18.708 Problem set 2

Due Thursday, March 17, 2022, in Canvas

Problem 1. Let k be a field. The **Hochschild homology** of an associative algebra A over k is

$$HH_i(A) = \mathrm{Tor}_i^{A-\mathrm{bimod}}(A, A).$$

(i) Prove the **Künneth formula**

$$HH_n(A \otimes B) = \oplus_{r+s=n} HH_r(A) \otimes HH_s(B).$$

(ii) Compute $HH_*(A)$ for $A = A_1 = k[x]$ using the resolution

$$A \otimes A \to A \otimes A \to A,$$

where the second map is the multiplication of components and the first map $k[x, y] \to k[x, y]$ is multiplication by $x - y$. What is the multiplicative structure of $HH_*(A)$?

(iii) Let $A = A_n = k[x_1, ..., x_n]$. Compute the Hochschild homology $HH_*(A)$ using (ii).

(iv) Define a resolution of A as an A-bimodule

$$P_n \to ... \to P_0 \to A$$

with $P_m := A \otimes \wedge^m(x_1, ..., x_n) \otimes A$ (the **bimodule Koszul resolution**), and use it to rederive (iii). Compute the multiplicative structure on $HH_*(A)$.

Problem 2. (i) Show that for any ring A, the groups $HH_*(A)$ are naturally modules over the center Z of A. In particular, if A is commutative then they are A-modules.

(ii) (optional) Let X be a smooth affine irreducible algebraic variety over k, and O_X be its algebra of regular functions. Show that there is an algebra isomorphism $HH_*(O_X) \cong \Omega^*(X)$, where $\Omega^*(X)$ is the algebra of regular differential forms on X (the **Hochschild-Kostant-Rosenberg** isomorphism).

Hint. Define a natural homomorphism of algebras $\phi : \Omega^*(X) \to HH_*(O_X)$ using a presentation of $\Omega^*(X)$ by generators and relations (this should allow you to define the homomorphism only in degrees 0 and 1). Then show that it is an isomorphism by localizing to (formal or étale) neighborhoods of points of X and using Problem 1.

Problem 3. (i) Let A be a $\mathbb{Z}_+$-filtered algebra. Explain why there is a spectral sequence starting with $HH_*(\mathrm{gr}A)$ which converges to $HH_*(A)$. Deduce that $\dim HH_*(\mathrm{gr}A) \geq \dim HH_*(A)$.

(ii) (optional) Let $D(X)$ be the algebra of differential operators on a smooth irreducible affine variety X of dimension n over $k = \mathbb{C}$. It has a filtration with $\mathrm{gr}D(X) = O_{T^*X}$, where T^*X is the cotangent bundle of X. Show that $HH_i(D(X)) \cong H^{2n-i}(X, \mathbb{C})$, the algebraic de Rham cohomology of X. By a theorem of Grothendieck, this coincides with the topological cohomology $H_c^i(X(\mathbb{C}), \mathbb{C}) = H^{2n-i}(X(\mathbb{C}), \mathbb{C})$. Conclude that $HH_i(D(X)) = 0$ unless $n \leq i \leq 2n$.