18.708, Homework for FI-modules

Due Tuesday May 10, 2022. Email your homework to me, or hand in during class.

1. In class we defined the notion of a concrete FI-module. For each $n \geq 0$, we have a $k[S_n]$-module M_n and transition maps $t_n : M_n \to M_{n+1}$ which are S_n-equivariant, satisfying the property that the image of $t_{n+m-1} \circ \cdots \circ t_n$ in M_{n+m} is S_m-invariant. Prove that the abelian category of FI-modules is equivalent to the abelian category of concrete FI-modules.

2. Given a $k[S_d]$-module V, show that the induced module $I(V)$ satisfies the desired universal property:

$$\mathrm{Hom}_{\mathrm{FI-mod}}(I(V), M) = \mathrm{Hom}_{k[S_d]}(V, M_d)$$

   for any FI-module M. Give an example, when k is a field of characteristic p, where $I(V)$ is not a projective object.

3. Given two FI-modules M and N over k, we can define the tensor product $M \otimes_k N$ to be the FI-module whose k-module in degree a is $M_a \otimes_k N_a$ with the natural transition maps between them. Prove that if M and N are finitely generated, then so is their tensor product.