Linear theory of superradiance in a free-electron laser

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This study is motivated by the analytical solutions of superradiance in a high-gain free-electron laser obtained by Bonifacio, Maroli, and Piovella, using the technique of Laplace transforms [Opt. Commun. **68**, 369 (1988)]. An error in these analytical solutions is remedied by a correct treatment of the boundary conditions on the electron beam, and the earlier theory is extended to allow for both electron shot noise and an optical pulse in the initial state. It is shown that, when the optical pulse is shorter than the electron pulse, superradiant behavior can also occur at the leading edge of the optical pulse.

I. INTRODUCTION

In a free-electron laser (FEL), a relativistic electron beam passes through the transverse periodic magnetic field of an undulator (or wiggler), transferring energy to a copropagating electromagnetic wave. For the device to generate coherent radiation, it is necessary to satisfy a resonance condition which requires the electrons to slip one wavelength behind the radiation as the electrons pass over one undulator period. When this resonance condition is satisfied, the electron beam, after traveling a distance z, lags behind the optical beam by a slippage distance given by

$$S=(\lambda_s/\lambda_w)z \ , \tag{1}$$

where λ_s and λ_w are, respectively, the wavelength of the radiation and the undulator. A standard approximation, frequently made in theoretical analyses, is to take the slippage distance S to be much smaller than the input electron pulse length L_B (or optical pulse length L). In this "long-pulse" approximation, one can follow electrons within one period of the ponderomotive potential well and assume that electrons in adjacent wells satisfy a periodic boundary condition. This approximation is clearly violated if $S \geq L_B$; even if $S < L_B$, the periodic boundary condition does not hold at the edge of the electron pulse.

Recently, Bonifacio, Maroli, and Piovella[1] have presented some interesting analytical solutions in the linear regime taking into account the effect of slippage. Two distinct regimes are identified in Ref. 1. In one regime, referred to hereafter as the steady-state regime, the effect of slippage can be neglected, and the intensity scales as $n_e^{4/3}$, where n_e is the electron density. In the other regime, the effect of slippage is crucial, and the peak intensity scales as n_e^2. This latter regime is shown to be significant when the slippage distance S is comparable with or larger than the electron pulse length. By analogy with laser physics,[2] Bonifacio and Casagrande have termed this regime the "superradiant" regime,[1,3] though it has been noted in Ref. 4 that perhaps the term "superfluorescence" is more appropriate. The occurrence of superradiance has been confirmed by recent numerical simulations.[4,5]

Analytical solutions describing superradiant behavior have been obtained in Ref. 1 by using Laplace transforms. Particular solutions have been given for the start-up of a high-gain FEL from initial conditions of zero optical field and electron shot noise. While these solutions predict correctly the existence of the superradiant regime for $S \leq L_B$, we show that the analysis given in Ref. 1 is not quite correct for $S > L_B$, which is precisely when superradiance is expected to be a dominant effect. In particular, we show that the evolution of the radiation field is qualitatively distinct in the two cases $S = L_B$ and $S > L_B$, a distinction that has not been made in the analytical treatment of Ref. 1. Thereafter, we give general solutions for the evolution of the radiation field in the presence of a finite optical pulse of length L at input. A new result that follows is the occurrence of superradiance at the leading edge of the optical pulse when $L < L_B$. (A preliminary account of these results was presented recently.[6])

II. DYNAMICAL EQUATIONS

We begin our analysis with the one-dimensional FEL equations[7,8]

$$\frac{d\gamma_j}{dz} = -\frac{k_s a_s a_w}{\gamma_j}\sin(\psi_j+\phi) \ , \tag{2a}$$

$$\frac{d\psi_j}{dz} = k_w\left[1-\frac{\gamma_r^2}{\gamma_j^2}\right] + \frac{k_s a_s a_w \cos(\psi_j+\phi)}{\gamma_j^2} \ , \tag{2b}$$

and

$$\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right]u = \frac{i a_w \omega_p^2}{2k_s c^2}\left\langle \frac{\exp(-i\psi)}{\gamma}\right\rangle \ , \tag{2c}$$

where z is the direction of propagation of the electron and optical beams and also coincides with the undulator axis; $\psi_j+\phi$ is the relative phase of the electron (of rest mass m) with respect to the radiation field, and $\psi_j mc^2$ is its energy; $\mathbf{A}=\mathbf{A}_w+\mathbf{A}_s$, $a_w = eA_w/mc^2$, $a_s = eA_s/mc^2$; $k_w, k_s=\omega/c$ are the wave numbers of the undulator and radiation fields, respectively; ω is the radiation frequency; $u = a_s \exp(i\phi)$ is the complex amplitude of the radiation field; and γ_r is the resonant electron energy factor, defined by the relation

$$\gamma_r^2 = \frac{k_s(1+a_w^2)}{2k_w} . \tag{3}$$

The symbol $\langle\ \rangle$ denotes an ensemble average over electrons.

We introduce the following variable definitions:

$$\Gamma_j = (\gamma_j - \gamma_0)/\gamma_0 , \tag{4a}$$

$$\delta = k_w(1-\gamma_r^2/\gamma_0^2) , \tag{4b}$$

$$\Psi_j = \psi_j - \delta z , \tag{4c}$$

$$A = u \exp(i\delta z) . \tag{4d}$$

Here γ_0 is the initial energy of the electron beam, taken to be monoenergetic. By linearizing Eqs. (2) around the equilibrium, $\Gamma_{j0}=0$, $\langle \exp(-i\Psi_0)\rangle = 0$, $A_0=0$, and introducing collective variables $x = -\langle i\delta\Psi \exp(-i\Psi_0)\rangle$ and $y = \langle \Gamma \exp(-i\Psi_0)\rangle$, where $\delta\Psi_j = \Psi_j - \Psi_{j0}$, Γ_j and A are small quantities, we obtain the system of equations

$$\frac{dx}{dz} = -ihy - ifA , \tag{5a}$$

$$\frac{dy}{dz} = ifA , \tag{5b}$$

$$\left[\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right] A = i\delta A + ig(x-y) , \tag{5c}$$

where $f \equiv k_s a_w/(2\gamma_0^2)$, $g \equiv \omega_p^2 a_w/(2k_s c^2 \gamma_0)$, and $h \equiv 2k_w \gamma_r^2/\gamma_0^2$. Note that in Eq. (5c) the parameter g depends on the density of the electron beam and is zero outside the beam. Inspection of the analogous equation in Ref. 1 [Eq. (4c)] shows that the explicit dependence of the radiation equation on the electron density is obscured by the dimensionless variables used. Since the phenomenon of superradiance occurs at the edges of an electron beam, it is important to track this density dependence explicitly in order that the boundary conditions on the beam can be imposed correctly.

We assume that the electrons are continuously distributed along the interaction region inside the pulse so that the electron beam can be treated as a fluid, each element of which moves at the average speed $\beta_\parallel = V_\parallel/c$. Hence the Lagrangian derivative along z in Eqs. (5a) and (5b) can be replaced by the Eulerian derivative, that is,

$$\frac{d}{dz} = \frac{\partial}{\partial z} + \frac{1}{\beta_\parallel c}\frac{\partial}{\partial t} . \tag{6}$$

It is convenient to transform to the coordinates,

$$\xi = z , \tag{7}$$

$$\tau = \frac{z - \beta_\parallel c t}{1-\beta_\parallel} , \tag{8}$$

where τ measures the position in the electron-beam frame. We choose the origin $z=0$ to coincide with the beginning of the undulator. Equations (5) can then be solved by Laplace transforms. For a function $F(\xi,\tau)$, the Laplace transform is defined by

$$\tilde{F}(p,\tau) = \int_0^\infty d\xi \exp(-p\xi) F(\xi,\tau) . \tag{9}$$

Equations (5) then yield the ordinary differential equation

$$\frac{d\tilde{A}}{d\tau} + \left[p - i\delta - \frac{2fg}{p} - \frac{ifgh}{p^2}\right]\tilde{A} = A_0 + ig\frac{x_0 - y_0}{p} + \frac{ghy_0}{p^2} , \tag{10}$$

where $A_0 \equiv A(z=0,\tau)$, $x_0 \equiv x(z=0,\tau)$, and $y_0 \equiv y(z=0,\tau)$ represent initial conditions at $z=0$. For simplicity we will take $y_0=0$, and assume that the initial electron and optical pulses to be rectangular and of lengths L_B and L, respectively, with their trailing edges aligned at $z=0$ when $\tau=0$. Note that the slippage is then given by $\tau=\xi$.

III. ANALYTICAL SOLUTIONS FOR ZERO INITIAL RADIATION FIELD

In order to compare the results of our analysis with that of Ref. 1, we first consider the simple case $A_0=0$. Then Eq. (10) can be integrated in τ to give

$$\tilde{A}(p,\tau) = \begin{cases} \dfrac{igx_0}{p\lambda(p)}\{1-\exp[-\lambda(p)\tau]\}, & 0 \le \tau \le L_B' \\ \dfrac{igx_0}{p\lambda(p)}\exp[-(p-i\delta)(\tau - L_B')] \\ \qquad \times\{1-\exp[-\lambda(p)L_B']\}, & L_B' < \tau \end{cases} \tag{11}$$

where $\lambda(p) \equiv p - i\delta - 2fg/p - ifgh/p^2$ and $L_B' \equiv L_B/(1-\beta_\parallel)$. In obtaining Eq. (11), we have imposed the condition that $\tilde{A}$ is continuous at $\tau = L_B'$.

The inverse Laplace transform of $\tilde{A}(p,\tau)$ is given by the standard formula

$$A(\xi,\tau) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} \exp(p\xi)\tilde{A}(p,\tau)dp , \tag{12}$$

where c_0 is chosen large enough so that all singularities of the integrand lie to the left of the straight line along which the integral is taken. From Eq. (11), we then obtain the following solutions for $A(\xi,\tau)$: for $0 \le \tau \le L_B'$ and $\xi > \tau$

$$A = \mathrm{Res}\left[-\frac{igx_0}{p\lambda(p)}\exp[p\xi - \lambda(p)\tau]\right]\Bigg|_{p=0} , \tag{13}$$

where $\mathrm{Res}(\ )|_{p=0}$ denotes the Cauchy residue at $p=0$. For $\xi \le \tau \le L_B'$

$$A = \sum_{l=1}^{3} \frac{igx_0 p_l}{(p_l - p_m)(p_l - p_n)}\exp(p_l \xi), \quad l \ne m \ne n \tag{14}$$

where p_l ($l=1,2,3$) are the three roots of the cubic equation $p^2\lambda(p)=0$. For $\xi > \tau > L_B'$

$$A=\operatorname{Res}\left\{-\frac{igx_0}{p\lambda(p)}\exp[-(p-i\delta)(\tau-L_B')-\lambda(p)L_B'+p\xi]\right\}\Bigg|_{p=0} . \tag{15}$$

For $\tau>\xi>\tau-L_B'$ and $\tau>L_B'$

$$A=\sum_{l=1}^{3}\frac{igx_0p_l}{(p_l-p_m)(p_l-p_n)}\exp[p_l\xi-(p_l-i\delta)(\tau-L_B')] ,\quad l\neq m\neq n . \tag{16}$$

Finally, for $\tau>\xi+L_B'$

$$A=0 . \tag{17}$$

Note that solutions (13) and (14) are similar to those obtained by Bonifacio *et al.*[1] Solution (13) is the superradiant solution.

On the basis of the solutions given above, we can depict schematically the optical amplitude $|A|$ as a function of τ at a fixed value of ξ. Figure 1(a) describes the case $S<L_B$ (or $\xi<L_B'$), which corresponds to the long-pulse limit. There are three regions: the exponentially growing region ($\xi+L_B'>\tau>L_B'$), the steady-state region ($L_B'>\tau>\xi$), and the superradiant region ($\xi>\tau>0$). Figure 1(b) describes the special case $S=L_B$, in which case the steady-state region disappears. Figures 1(a) and 1(b) are similar to that given in Ref. 1.

Figure 1(c) describes the case $S>L_B$ (or $\xi>L_B'$), which corresponds to a short electron pulse. In this case our results are significantly different from the results of Ref. 1, in which no distinction is made between the cases $S=L_B$ and $S>L_B$. Contrasting Figs. 1(a) and 1(b), we note that for $S>L_B$ a new region appears for τ in the range $L_B'<\tau<\xi$. The solution for this region is given by Eq. (15), and is of the superradiant type. The physical mechanism for the occurrence of this region is as follows. As the radiation pulse interacts with and eventually passes over the leading edge of the electron pulse, the superradiance within the electron pulse grows and eventually slips out of the leading edge of the electron pulse. However, once the radiation slips out of the electron pulse, it can no longer grow, and merely oscillates with the phase $\exp(i\delta\tau)$, with an amplitude that depends on $(\xi-\tau)$.

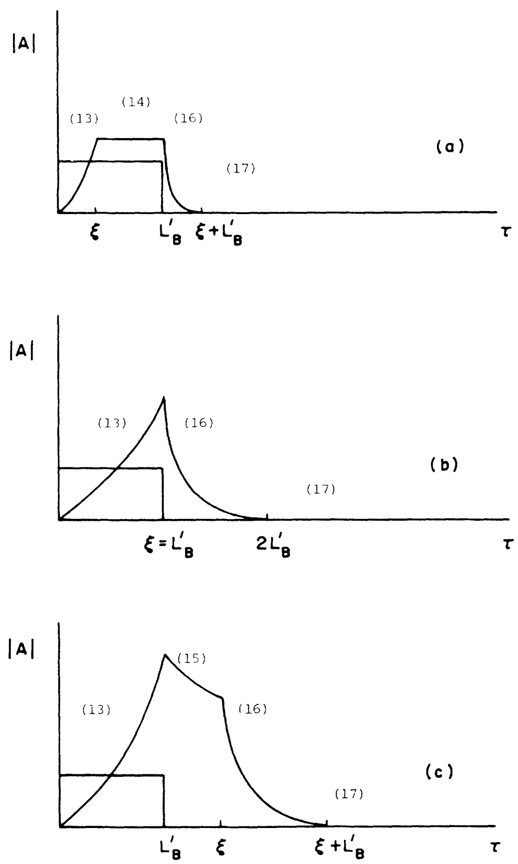

FIG. 1. Schematic plots of the optical amplitude as a function of τ, in the case of zero initial optical field. The amplitude is viewed at three positions in the undulator: (a) $S<L_B$ (or $\xi<L_B'$), (b) $S=L_B$ (or $\xi=L_B'$), and (c) $S>L_B$ (or $\xi>L_B'$). The rectangles stand for the electron pulse and the number in parentheses above each sector of a curve is the equation number in the text that describes that sector.

IV. ANALYTICAL SOLUTIONS WITH NONZERO RADIATION FIELD

We now study the case when $A_0\neq 0$. For specificity, we assume that the electron and optical pulses are rectangular, as is shown in Fig. 2. In general $L_B\neq L$, and we allow for both possibilities, $L_B>L$ or $L_B<L$. We define $L'\equiv L/(1-\beta_\parallel)$.

We first look at the case $L>L_B$. In this case, the solution to Eq. (10) is

$$\widetilde{A}=\begin{cases}\dfrac{A_0+igx_0/p}{\lambda(p)}\{1-\exp[-\lambda(p)\tau]\}, & 0\le\tau\le L_B'\\[2ex] \dfrac{A_0+igx_0/p}{\lambda(p)}\exp[-(p-i\delta)(\tau-L_B')]\\ \quad\times\{1-\exp[-\lambda(p)L_B']\}\\ \quad+\dfrac{A_0}{p-i\delta}\{1-\exp[-(p-i\delta)(\tau-L_B')]\} , & L_B'<\tau\le L'\\[2ex] \dfrac{A_0+igx_0/p}{\lambda(p)}\exp[-(p-i\delta)(\tau-L_B')]\\ \quad\times\{1-\exp[-\lambda(p)L_B']\}\\ \quad+\dfrac{A_0}{p-i\delta}\{1-\exp[-(p-i\delta)(L'-L_B')]\}\\ \quad\times\exp[-(p-i\delta)(\tau-L')], & L'<\tau .\end{cases} \tag{18}$$

The inverse Laplace transform gives the following solutions: for $0\le\tau\le\min(L_B',\xi)$

$$A=\operatorname{Res}\left\{-\frac{A_0+igx_0/p}{\lambda(p)}\exp[-\lambda(p)\tau+p\xi]\right\}\Bigg|_{p=0} , \tag{19}$$

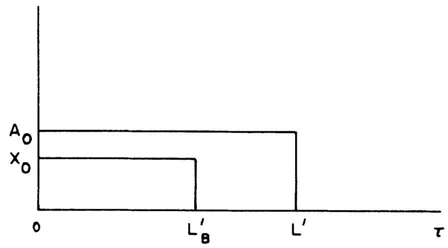

FIG. 2. Initial electron shot noise and optical input pulse. Here the electron pulse length L_B' can be either greater or smaller than the optical pulse length L'.

which is again the superradiant solution. For $\xi \le \tau < L_B'$

$$A = \sum_{l=1}^{3} \frac{(p_l A_0 + igx_0)p_l}{(p_l - p_m)(p_l - p_n)} \exp(p_l \xi), \quad l \ne m \ne n \tag{20}$$

which is the steady-state solution. For $L_B' \le \tau < \xi$

$$A = \mathrm{Res}\left[-\frac{A_0 + igx_0/p}{\lambda(p)} \times \exp[p\xi - \lambda(p)L_B' - (p - i\delta)(\tau - L_B')] \right] \Bigg|_{p=0} , \tag{21}$$

which is the superradiance that has slipped out of the leading edge of the electron pulse. For $\max(\xi, L_B') \le \tau < \xi + L_B'$

$$A = \exp[i\delta(\tau - L_B')] \sum_{l=1}^{3} \frac{(p_l A_0 + igx_0)p_l}{(p_l - p_m)(p_l - p_n)} \times \exp[p_l(\xi - \tau + L_B')] , \quad l \ne m \ne n . \tag{22}$$

For $\xi + L_B' \le \tau < \xi + L'$

$$A = A_0 \exp(i\delta\xi) , \tag{23}$$

which is the part of the initial optical pulse that does not interact with the electron pulse, so that $u = A \exp(-i\delta z) = A_0$ remains constant. Finally, for $\xi + L' < \tau$

$$A = 0 . \tag{A24}$$

The solutions (19)–(24) are schematically drawn in Fig. 3. Figure 3(a) describes the case $\xi < L_B'$, Fig. 3(b) the case $\xi = L_B'$, and in Fig. 3(c) the case $\xi > L_B'$. Note that these solutions reduce to the special case discussed in Sec. III if we set $A_0 = 0$.

We now show that the radiation growing at the trailing edge has indeed the scaling properties of superradiance. To see this, we look at the first term in (19), which we write as

$$A_1 = \mathrm{Res}\left[-\frac{A_0}{\lambda(p)} \exp[-\lambda(p)\tau + p\xi] \right] \Bigg|_{p=0} . \tag{25}$$

To make the problem simpler, we assume that the FEL is perfectly tuned, so that $\delta = 0$, and we neglect the small term $2fg/p$ in $\lambda(p)$. Then A_1 becomes

$$A_1 = \mathrm{Res}\left[-\frac{A_0 p^2}{p^3 - ifgh} \times \exp\left[p(\xi - \tau) + \frac{ifgh}{p^2}\tau \right] \right] \Bigg|_{p=0} = \frac{A_0}{ifgh} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{1}{ifgh} \right]^l \frac{(\xi - \tau)^n (ifgh)^m}{n!m!} , \tag{26}$$

where $3l + n - 2m = -3$. In the short-pulse limit, $fgh\tau \ll 1$, and at $\tau = \xi$, only $N = 0$ terms contribute. Therefore the leading term in (26) is the term with $n = 0$, $m = 3$, and $l = 1$. Then

$$A_1 \sim \frac{A_0}{ifgh} \frac{1}{ifgh} \frac{(ifgh\tau)^3}{6} = \frac{iA_0}{6} fgh\tau^2 \sim n_e , \tag{27}$$

which implies that the radiation intensity scales as n_e^2.

We now consider the complementary case $L < L_B$. This case describes what happens in an FEL oscillator in

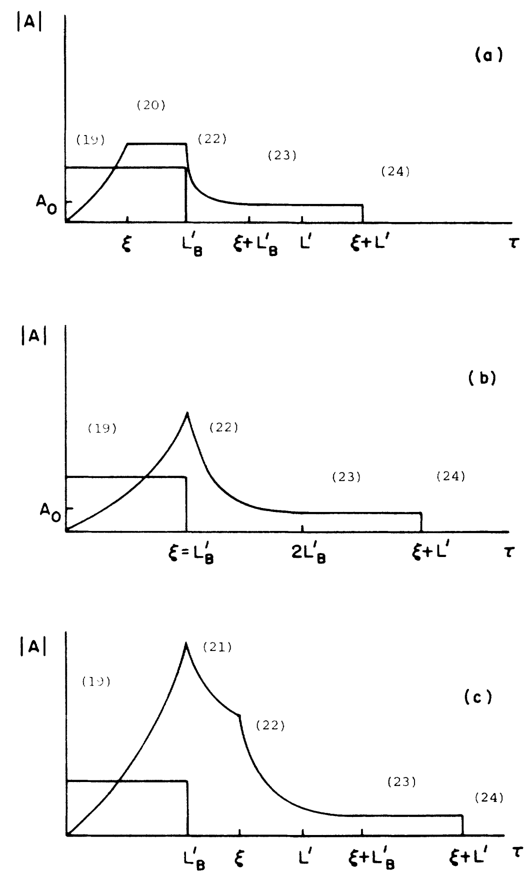

FIG. 3. Schematic plots of the optical field amplitude as a function of τ, in the case of a nonzero initial optical pulse and $L > L_B$ (or $L' > L_B'$). The amplitude is viewed at three positions in the undulator: (a) $S < L_B$ (or $\xi < L_B'$), (b) $S = L_B$ (or $\xi = L_B'$), and (c) $S > L_B$ (or $\xi > L_B'$).

which the optical pulse is shorter than the electron pulse and their trailing edges are aligned at the entry of the undulator. In this case, the solution to Eq. (10) is

$$\tilde{A}=\begin{cases}\dfrac{A_0+igx_0/p}{\lambda(p)}\{1-\exp[-\lambda(p)\tau]\}, \quad 0\le\tau\le L' \\ \dfrac{A_0+igx_0/p}{\lambda(p)}\{\exp[-\lambda(p)(\tau-L')] \\ \qquad -\exp[-\lambda(p)\tau]\} \\ \quad +\dfrac{igx_0/p}{\lambda(p)}\{1-\exp[-\lambda(p)(\tau-L')]\}, \\ \qquad\qquad L'<\tau\le L_B' \\ \left[\dfrac{A_0+igx_0/p}{\lambda(p)}\{\exp[-\lambda(p)(L_B'-L')] \right.\\ \qquad -\exp[-\lambda(p)L_B']\} \\ \quad \left.+\dfrac{igx_0/p}{\lambda(p)}\{1-\exp[-\lambda(p)(L_B'-L')]\}\right] \\ \quad \times\exp[-(p-i\delta)(\tau-L_B')], \quad L_B'\le\tau\,.\end{cases} \tag{28}$$

The inverse Laplace transformation of (28) gives the following results: for $0\le\tau<\min(\xi,L')$

$$A=\operatorname{Res}\left[-\frac{A_0+igx_0/p}{\lambda(p)}\exp[-\lambda(p)\tau+p\xi]\right]\Bigg|_{p=0}, \tag{29}$$

which is the same as Eq. (19) and is the superradiant solution. For $\xi\le\tau<L'$

$$A=\sum_{l=1}^{3}\frac{(A_0+igx_0/p_l)p_l^2}{(p_l-p_m)(p_l-p_n)}\exp(p_l\xi), \quad l\ne m\ne n\,, \tag{30}$$

which is the same as (20) and is the steady-state solution. For $L'\le\tau<\min(\xi,L_B')$ and $L'<\xi$

$$A=\operatorname{Res}\left[\frac{A_0+igx_0/p}{\lambda(p)}\{\exp[-\lambda(p)(\tau-L')] -\exp[-\lambda(p)\tau]\}\exp(p\xi) -\frac{igx_0}{p\lambda(p)}\exp[-\lambda(p)(\xi-L')+p\xi]\right]\Bigg|_{p=0}. \tag{31}$$

For $\max(\xi,L')\le\tau<\min(\xi+L',L_B')$ and $\xi<L_B'$

$$A=\operatorname{Res}\left[\frac{A_0}{\lambda(p)}\exp[-\lambda(p)(\tau-L')+p\xi]\right]\Bigg|_{p=0} +\sum_{l=1}^{3}\frac{(p_lA_0+igx_0)p_l}{(p_l-p_m)(p_l-p_n)}\exp(p_l\xi), \quad l\ne m\ne n \tag{32}$$

which is a new solution. The first term in (32) is of the same form as Eq. (25), describing superradiance growing from A_0. Therefore (32) is a combination of the superradiant and steady-state solutions. For $\xi+L'\le\tau<\xi+L_B'$ and $\xi<L_B'-L'$

$$A=\sum_{l=1}^{3}\frac{igx_0p_l}{(p_l-p_m)(p_l-p_n)}\exp(p_l\xi), \quad l\ne m\ne n \tag{33}$$

is another steady-state solution. For $L_B'\le\tau<\xi$

$$A=\operatorname{Res}\left[\frac{A_0}{\lambda(p)}\{\exp[-\lambda(p)(L_B'-L')]-\exp[-\lambda(p)L_B']\} -\frac{igx_0}{p\lambda(p)}\exp[-\lambda(p)(L_B'-L')]\exp[-(p-i\delta)(\tau-L_B')+p\xi]\right]\Bigg|_{p=0}. \tag{34}$$

For $\max(\xi,L_B')\le\tau<\xi+L'$ and $\xi>L_B'-L'$

$$A=\sum_{l=1}^{3}\frac{(p_lA_0+igx_0)p_l}{(p_l-p_m)(p_l-p_n)}\exp[-(p_l-i\delta)(\tau-L_B')+p_l\xi] +\operatorname{Res}\left[\frac{A_0}{\lambda(p)}\exp[-\lambda(p)(L_B'-L')-(p-i\delta)(\tau-L_B')+p\xi]\right]\Bigg|_{p=0}, \tag{35}$$

which is again a combination of the steady-state and superradiant solutions. For $\max(\xi+L',L_B')\le\tau<\xi+L_B'$

$$A=\sum_{l=1}^{3}\frac{igx_0p_l}{(p_l-p_m)(p_l-p_n)} \times\exp[-(p_l-i\delta)(\tau-L_B')+p_l\xi]\,. \tag{36}$$

For $\xi+L_B'<\tau$

$$A=0\,. \tag{37}$$

Solutions (29)–(37) for the amplitude of the optical pulse are represented schematically in Figs. 4–6. In Fig. 4, we plot the case $S<L$ (or $\xi<L'$), which describes a long electron pulse. When the slippage is small so that $S+L<L_B$ (or $\xi+L'<L_B'$), there are, as shown in Fig. 4(a), two regions of steady-state behavior separated by a region that contains a combination of steady-state and superradiant behavior. However, as shown in Fig. 4(b), when the slippage increases so that $S+L>L_B$, the steady-state behavior in the leading edge disappears. Fig-

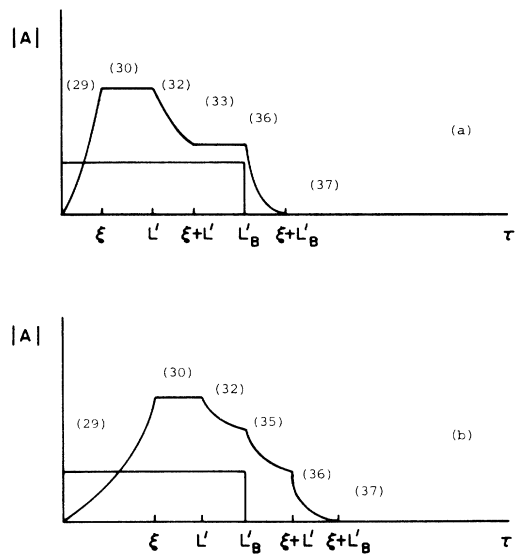

FIG. 4. Schematic plots of the optical field amplitude as a function of τ, in the case $x_0 \neq 0$, $A_0 \neq 0$, and $L < L_B$. The amplitude is viewed at positions $S < L$ (or $\xi < L'$) and for (a) $S+L < L_B$ (or $\xi+L' < L_B'$), (b) $S+L > L_B$.

ure 5 shows similar plots except that now $L < S < L_B$ (or $L' < \xi < L_B'$), and therefore a superradiant region replaces the steady-state region at the trailing edge. Figure 6 shows the case $L_B < S$ (or $L_B' < \xi$), which corresponds to the short-pulse limit.

Note an interesting new feature in each of the Figs. (4)–(6): there exists a region that is a combination of the steady-state and superradiant solutions occurring at the leading edge of the optical pulse. The physical explanation for the occurrence of this "semi-superradiant" region is as follows. In this region, the steady-state part of the radiation field is generated by the standard FEL mechanism from the noise sources A_0 and x_0. However,

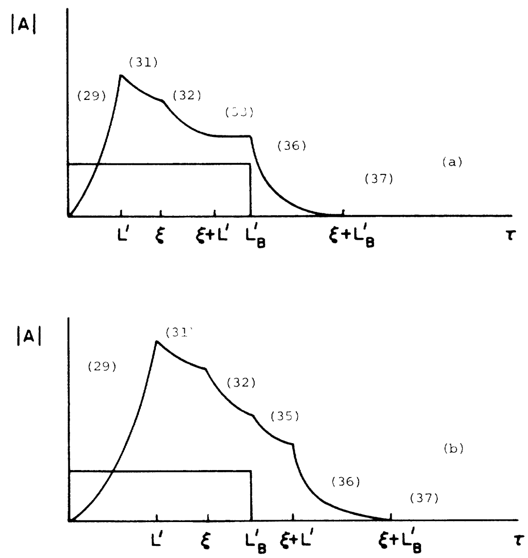

FIG. 5. Schematic plots of the optical field amplitude at positions $L < S < L_B$ (or $L' < \xi < L_B'$) as a function of τ, in the case $x_0 \neq 0$, $A_0 \neq 0$, and $L < L_B$. (a) is for $S+L < L_B$ (or $\xi+L' < L_B'$) and (b) is for $S+L > L_B$ (or $\xi+L' > L_B'$).

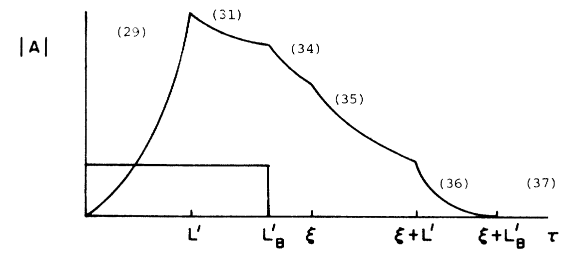

FIG. 6. Schematic plot of the optical field amplitude at positions $S > L_B$ as a function of τ when $x_0 \neq 0$, $A_0 \neq 0$, and $L < L_B$.

as the electrons in front of the initial optical pulse slip into the pulse, they experience rapidly varying external fields and emit spontaneous radiation which contributes to superradiance, the magnitude of which depends on A_0. However, after this region slips over the entire electron pulse, the growth of the superradiant component is stopped, with the consequence that the radiation amplitude in the leading edge is usually smaller than that at the trailing edge.

V. SUMMARY

In this paper, we have given a linear theory of superradiance for a free-electron laser in the high-gain Compton regime. One of the aims of this paper has been to describe the leading as well as the trailing edges of the optical pulse, and to fix correctly the boundary conditions at the edges of the electron pulse. We caution that computational methods that do not correctly incorporate these boundary conditions may produce spurious behavior in the radiation field dynamics.

When the FEL evolves from the initial conditions of zero initial optical field, which is the case considered in Ref. 1, we show that the evolution of the radiation pulse proceeds in qualitatively different ways in the three cases $S < L_B$, $S = L_B$, and $S > L_B$. We then extend the calculation of Ref. 1 to allow for the presence of an initial optical pulse of length L, which may be greater or smaller than L_B. Whereas superradiance is a persistent feature of the radiation field at the trailing edge of the electron beam, we find that it can also occur at the leading edge of the optical pulse when $L < L_B$.

The one-dimensional linear theory presented here is a first step, but leaves several interesting questions unanswered. The nonlinear evolution of superradiance is a subject of considerable interest. Though superradiance and sidebands differ in their growth rates, numerical simulations seem to suggest that they are essentially indistinguishable in their nonlinear states. This feature deserves closer scrutiny. We also note that the optical "spikes" seen in numerical simulations are sufficiently singular that the "slowly varying" approximation routinely used for the radiation field in a high-gain FEl is open to question.

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