0 Main References


1 Superalgebra

Work over a field $k$ of characteristic $p \neq 2$ (assume $p \geq 3$ unless say otherwise).

- Super vector spaces $V = V_0 \oplus V_1$
- Sign convention
- Tensor products $V \otimes W \cong W \otimes V$, $v \otimes w \mapsto (-1)^{\pi_1}w \otimes v$
- Linear maps $\text{Hom}_k(V, W) = \text{Hom}_k(V, W)_0 \oplus \text{Hom}_k(V, W)_1$
- Underlying abelian subcategory $\text{Hom}_{\text{sec-ev}}(V, W) = \text{Hom}_k(V, W)_{\overline{1}}$
- Parity change functor(s) $\Pi(V) = V \otimes k^{0|1}$, $\Pi(V) = k^{0|1} \otimes V$
- Commutative superalgebras e.g., symmetric superalgebra $S(V) \cong S(V_0) \otimes \Lambda(V_1)$
- Anti-commutative superalgebras e.g., exterior superalgebra $\Lambda(V) \cong \Lambda(V_0) \otimes \Lambda(V_1)$
- Hopf superalgebras e.g., $\Lambda(V_1)$

2 Supergroups

- Affine supergroup schemes $G(A) = \text{Hom}_{\text{alg}}(k[G], A)$
- General linear supergroup $GL(m|n)$
- Underlying purely even subgroup $k[G]_{\text{ev}} = k[G]/\langle k[G]_1 \rangle$, $G_{\text{ev}}(A) = G(A_{\overline{1}})$
- Frobenius morphism $F^r : G \rightarrow (G_{\text{ev}})^{(r)}$
- Algebraic, finite, infinitesimal (height $\leq r$) supergroup schemes
- Structure of finite supergroup schemes [2, Lemma 5.3.1] $G \cong G^0 \rtimes \pi_0(G)$
- As in FS, reduce CFG to the infinitesimal case.
3 Height one: restricted Lie superalgebras

- Lie superalgebra \( \mathfrak{g} = \mathfrak{g}_\pi \oplus \mathfrak{g}_\Gamma \)
- Sidebar: CFG for ordinary Lie superalgebras
  - Koszul resolution \( \mathcal{Y}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}) \otimes \Lambda(\mathfrak{g}_0) \otimes \Gamma(\mathfrak{g}_1) \)
  - Cochain complex \( \text{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{Y}(\mathfrak{g}), M) \cong M \otimes \Lambda(\mathfrak{g}_0^*) \otimes S(\mathfrak{g}_T^*) \)
  - \( H^*(\mathfrak{g}, M) = H^*(\mathcal{U}(\mathfrak{g}), M) \) finite over \( S(\mathfrak{g}_T^*[1]) \)
- Restricted Lie superalgebra, restricted enveloping algebra \( V(\mathfrak{g}) \)
- May spectral sequence \( E_0^{2i,j}(M) = M \otimes \Lambda^j(\mathfrak{g}_0^*) \otimes S^i(\mathfrak{g}_1^*) \Rightarrow H^{i+j}(V(\mathfrak{g}), M) \)
- Prove terms are permanent cycles using ‘explicit’ projective resolution \( X(\mathfrak{g}) \) described by May [6] and Iwai and Shimada [5].
- CFG for arbitrary infinitesimal supergroup schemes
  - For each \( r \), exhibit certain cohomology classes
    \[
    c_r^{m,n} \in H^{2p-1}(GL(m|n), \mathfrak{gl}(m|n)(r)_\pi),
    \]
    \[
    c_r^{m,n} \in H^p(GL(m|n), \mathfrak{gl}(m|n)(r)_\Gamma)
    \]
    whose restrictions to \( GL(m|n)_{(1)} \) define nonzero maps
    \[
    c_r^{m,n} : S(\mathfrak{gl}(m|n)^*[2p-1])^{(r)} \to H^*(GL(m|n)_{(1)}, k)
    \]
    \[
    c_r^{m,n} : S(\mathfrak{gl}(m|n)^*[p^r])^{(r)} \to H^*(GL(m|n)_{(1)}, k)
    \]
    admitting nice interpretations in terms of the edge maps of the May spectral sequence.
  - Tensor product of Frobenius twists of maps \( (\bigotimes_{i=1}^r (e_i^{m,n})^{(r-i)}) \otimes c_r^{m,n} \) defines
    \[
    \left( \bigotimes_{i=1}^r S(\mathfrak{gl}(m|n)^*[2p-1])^{(r)} \right) \otimes S(\mathfrak{gl}(m|n)^*[p^r])^{(r)} \to H^*(GL(m|n)_{(r)}, k).
    \]

4 Superizing FS

4.1 Strict polynomial superfunctors
- Categories \( \mathcal{V} \) and \( \Gamma^m\mathcal{V} \)
- Objects and homomorphisms in \( \mathcal{P}_n \)
• Define ‘ordinary’ functor on $\mathbf{V}_{ev}$
• Restriction from $\mathcal{P}_n$ to $\mathcal{P}_n$ (but no lifting from $\mathcal{P}_n$ to $\mathcal{P}_n$)
• Examples
  – parity change functors $\Pi \cong \Pi$
  – symmetric powers $S(V) \cong S(V_0) \otimes \Lambda(V_T)$
  – exterior powers $\Lambda^n(V) \cong \Lambda(V_T) \otimes S(V_T)$
  – divided powers $\Gamma^n(V) \cong \Gamma(V_T) \otimes \Lambda(V_T)$
  – alternating powers $A^n(V) \cong \Lambda(V_T) \otimes \Gamma(V_T)$
  – Frobenius twists $I^{(r)} = I_0^{(r)} \oplus I_1^{(r)}$
  – If $F \in \mathcal{P}_n$, then $F^{(r)} := F \circ I^{(r)}$, $F_0^{(r)} := F \circ I_0^{(r)}$, and $F_1^{(r)} := F \circ I_1^{(r)}$ are in $\mathcal{P}_{p^{nr}}$.
• Conjugation by $\Pi$ (or $\Pi$)
• Duality
• Projectives and injectives

4.2 Calculating $\text{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$
• The decomposition $I^{(r)} = I_0^{(r)} \oplus I_1^{(r)}$ leads to a matrix ring decomposition

$$\text{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}) = \begin{pmatrix}
\text{Ext}^*_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) & \text{Ext}^*_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)}) \\
\text{Ext}^*_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}) & \text{Ext}^*_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)})
\end{pmatrix}$$

– $\text{Ext}^*_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) \cong \text{Ext}^*_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)}) \cong k$ in even degrees (zero otherwise)
– $\text{Ext}^*_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}) \cong \text{Ext}^*_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}) \cong k$ in odd degrees $\geq p^r$ (zero otherwise)
• Contrast to classical situation: Frobenius twist is not injective on cohomology
• $\text{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$ generated as a $k$-algebra by certain (even superdegree) extension classes

$$e_i^{(r-i)} \in \text{Ext}^{2p^{i-1}}_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)})$$

$$\Pi \in \text{Ext}^{2p^{i-1}}_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)})$$

for $1 \leq i \leq r$, and

$$c_r \in \text{Ext}^{p^r}_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}),$$

$$c_r^{\Pi} \in \text{Ext}^{p^r}_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}),$$

subject only to the relations imposed by the matrix ring decomposition and:

1. $(e_r)^p = c_r \circ c_r^{\Pi}$ and $(e_r^{\Pi})^p = c_r^{\Pi} \circ c_r$.
2. For each $1 \leq i < r$, $(e_i^{(r-i)})^p = [(e_i^{(r-i)})^\Pi]^p = 0$.
3. For each $1 \leq i \leq r$, $e_i^{(r-i)} \circ c_r = c_r \circ (e_i^{(r-i)})^\Pi$ and $(e_i^{(r-i)})^\Pi \circ c_r^{\Pi} = c_r^{\Pi} \circ e_i^{(r-i)}$.
4. The subalgebra generated by $e_1^{(r-1)}, \ldots, e_r, (e_1^{(r-1)})^\Pi, \ldots, e_r^\Pi$ is commutative.

- Strategy for computing the vector space structure of $\text{Ext}_P^*(I^{(r)}, I^{(r)})$ is based on hypercohomology spectral sequences

$$\begin{align*}
\text{I}E_{s,t}^1 &= \text{Ext}_P^t(I^{(r)}, C^s) \Rightarrow \text{Ext}_P^{s+t}(I^{(r)}, C) \\
\text{II}E_{s,t}^2 &= \text{Ext}_P^s(I^{(r)}, H_t(C)) \Rightarrow \text{Ext}_P^{s+t}(I^{(r)}, C)
\end{align*}$$

for $\ell \in \{0, 1\}$ and for the following choices for $C$:

- Super Koszul–de Rham complex $\Omega_p = S^n \otimes A^i, \kappa : \Omega_p \to \Omega_n \otimes A^i, \partial : \Omega_p \to \Omega_p \otimes A^i$
- Super Koszul kernel complex $(K_{p,n}, d), K_{p,n} = \ker(\kappa : \Omega_{p,n} \to \Omega_{p,n})$
- Frobenius twists of the ordinary Koszul–de Rham complex $\Omega^{(j)} (j \geq 1)$

- Cohomology of the Super de Rham complex: $H^*(\Omega_{p,n}) \cong \Omega^{(1)}$, but spread across more cohomology degrees than in the classical situation:

$$H^i(\Omega_{p,n}) = \bigoplus_{a+b+c+d=n, b(p-1) + c + pd = t} (S_0^{a(1)} \otimes \Lambda_1^{b(1)}) \otimes (\Lambda_0^{c(1)} \otimes \Gamma_1^{d(1)})$$

- Play spectral sequences for $\ell = 0$ and $\ell = 1$ off of each other.

### 4.3 Restrictions of the extension classes

- Argue by induction on $r$ that $c_r, c_r^\Pi, e_r, e_r^\Pi$ restrict in the desired manner to $GL(m|n)(1)$.

- Case $r = 1$: explicit resolutions representing $e_1$ and $c_1$:

$$\begin{align*}
e_1 : 0 &\to I_0^{(1)} \to S^p \to \Gamma_p \to I_0^{(1)} \to 0 \\
c_1 : 0 &\to I_0^{(1)} \to K_0^p \to K_1^p \to \cdots \to K_{p-2}^p \to K_{p-1}^p \to I_1^{(1)}
\end{align*}$$

- Induction step: In FS, they showed that $(e_1^{p-1})^{(p-1)}$ and $e_r^{p-1}$ have the same image in a particular cohomology group, and use this to relate their restrictions to $(GL_n)(1)$. But the ‘compatibility diagram’ of [FS, §5] doesn’t seem to generalize for $c_r$. 
Diagram 5.1.

\[
\begin{array}{ccc}
\text{Ext}^{2(p-1)p'^{-1}}(I^{(r)}, I^{(r)}) = k \cdot e_{r}^{p^{-1}} & \simeq & \\
\downarrow & & \downarrow \\
\text{Ext}^{2(p-1)p'^{-1}}(S^{p'}, S^{p'^{-1}(1)}) & \longrightarrow & \text{Ext}^{2(p-1)p'^{-1}}(f^{(r)}, S^{p'^{-1}(1)}) = k \\
\uparrow \simeq ? & & \uparrow \simeq ?
\end{array}
\]

\[
\begin{array}{ccc}
\text{Ext}^{2(p-1)p'^{-1}}((S^{p}) \otimes p'^{-1}, S^{p'^{-1}(1)}) & \longrightarrow & \text{Ext}^{2(p-1)p'^{-1}}(f^{(r)}(1), S^{p'^{-1}(1)}) \\
\uparrow \simeq ? & & \uparrow \simeq ?
\end{array}
\]

\[
\begin{array}{ccc}
\text{Ext}^{2(p-1)p'^{-1}}(S^{p}, I^{(1)}) \otimes p'^{-1} & \longrightarrow & \text{Ext}^{2(p-1)p'^{-1}}(f^{(1)}(1), I^{(1)}) \otimes p'^{-1} = k
\end{array}
\]

- Instead take a cue from Franjou–Friedlander–Scorichenko–Suslin [4].
  - Cup product map
    \[
    \text{Ext}^{p'}_{P}(I^{(r)}, I^{(r)}) \otimes_{P} \longrightarrow \text{Ext}^{p'+1}_{P}(\Gamma^{p(r)}, S^{p(r)}), \quad (c_{r}) \otimes_{P} \rightarrow c_{r}^{\cup p}
    \]
  - Maps \( I^{(r+1)}_{0} \rightarrow S^{p(r)}_{0} \) and \( \Gamma^{p(r)} \rightarrow I^{(r+1)}_{1} \) induce
    \[
    \text{Ext}^{p'+1}_{\mathcal{P}}(I^{(r+1)}_{1}, I^{(r+1)}_{0}) \rightarrow \text{Ext}^{p'+1}_{\mathcal{P}}(\Gamma^{p(r)}_{1}, S^{p(r)}_{0})
    \]
  - Show that \( c_{r}^{\cup p} \) and (the image of) \( c_{r+1} \) span the same subspace.

### 4.4 Odds and ends

- Only able to deduce that the Yoneda product \( e_{r}^{p} \neq 0 \) after computing restriction of the cohomology class to an particular subgroup of \( GL(m|n)_{(1)} \).
- Accidental byproduct of calculations [3, 5.6.1]:
  \[
  \text{Ext}^{2}_{GL(m|n)}(k, k) \neq 0
  \]
- What is the structure of the rational cohomology ring \( H^{*}(GL(m|n), k) \)?
References


