This are my notes for the (second part of the) eleventh lecture in a learning seminar organized by Pavel Etingof and myself on finite generation of cohomology rings in the summer and fall of 2021.

1 Cohomology for Lie superalgebras

Recall that currently, our goal is the main result of [3]: finite generation for cohomology rings of all finite supergroup schemes. We've now seen a great introduction in the previous two lectures to supergroup schemes in general, and the outline of this proof in specific.

The base case of the inductive proof will be supergroup schemes of height 1, e.g. for restricted enveloping algebras $V(L)$ of Lie superalgebras $L$. This is the goal of Section 3 of [2], which will be our focus today.

Setup: $L = L_0 \oplus L_1$ a finite-dimensional Lie superalgebra, such that $L_0$ has a basis $\{x_1, \ldots, x_s\}$ and $L_1$ has a basis $\{y_1, \ldots, y_t\}$. Begin by constructing a resolution that will allow us to compute Lie superalgebra cohomology.

Recall the way we compute usual Lie algebra cohomology, via the Chevalley-Eilenberg resolution (see [8, Section 7.7]). If $L$ is a usual Lie algebra, we have a free resolution of the trivial module $k$ by

$$\ldots U(g) \otimes \Lambda^n g \to \ldots \to U(g) \otimes \Lambda^2 L \to U(g) \otimes L \to U(g) \xrightarrow{\epsilon} k \to 0.$$ 

Here the maps are given by

$U(g) \otimes g \to U(g)$

$u \otimes x \mapsto ux,$

and more generally

$U(g) \otimes \Lambda^n g \to U(g) \otimes \Lambda^{n-1} g$

$u \otimes x_1 \wedge \ldots \wedge x_n \mapsto \sum_{i=1}^n (-1)^{i+1} u x_i \otimes x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots$

$+ \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \ldots \wedge \hat{x}_i \ldots \wedge \hat{x}_j \ldots \wedge x_n.$

Hence to get cohomology of a module $M$, apply $\text{Hom}_g(-, M)$ to this complex. This gives

$\text{Hom}_g(U(g) \otimes \Lambda g, M) \cong \text{Hom}_k(\Lambda g, M) \cong \Lambda g^* \otimes M.$

Now we consider the superalgebra case. Define $\overline{V}(L) := \Lambda(L_0) \otimes \Gamma(L_1)$. Both $\Lambda(L_0)$ and $\Gamma(L_1)$ have right $L$-actions by multiplication: for $x \in L_0$ and $u \in L$, the action is $x \cdot u = [x, u]$, and for $y \in L_1$, the action is $\gamma_r(y) \cdot u = \gamma_{r-1}(y)[y, u]$. This gives a $U(L)$ action on $\overline{V}(L)$. Now define $Y(L) := U(L) \# \overline{V}(L)$. As a graded superspace, this is

$U(L) \otimes \overline{V}(L) = U(L) \otimes \Lambda(L_0) \otimes \Gamma(L_1).$
This will play the role of the Chevalley-Eilenberg resolution. Note that \( Y(L) \) is spanned by objects of the form
\[
u(x_{i_1} \cdots x_{i_b})\gamma_{a_1}(y_1) \cdots \gamma_{a_t}(y_t)
\]
where \( 1 \leq i_1 < \cdots < i_b \leq s \) and \( a_j \in \mathbb{N} \).

The next three results now parallel the ordinary Lie algebra case.

**Theorem 1.0.1.** ([2, Theorem 3.1.1, Lemma 3.2.1, Theorem 3.2.4]). We have the following.

1. There exists a differential giving \( Y(L) \) the structure of a differential graded superbialgebra, and a free resolution of the trivial module.
2. \( \text{Hom}_{U(L)}(Y(L),\mathbb{k}) \cong \Lambda s(L^*) \).
3. \( H^*(L,\mathbb{k}) \) is a finitely generated graded superalgebra, and \( H^*(L,M) \) is finitely generated over \( H^*(L,\mathbb{k}) \).

**Proof.** We first consider (1). The differential on \( Y(L) \) is defined via
\[
d(u) = 0, \\
d(\langle x \rangle) = x, \\
d(\gamma_r(y)) = y\gamma_{r-1}(y) - \frac{1}{2}([y,y])\gamma_{r-2}(y).
\]
It can be checked that this gives \( Y(L) \) the structure of a differential graded bialgebra.
We wish to show that it defines a resolution of \( \mathbb{k} \), i.e. it is exact. Consider filtration on \( Y(L) \) defined by
\[
F_nY(L) = \bigoplus_{i+j} F_iU(L) \otimes \gamma_j(L)
\]
where \( F_iU(L) \) is the standard PBW filtration. In other words, \( F_nY(L) \) is spanned by monomials of the form
\[
z_{\alpha_1} \cdots z_{\alpha_i} \langle x_{\beta_1} \cdots x_{\beta_j} \rangle \gamma_{a_1}(y_1) \cdots \gamma_{a_t}(y_t)
\]
with
\[
i + j + \sum a_k \leq n, \\
\alpha_1 \leq \cdots \leq \alpha_i, \\
\beta_1 < \cdots < \beta_j.
\]
where the set \( \{z_i\} \) is defined by \( z_1 = x_1, \ldots, z_s = x_s, z_{s+1} = y_1, \ldots, z_{s+t} = y_t \). Consider the associated graded of \( Y(L) \) with respect to this filtration:
\[
gr Y(L) = \bigoplus_{n \geq 0} F_nY(L)/F_{n-1}Y(L) \cong Y(L_{ab}),
\]
where \( L_{ab} \) is the abelian Lie superalgebra with underlying vector space same as \( L \). We can check that \( Y(L_{ab}) \) is exact by the following argument. If the dimension of \( L \) is 1, we have two cases: the basis element \( x \) is degree 0, or the basis \( y \) is degree 1. In the first case, we get
\[
U(L_{ab}) \otimes \Lambda^2(L_{ab}) \cong 0 \to U(L_{ab}) \otimes L \to U(L_{ab}) \to \mathbb{k} \to 0.
\]
\( U(L_{ab}) \) is a polynomial ring in \( x \). By the Leibniz rule, \( d \) sends any positive degree element of \( U(L_{ab}) \) to 0, and so exactness at \( \mathbb{k} \) is straightforward. The map \( U(L_{ab}) \otimes L \to U(L_{ab}) \) is given by \( u(x) \mapsto ux \) by the Leibniz rule, which is injective and has image equal to the positively graded elements of \( U(L_{ab}) \), so exactness of the whole complex now follows.
Similarly, the second case, where $x$ is odd, can also be checked directly. The general case follows as

$$Y(L_{ab}) \cong Y(k_0) \otimes \ldots Y(k_0) \otimes Y(k_1) \otimes \ldots Y(k_1).$$

The Kunneth formula (see [8, Theorem 3.6.1]) says in general that:

**Theorem 1.0.2.** If $P$ is a chain complex of flat $R$-modules such that each submodule $d(P_n)$ of $P_{n-1}$ is also flat, then for every $R$-module $M$ there is an exact sequence

$$0 \to H_n(P) \otimes_R M \to H_n(P \otimes_R M) \to \text{Tor}_R^1(H_{n-1}(P), M) \to 0.$$ 

In our case, $R$ is the field $k$, and so since the outer two terms are inductively 0, the cohomology of $Y(L_{ab})$ is 0 as well. Hence, we have obtained that $Y(L_{ab})$ is a free resolution for $k$. To get the general case, just consider the short exact sequence

$$0 \to F_{i-1}Y(L) \to F_iY(L) \to Y(L_{ab}) \to 0.$$ 

This gives a long exact sequence

$$H^j(F_{i-1}Y(L)) \to H^j(F_iY(L)) \to H^j(Y(L_{ab})) = 0.$$ 

Since $F_{i-1}Y(L) = 0$, we have that $F_iY(L)$ is exact. Since $Y(L) = \bigcup F_iY(L)$, it follows that $Y(L)$ is exact.

(2) is straightforward. As superspaces,

$$C^n(L, k) := \text{Hom}_{U(L)}(Y_n(L), k)$$

$$= \text{Hom}_{U(L)}(\bigoplus_{i+j=n} U(L) \otimes \Lambda^i(L_0) \otimes \Gamma_j(L_1), k)$$

$$\cong \text{Hom}_k(\bigoplus_{i+j=n} \Lambda^i(L_0) \otimes \Gamma_j(L_1), k)$$

$$\cong \bigoplus_{i+j=n} \Lambda^i(L_0)^* \otimes \Gamma_j(L_1)^*$$

$$\cong \bigoplus_{i+j=n} \Lambda^i(L_0) \otimes S^j(L_1^*)$$

$$\cong \Lambda^n(L^*).$$

Now for (3). Note that

$$f \circ \partial(f) = (-1)^{m(m-1)}(-1)^{mm} \partial(f) \circ f = (-1)^m \partial(f) \circ f.$$ 

Then inductively we have $\partial(f \otimes n) = n \partial(f) \circ f \otimes (n-1)$ by the following argument:

$$\partial(f \otimes n) = \partial(f) \circ f \otimes (n-1) + (-1)^m f \circ \partial(f \otimes (n-1))$$

$$= \partial(f) \circ f \otimes (n-1) + (-1)^m f \circ (n-1) \partial(f) \otimes f \otimes (n-2)$$

$$= \partial(f) \circ f \otimes (n-1) + (-1)^m (n-1)(-1)^m \partial(f) \circ f \otimes (n-1)$$

$$= n \partial(f) \circ f \otimes (n-1).$$

Therefore, $\partial(f \otimes p) = p \partial(f) \circ f \otimes (p-1) = 0$, and so $f \otimes p$ is a cocycle in $C^*(L, k)$.

Let $S(L_1^*)^p$ denote the subalgebra of $p$ powers in $S(L_1^*)$. We know now that $S(L_1^*)^p$ consists of cocycles. Now both

- $C^*(L, M) \cong M \otimes C^*(L, k)$ and
- $C^*(L, k) = \Lambda(L_0^*) \otimes S(L_1^*)$

are finitely generated over $S(L_1^*)^p$. Since $S(L_1^*)^p$ consists of cocycles, the cohomologies
\[ \bullet \text{H}^*(L, M) \text{ and } \text{H}^*(L, k) \]

are also finitely generated over \( S(L^*_1)^p \).

Note that unlike in the ordinary Lie algebra case, the cohomology of a Lie superalgebra does not have to be finite-dimensional: e.g. if \( L = L_1 \) and is abelian, then

\[ \text{H}^*(L, k) = C^*(L, k) = \Lambda_s(L^*) \cong S(L^*). \]

In this case, \( U(L) \cong \Lambda(L) \). In this case, then, we recover the classic Koszul duality result

\[ \text{H}^*(\Lambda(L), k) \cong S(L^*). \]

## 2 Finite generation for restricted Lie superalgebras

At this point, we move from the usual enveloping algebra \( U(L) \) to the restricted enveloping algebra

\[ V(L) := U(L)/(x^p - x^{[p]} : x \in L_0). \]

In this section, we will assume \( k \) is a perfect field of characteristic \( p > 2 \). We first construct a version of the Koszul resolution for \( V(L) \)-- since this is similar to the non-restricted version above, we will omit the proofs. Define

\[ X(L) := W(L) \otimes \Gamma'(L_0) := V(L) \otimes \overline{V}(L) \otimes \Gamma'(L_0) := V(L) \otimes \Lambda(L_0) \otimes \Gamma(L_1) \otimes \Gamma'(L_0) \]

(as a graded \( V(L) \)-supermodule). \( W(L) := V(L)\#\overline{V}(L) \) inherits the structure of a differential graded superbialgebra. We consider \( X(L) \) as a complex with differential

\[ d : X(L) \rightarrow X(L) \]

\[ w \otimes b \mapsto d(w) \otimes b \]

for \( w \in W(L) \) and \( b \in \Gamma'(L_0) \).

We will next twist the differential in order to give \( X(L) \) the structure of a free resolution of \( k \):

**Theorem 2.0.1.** ([2, Theorem 3.3.1, Theorem 3.3.3]). There exists \( t \in \bigoplus_{i \geq 0} \text{Hom}_k(\Gamma'_i(L_0), Y_{i-1}(L_0)) \) such that:

1. The map

   \[ d_t : X(L) \rightarrow X(L) \]

   \[ w \otimes b \mapsto d(w) \otimes b + (-1)^{\deg(w)\otimes b}(w \otimes b) \cap t \]

   gives \( X(T) \) the structure of a complex of free \( V(L) \)-modules, where

   \[ (w \otimes b) \cap t = (-1)^{\deg(r)\deg(w)\otimes b}(\sigma \otimes 1) \circ (1 \otimes t \otimes 1) \circ (1 \otimes \Delta)(w \otimes b), \]

   and \( \sigma : W(L) \otimes Y(L_0) \rightarrow W(L) \) is the right action, and

2. If \( t_i \) is defined as the restriction of \( t \) to \( \Gamma'_i(L_0) \), then

   \[ t_2(\gamma'_i(x_i)) = x_i^{p-1}(x_i) - (x_i^{[p]}), \]

   and for \( n > 1 \),

   \[ t_{2n}(\Gamma_{2n}(L_0)) \subset F_{np-1}Y_{2n-1}(L_0). \]

   (Recall here \( \{x_i\} \) are a basis for \( L_0 \) and \( \{\gamma'_i(x_i)\} \) are a basis for \( \Gamma'_2(L_0) \).)

3. Under \( d_t \), \( X(L) \) is a free \( V(L) \)-resolution of \( k \).
The $t$ constructed in (1) and (2) is done by induction. It depends on our choice $\{x_i\}$ of a basis of $L_0$, and hence the construction of $X(L)$ is not functorial.

We now consider two spectral sequences: one arising from the bar complex, and one from the Koszul resolution $X(L)$. We will denote $V := V(L)$. Recall that the bar resolution is

$$B(V) = V \otimes_{n+1},$$
$$d : v_0[v_1 ... v_n] \mapsto a_0a_1[a_2]... + (-1)^n a_0[a_1 ... a_{n-1}]e(a_n) + \sum_{i=1}^{n-1} (-1)^i a_0[a_1 ... a_ia_{i+1} ...].$$

Define $C^n(V, M) = \text{Hom}_V(B_n(V), M)$. Then the monomial length filtration on $V$ induces a filtration on $B_n(V)$ with

$$F_iB_n(V) = \sum_{j_0 + ... + j_n \leq i} F_jV \otimes ... \otimes F_jV,$$

which induces a filtration on $C^n(V, M)$ by

$$F^iC^n(V, M) = \text{Hom}_V(B_n(V)/(V.F_{i-1}B_n(V), M).(Note here that \overline{B} = 1 \otimes A^{\otimes n}).$$

By [7, Theorem 2.6], since this filtration is bounded (e.g.

$$F^iC^n(V, M) = 0 \forall i > n),$$

there exists a spectral sequence

$$E^1(M) = H^*(V, M),$$

$$E^i_{0,j} = \text{gr}_i C^{i+j}(V, M), \text{ and } E^i_{1,j}(M) \cong H^{i+j}(\text{gr}_i C^*(V, M)).$$

This is the May spectral sequence. We have ([2, Proposition 3.4.2]) that

$$E^1_{ij}(k) \cong \bigoplus_{\{a, b \in \mathbb{N}: a + pb = i, a + 2b = i + j\}} \Lambda^a_k(L^*) \otimes S^b(L^0_0).$$

E.g. if $p = 3$, this page looks like

$$\begin{align*}
\mathbb{N} &\rightarrow \Lambda^1_k(L^*) \rightarrow \Lambda^2_k(L^*) \rightarrow \Lambda^3_k(L^*) \rightarrow \Lambda^4_k(L^*) \\
\downarrow &\downarrow &\downarrow &\downarrow \\
0 &\rightarrow 0 &\rightarrow 0 &\rightarrow S^1(L^0_0) \rightarrow S^1(L^0_0) \otimes \Lambda^1_k(L^*) \\
\downarrow &\downarrow &\downarrow &\downarrow \\
0 &\rightarrow 0 &\rightarrow 0 &\rightarrow 0 &\rightarrow 0 \\
&\downarrow &\downarrow &\downarrow &\downarrow &\downarrow \\
&0 &\rightarrow 0 &\rightarrow 0 &\rightarrow 0 &\rightarrow 0 &\rightarrow 0 \\
\end{align*}$$

This can be relabeled in a way that more closely resembles the classical case (see [6] Jantzen):

$$E^0_{i,j}(k) = \Lambda^j_k(L^*) \otimes S^i(L^0_0).$$

The proof amounts to:

1. Ignoring the bigradings, we have $E^1_1(k) \cong H^*(V(L_{ab}), \mathbb{N})$.
2. Compute the right hand side: $V \cong V(L_0) \otimes U(L_1) \cong V(L_0) \otimes \Lambda(L_1)$. We have already computed $H^*(U(L_1), \mathbb{N}) \cong S(L^*_1)$, and by classical theory [6], we have the usual cohomology computation $H^*(V(L_0), \mathbb{N}) \cong \Lambda(L^0_0) \otimes S^0(L^*_0)$. Now using
Bergh-Opperman’s theorem for cohomology of twisted tensor products [1], we compute
\[ H^*(V, \mathbb{k}) \cong \Lambda(L_0^p) \otimes S(L_1^p) \otimes S'(L_0^s) \cong \Lambda_s(L^*) \otimes S'(L_0^s). \]

(3) Hard part: identify the summands \( E_1^{ij} \) in \( E_1(\mathbb{k}) \cong H^*(V, \mathbb{k}) \).

A second spectral sequence can also be defined, using \( X(L) \) this time: we define a decreasing filtration on \( \text{Hom}_V(X(L), \mathbb{k}) \) by
\[ F^i \text{Hom}_V(X_n(L), \mathbb{k}) = \text{Hom}_V(X_n(L)/(V.F_i - 1 X_n(L)), \mathbb{k}). \]
(Here, recall that \( X(L) = V(L) \otimes \overline{Y}(L) \otimes \Gamma'(L_0) \), and \( \overline{X}(L) := \overline{Y}(L) \otimes \Gamma'(L_0). \)) Then just as before, we get a spectral sequence
\[ D_r(\mathbb{k}) \Rightarrow H^*(V, \mathbb{k}) \]
\[ D_r^0(\mathbb{k}) = F^i \text{Hom}_V(X_{i+j}(L), \mathbb{k})/F^{i+1} \text{Hom}_V(X_{i+j}(L), \mathbb{k}) \cong \text{Hom}_{\mathbb{k}}(F_i X_{i+j}(L)/F_{i-1} X_{i+j}(L), \mathbb{k}). \]

\( E_1(\mathbb{k}) \cong H^*(V(Lab), \mathbb{k}) \) and \( E_1(1) \cong M \otimes H^*(V(Lab), \mathbb{k}) \) are finitely generated right supermodules over \( S(L_1^p) \otimes S'(L_0^s) \) in \( E_1(\mathbb{k}) \). It is therefore enough to prove that this commutative subalgebra is generated by permanent cycles. To prove this, we show that there exists a map \( E_r(\mathbb{k}) \rightarrow D_r(\mathbb{k}) \) such that. This map is constructed in the following way:

\[
\begin{align*}
\xymatrix{
X(L) = V \otimes \Lambda(L_0) \otimes \Gamma(L_1) \otimes \Gamma'(L_0) \ar[r]^-{\mu'} & \text{Hom}(X(L), \mathbb{k}) \ar[r] & D_r(\mathbb{k}) \\
N(V) = \bigoplus V \otimes I(V) \otimes \ldots \otimes I(V) \ar[r]^-{\psi} & \text{Hom}(B(V), \mathbb{k}) \ar[r] & E_r(\mathbb{k})
}
\end{align*}
\]

The map \( \mu' \) exists by a theorem of May. Here \( I(V) \) denotes the cokernel of the unit map \( \mathbb{k} \rightarrow V \). The map \( N(V) \rightarrow B(V) \) corresponds to the map \( I(V) \rightarrow V \) given by \( v + \mathbb{k} \mapsto v - \epsilon(v) \). This chain map \( N(V) \rightarrow B(V) \) is a quasi-inverse to the projection map \( B(V) \rightarrow N(V) \). The composition \( \mu = \psi \circ \mu' \) induces then the maps in the right columns, first \( \text{Hom}(B(V), \mathbb{k}) \rightarrow \text{Hom}(X(L), \mathbb{k}) \), and then to the spectral sequences defined in terms of these complexes, \( E_r(\mathbb{k}) \rightarrow D_r(\mathbb{k}) \).

To complete the proof, it suffices to show that \( S(L_1^p) \otimes S'(L_0^s) \subset E_r(\mathbb{k}) \) map to permanent cycles in \( D_r(\mathbb{k}) \). Drupieski shows that the generators \( f_i \) of \( S(L_1^p) \), defined by sending \( [y_i] \mapsto 1 \) and all other standard monomials to 0, map to the elements of \( \text{Hom}(X_\rho(L), \mathbb{k}) \) defined by \( \gamma_\rho(y_i) \mapsto 1 \), and other monomials map to 0. On the other hand, \( S'(L_0^s) \) is generated by the piece \( L_0^s \) which sits in \( E^{p,2-p}_1(\mathbb{k}) \) and therefore maps to \( D^{p,2-p}_1(\mathbb{k}) \cong \text{Hom}(\Gamma^2_2(L_0), \mathbb{k}) \). This is generated by the elements \( \gamma_1'(x_j) \mapsto 1 \) and other monomials map to 0. To summarize, we have the diagram
The argument is now an explicit calculation, showing that the generators in $D_r(k)$ outlined here are all permanent cycles. It then follows that:

**Theorem 2.0.2.** ([2, 3.5.4]). If $k$ is a perfect field of characteristic $p > 2$, then

1. $H^*(V(L), k)$ is finitely generated as a superalgebra, and
2. $H^*(V(L), M)$ is finitely generated as a supermodule for $H^*(V(L), k)$, for any finitely generated $V(L)$ supermodule $M$.

**References**