times \vec{B} = 0 \tag{1}$$

or $E = vB$. Since $E = V/d$, we then have

$$\beta = v/c = E/(cB) = V/(cBd). \tag{2}$$

The spread of electron energies about this value is assumed to be symmetric, so that we may simply find an approximate center, which is also the peak value. If this assumption is inaccurate, the calculations which use energies are slightly more inaccurate than the analysis in this paper would lead one to believe.

Finally, the measured energy is hypothesized to be related to the momentum of the electron via either the classical formula $K = p^2/2m$ or the relativistic formula(s)

$$K = (\gamma - 1)mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2. \tag{3}$$

We may thus determine goodness-of-fit for the momentum-velocity relationships and for the energy-momentum relationships (if we take the textbook value of $m_e$), and we may calculate the ratio $e/m_e$ under each regime; classically, we get $e/m_e = V/(B^2\rho d)$, and relativistically, we get $e/m_e = cV/(B\rho\sqrt{B^2c^2d^2 - V^2})$. If we assume the approximate accuracy of relativity (i.e., give up classical fitting), we may determine the ratio of $e/m_e$ by comparing $p = eB\rho$ to $K = (\gamma - 1)mc^2$ and we may use $K = \sqrt{p^2c^2 + m^2c^4} - mc^2$ to determine $m_e$ and $e$ simultaneously.

## IV. PROCEDURE

### IV.1. Calibration

In order to get a meaningful reading from either the Gauss-meter or the MCA, we needed to calibrate the former to the magnetic field and the latter to a known energy spectrum.

The Gauss-meter calibration was mostly straightforward; a description can be found in [1]. We found a variation of approximately 2 G along the track. Estimation of this variation was made less accurate by the fact that the $\vec{B}$ field tends to decrease over time as the wires heat up, and we only had one Gauss-meter; we were unable to fix $B$ at a point while we measured variation. Repeated sweeps, however, suggested that 2 G is roughly accurate.

The MCA was calibrated against a $^{133}$Ba source. The energy spectrum of this source was obtained from [2] and [3]. Compton edges were calculated via the formula

$$E_{\text{Compton}} = \frac{2E^2}{m_ec^2 + 2E} \tag{4}$$

from [4]. To get an accurate energy calibration, we took an overnight reading of the energy spectrum from the $^{133}$Ba source. A rough fit of energy to channel values was obtained by manually fiddling with which energies to match to the first two prominent peaks. This determined an approximate linear fit, and these channel values for peaks and Compton edges were used as seed values in the following more precise fitting routine. At each peak, I attempted to fit a Gaussian, plus an offset, to the 20 channels surrounding the approximate peak. If successful, the mean and standard deviation were used as the channel number and channel number uncertainty for that peak. Otherwise, I maximized the interpolation function in a small region surrounding the approximate peak, and assigned an uncertainty of 5 channels. To match Compton edges, I took the simple but slightly inaccurate method of minimizing the derivative of a smoothed interpolation function in the region of the edge. Because some of the peaks and most of the Compton edges did not admit simple fitting, and I was worried that my fitting method could be overly sensitive to the seed value when the peak was not clear, I looked at a linear regression of energy vs. channel number both using only the six most prominent peaks, and using all peaks and Compton edges. The results were virtually identical, both giving a $\chi^2_\nu$ value of about 0.05 and a formula

$$K = (\text{channel \#})(0.456 \pm 0.002)\text{keV} + (27 \pm 1)\text{keV} \tag{5}$$

give or take one volt in the slope. The linear fit and resulting peaks are shown in Figure 2.

On subsequent days, we took calibration data for 15-30 minutes. The first 500 data points were fit against an interpolation of the long calibration run to obtain a constant offset and linear scaling of the channel to energy equation. The constant offset was less than half a channel, and the scaling was less than one part in a thousand channels, with a $\chi^2_\nu$ value of 1–1.5.

### IV.2. Energy Peaks

In order to determine the potential which gave the peak count-rate for a particular magnetic field strength, we counted the number of events in 30 s periods at intervals of 0.5 kV, starting from 0 kV and working up to approximately 5 kV. We eyeballed an approximate center and took 30 s long counts in a range of 1 kV around the peak, with a step size of 0.1–0.2 kV. Due to a lack of foresight, we simply eyeballed a center and took a 3–5 minute-long data collection at the approximate center.

To determine the potential which gave the peak count-rate for a particular magnetic field strength, I fit a Gaussian plus and offset to each set of data. Using Poisson count errors and $\pm 1$ in the last digit displayed on the multimeter for $V$ readings, the $\chi^2_\nu$ values for the fits ranged from 0.5 to 3.

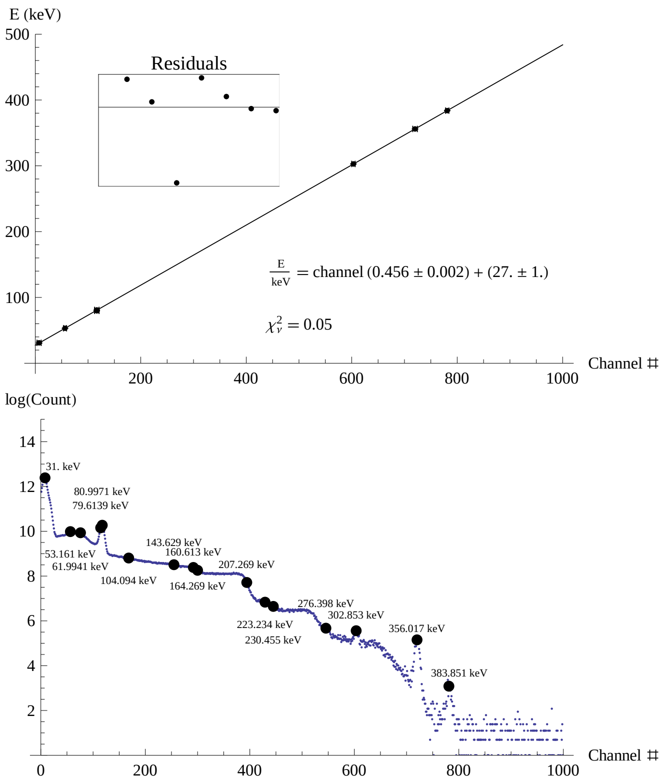


FIG. 2. Linear fit to the most prominent energy peaks, along with the log-scale spectrum collected overnight with peaks and Compton edges shown and labeled.

## V. ANALYSIS AND RESULTS

### V.1. Comparing Relativity and Newton

#### *V.1.1. Momentum-Velocity Relationships*

Since $p = eB\rho$, Newtonian dynamics predicts that $\beta = eB\rho/(m_e c)$, while relativity predicts that $\beta = \sqrt{1-\beta^2}eB\rho/(m_e c)$. Plotting our data and these two curves gives Figure 4.

Although relativity is clearly much better, there's a systematic error. I hypothesize that it comes from a constant offset in some combination of $V$, $B$, $d$ and $\rho$. I fit the data to the classical and relativistic curves using each of these as fit parameters in succession. For each parameter except for $d$, the residuals retained a strong systematic correlation with $B$. Using $d$ as a fit parameter and assuming the textbook values of $e$ and $m_e$ gives a $\chi^2_\nu$ value of about 0.05. This suggests that I overestimated the error in my values. The most likely candidate is the $B$ field, for which I used an uncertainty of 2 G; we measured 2 G variation along the path, but fixed the $B$ field at a point to within 0.02 G. Using 0.02 G as the uncertainty in $B$ gives a much more reasonable $\chi^2_\nu$ of 0.5, and a value of $d = 0.1696(4)\,\mathrm{cm}$ (as compared with the value of $d = 0.180(3)\,\mathrm{cm}$ given in [1]).

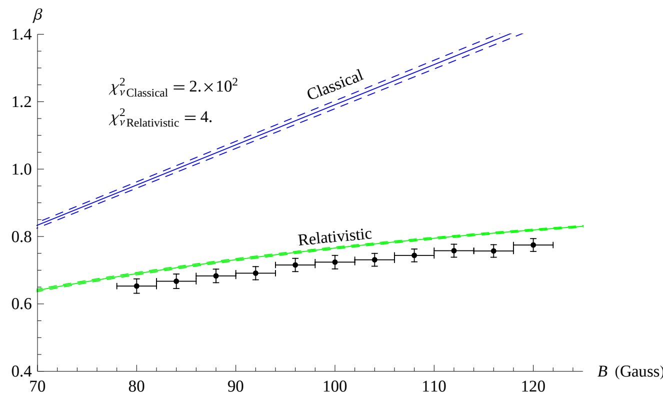


FIG. 3. Plot of $\beta$ vs. $B$, classical and relativistic predictions, assuming textbook values of electron charge and mass. Dotted lines show uncertainty in $\beta$ due to uncertainty in $\rho$.

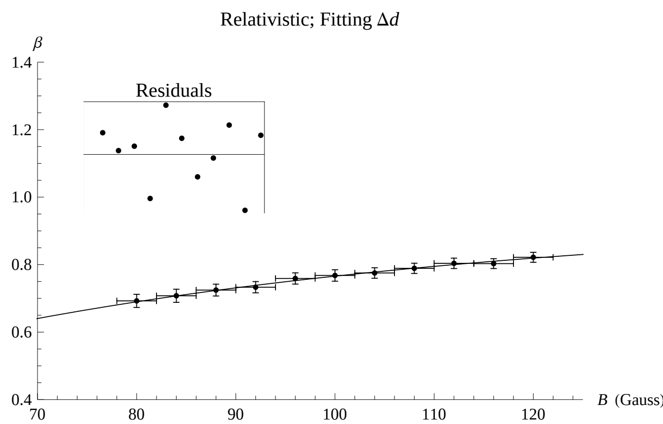


FIG. 4. Plot of $\beta$ vs. $B$, using $d$ as a fit parameter. $\chi^2_\nu = 0.05$

### V.2. Calculating $e/m_e$

Instead of comparing the plot of $\beta$ vs. $B$ to the predictions of relativity, we may instead calculate $e/m_e = \beta c/(B\rho)$ and fit this to a constant. The plot of $e/m_e$ vs. $B$ is shown in Figure 5. The classical value is $e/m_e = (1.04 \pm 0.01)\cdot 10^{11}\,\mathrm{C/kg}$ with $\chi^2_\nu = 3$. The relativistic value is $e/m_e = (1.54 \pm 0.03)\cdot 10^{11}\,\mathrm{C/kg}$ with $\chi^2_\nu = 0.1$. If we instead use the revised uncertainty of $\pm 0.02$ G in $B$ we get the classical value of $(1.053 \pm 0.008)\cdot 10^{11}\,\mathrm{C/kg}$ ($\chi^2_\nu = 10$) and the relativistic value of $(1.54 \pm 0.02)\cdot 10^{11}\,\mathrm{C/kg}$ ($\chi^2_\nu = 0.3$). The textbook value is $1.758\,820\,088(39)\cdot 10^{11}\,\mathrm{C/kg}$. [5]

### V.3. Calculating $m_e$; $K$ vs. $\beta$

We may calculate $\beta = V/(Bcd)$. Relativity predicts that $K = ((1-\beta^2)^{-1/2} - 1)m_e c^2$. Newtonian mechanics predicts that $K = \frac{1}{2}m_e c^2\beta^2$. A fit to the classical equation gives $m_e = (1.95 \pm 0.02)\cdot 10^{-32}\,\mathrm{kg}$ ($\chi^2_\nu = 3$) A fit to the relativistic equation gives $m_e =$

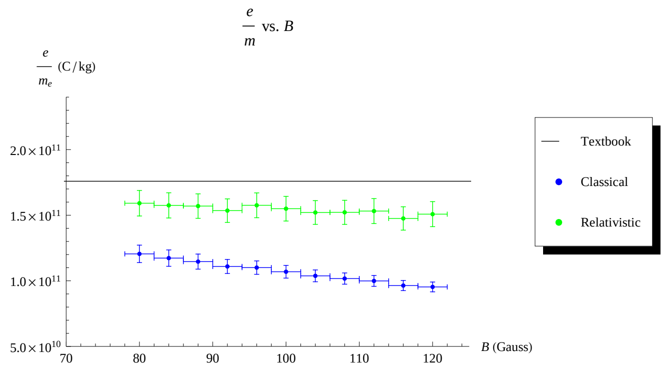


FIG. 5. Plot of $e/m_e$ vs. $B$. Classically, $e/m_e = (1.04 \pm 0.01) \cdot 10^{11}\ \mathrm{C/kg}$ ($\chi^2_\nu = 3$). Relativistically, $e/m_e = (1.54 \pm 0.03) \cdot 10^{11}\ \mathrm{C/kg}$ ($\chi^2_\nu = 0.1$).

$(1.13 \pm 0.04) \cdot 10^{-30}$ kg ($\chi^2_\nu = 0.2$). If instead we use the corrected value of $d$ and an uncertainty in $B$ of 0.02 G (see section V.1.1), we instead get $m_e = (9.1 \pm 0.3) \cdot 10^{-31}$ kg ($\chi^2_\nu = 0.2$) for the relativistic prediction. The textbook value is $9.109\,382\,91(40) \cdot 10^{-31}$ kg. [6] A plot of the relativistic and classical predictions, along with the corrected and uncorrected data, is shown in Figure 6.

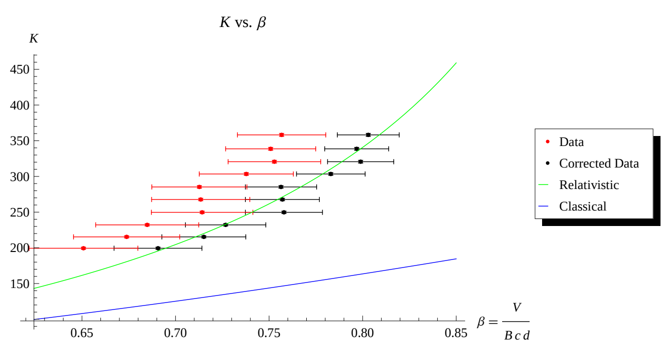


FIG. 6. Plot of $K$ vs. $\beta$. Plot includes the classical prediction, the relativistic prediction, the measured data, and the data after correction for the hypothesized systematic error in $d$.

### V.4. Calculating $e$ and $m_e$ simultaneously

A plot of $K$ vs. $p = Be\rho$ allows for calculation of $e$ and $m_e$ simultaneously. Figure 7 shows a plot of the data, along with classical ($K = p^2/2m$; $\chi^2_\nu = 200$), ultra-relativistic ($v \approx c$; $K = pc$; $\chi^2_\nu = 3000$), and relativistic ($K = \sqrt{p^2c^2 + m^2c^4} - mc^2$; $\chi^2_\nu = 10$) predictions.

The systematic error in the plot of $K$ vs. $p$ cannot be explained by an offset in any one measured value or parameter; the residuals remain systematically correlated. I hypothesize that the error is due to a combination of a systematic error in $\rho$ (order of magnitude is three times the stated uncertainty on diameter) and a systematic error in $K$ due to the fact that we eyeballed the the center of the $V$ readings (order of magnitude 5 keV; 0.05 to 0.1

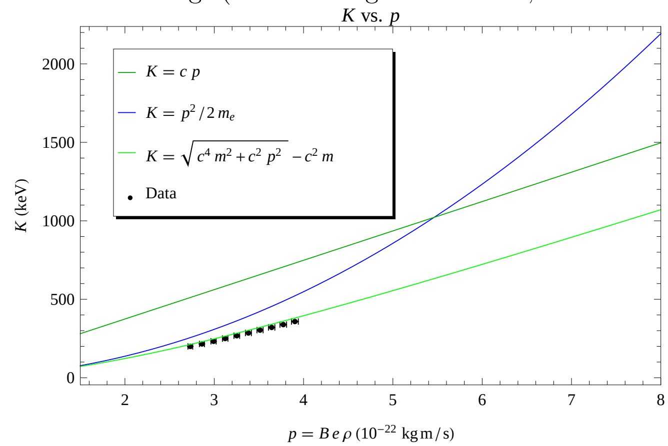


FIG. 7. Plot of $K$ vs. $p$, along with the classical, relativistic, and ultra-relativistic predictions.

kV). The $\chi^2_\nu$ value associated with using these variables as fit parameters is 110; it's a terrible fit, (though the residuals are not systematic) but I don't know what else might be causing the error.

I fit the data of $K$ vs. $p/e = B\rho$ to the relativistic model and hence calculated $e$ and $m_e$ simultaneously: $m_e = (9 \pm 2) \cdot 10^{-31}$ kg and $e = (1.5 \pm 0.1) \cdot 10^{-19}$ C ($\chi^2_\nu = 0.01$). The textbook values are $9.109\,382\,91(40) \cdot 10^{-31}$ kg and $1.602\,176\,565(35) \cdot 10^{-19}$ C, respectively. [6, 7] These values give an electron charge-to-mass ratio of $(1.75 \pm 0.45) \cdot 10^{11}\ \mathrm{C/kg}$.

The ultra-relativistic model permits a calculation of $e = (7.09 \pm 0.06) \cdot 10^{-20}$ C with $\chi^2_\nu = 40$ from $K$ vs. $p/e$; the residuals show a strong systematic correlation with $p$, though the plot is not included for lack of space.

The classical model permits a calculation of $e^2/m_e = (2.06 \pm 0.02) \cdot 10^{-8}\ \mathrm{C^2/kg}$ with $\chi^2_\nu = 4$ from $K$ vs. $p/e$; the residuals show a strong systematic correlation with $p$, though the plot is not included for lack of space. The textbook value is $2.817\,940\,3(12) \cdot 10^{-8}\ \mathrm{C^2/kg}$.

## VI. CONCLUSIONS

The data strongly support the relativistic model over the Newtonian model of dynamics for speeds 60%–80% the speed of light. If we use the energy and magnetic field strength data, we get good values for $e$ and $m_e$; the textbook values are within $1\sigma$. If we use the electric field strength data, we get bad values for $e/m_e$ and $m_e$; the textbook values are $4\sigma$ and $25\sigma$ away, respectively. I hypothesize an error in the (stated) measurement of $d$ of approximately three times the stated uncertainty as the most plausible explanation.

[1] N. R. Hunter-Jones, "Relativistic dynamics," (2010).
[2] T. Pantazis, "Scintillation detector and multichannel analyzer (mca) (homeland security applications)," (2006).
[3] R. B. Firestone and L. P. Ekström, "Table of isotopes decay data," (1999).
[4] "Compton edge," (2011).
[5] "CODATA value: electron charge to mass quotient," (2010).
[6] "CODATA value: electron mass," (2010).
[7] "CODATA value: electron charge," (2010).

## ACKNOWLEDGMENTS

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### Appendix A: Notational Conventions

Throughout this paper, $\vec{a}$ denotes a polar vector with unsigned magnitude $a$ pointing in the direction of the unit vector $\hat{a}$. I use SI units and equations, and the following variables:
$m$ is rest mass; $m_e$ is the mass of an electron
$q$ is signed charge; $-e$ is the charge of an electron
$\vec{p}$ is momentum
$\vec{F}$ is force
$\vec{v}$ is velocity
$\vec{E}$ is the electric field
$\vec{B}$ is the magnetic field
$V$ is an electric potential
$c$ is the speed of light
$E$ is total energy; $K$ is kinetic energy
$\vec{\beta} = \vec{v}/c$
$\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor
$t$ is time
$\rho$ is a radius of circular motion
$d$ is a separation between plates of a capacitor
$\vec{\omega}$ is angular velocity