This are my notes for the second lecture in a learning seminar organized by Pavel Etingof and myself on the paper “Cohomology of finite group schemes over a field” by E. M. Friedlander and A. Suslin [2] in the summer and fall of 2021.

1 Reduction to the Existence of Universal Classes (continued from last week)

Recall that the main theorem of Friedlander-Suslin [2, Theorem 1.1] is the following:

**Theorem 1.0.1. Friedlander-Suslin.** Let $G$ be a finite group scheme and $M$ a finite-dimensional rational $G$-module. Then $H^* (G,k)$ is a finitely-generated $k$-algebra and $H^* (G,M)$ is a finite $H^* (G,k)$-module.

Recall that our goal in Section 1 of [2] is the reduction of the main theorem to the existence of universal classes. In other words, we will assume the following, and prove that finite generation follows from it [2, Theorem 1.2]:

**Theorem 1.0.2.** Assume that $k$ is a field of characteristic $p > 0$. For any $n > 1$ and $i ≥ 1$, there exist rational cohomology classes defined over $F_p$

$$e_i \in H^{2p^i-1} (G_{n,k}, gl^{(i)}_n)$$

which restrict nontrivially to

$$H^{2p^i-1} ((G_{n,k})_{(1)}, gl^{(i)}_n) \cong H^{2p^i-1} ((G_{n,k})_{(1)}, k) \otimes gl^{(i)}_n.$$  

Here is a big-picture sketch of the proof that Theorem 1.0.1 reduces to Theorem 1.0.2.

1. Given an infinitesimal (recall: the augmentation ideal of $k[G]$ is nilpotent, or alternately $G$ is connected) group scheme $G$ of height $r$ with embedding $G \to (GL_{n,k})_{(r)}$ (such an embedding for some $n$ always exists), the universal class $e_i$ can be used to produce a homomorphism of graded $k$-algebras

$$S^* ((gl^{(r)}_n)^\# (2p^i-1)) \to H^* (G,k).$$

The $(2p^i-1)$ in the above algebra denotes the degree in which the generators are placed.

2. Under the product of these morphisms, $H^* (G,M)$ is finitely generated over

$$\bigotimes_{i=1}^r S^* ((gl^{(r)}_n)^\# (2p^i-1)).$$

Since the latter algebra is finitely generated, this in particular implies that $H^* (G,k)$ is a finitely generated algebra, and that each $H^* (G,M)$ is finitely generated over $H^* (G,k)$. This has two steps:

(a) First, we prove for height 1 infinitesimal group schemes.

(b) Then, use induction to prove for infinitesimal group schemes of arbitrary height.
(3) By passing to an appropriate field extension if necessary, a general group scheme $G$ may be written as a semidirect product $\pi \times G_0$ of an infinitesimal group scheme $G_0$ (the connected component) and a finite group $\pi$ (the group of $k$-points). The finite-generation of cohomology for both $G_0$ and $\pi$ can then be used to prove finite-generation for $G$.

We begin by explaining (1). The group homomorphism $F : \text{GL}_{n,k} \to \text{GL}_{n,k}$ gives a functor from $\text{Rep}(\text{GL}_{n,k}) \to \text{Rep}(\text{GL}_{n,k})$ by pullback along $F$ (e.g. for a representation $M$, we consider the new representation where $\text{GL}_{n,k}$ acts by composing the original action with $F$). This functor sends $\mathfrak{g}_n^{(i)}$ to $\mathfrak{g}_n^{(i+1)}$. Via the pullback-long-$F$ functor, we get a natural graded homomorphism

$$H^*(G, \mathfrak{g}_n^{(i)}) \to H^*(G, \mathfrak{g}_n^{(i+1)}).$$

In general, we will denote the image of elements under this map by

$$\zeta \mapsto \zeta^{(i)}.$$

Applying this homomorphism to the universal classes of Theorem 1.0.2, we get

$$H^{2p^r-1}(\text{GL}_{n,k}, \mathfrak{g}_n^{(i)}) \to H^{2p^r-1}(\text{GL}_{n,k}, \mathfrak{g}_n^{(i+j)})$$

$$e_i \mapsto e_i^{(j)}.$$

Let $G$ be an infinitesimal group scheme. By definition, the augmentation ideal, e.g. the kernel of the counit $\epsilon : k[G] \to k$, is nilpotent. The minimal integer $N$ such that $x^{p^N} = 0$ for all $x$ in the augmentation ideal is called the height of $G$, denoted $\text{ht}(G)$. The height may also be defined relative to the Frobenius kernel of $\text{GL}_{n,k}$, as in [2, Lemma 1.3]:

**Lemma 1.0.3.** Let $G$ be an infinitesimal group scheme over $k$ and $r \geq 0$ an integer. Then the following are equivalent:

1. $r \geq \text{ht}(G)$.
2. There exists a closed embedding $G \hookrightarrow (\text{GL}_{n,k})_{(r)}$.
3. For any closed embedding $G \hookrightarrow \text{GL}_{n,k}$, the image of $G$ is contained in $(\text{GL}_{n,k})_{(r)}$.

This can be observed at the level of coordinate rings: in $k[\text{GL}_{n,k}]$, the counit is defined on the generators $x_{ij}$ as $\epsilon(x_{ij}) = \delta_{ij}$. Hence, $x_{ij} - \delta_{ij}$ is in the augmentation ideal of $k[\text{GL}_{n,k}]$. We know that there exists a surjective map $k[\text{GL}_{n,k}] \to k[G]$, and so if $r \geq \text{ht}(G)$, we have that the images of $x_{ij} - \delta_{ij}$ are 0 in $k[G]$. Therefore, the map factors

$$k[\text{GL}_{n,k}] \to k[G]$$

$$k[(x_{ij})_{i,j=1}^n]/(x_{ij}^{p^r} - \delta_{ij}) \cong k[(\text{GL}_{n,k})_{(r)}]$$

On the other hand, if there exists a surjective map $k[(\text{GL}_{n,k})_{(r)}] \to k[G]$, then since the elements $x_{ij} - \delta_{ij}$ generated the augmentation ideal of $k[(\text{GL}_{n,k})_{(r)}]$, it is clear that every element of the augmentation ideal of $k[G]$ raised to the $p^r$ power must be 0.

Now, if $G$ is an infinitesimal group scheme of height $r$, we fix an embedding $G \hookrightarrow (\text{GL}_{n,k})_{(r)}$. Via this embedding, the induced action of $G$ on $\mathfrak{g}_n^{(r)}$ is trivial, and so for any
Assume we produce graded ring homomorphisms

\[ \text{Lemma } 1.0.5. \quad \text{Let } A \subset (\text{GL}_{n,k})_{(r)} \text{ be an infinitesimal group scheme over } k \text{ of height } r. \text{ Let } C \text{ be a commutative } k\text{-algebra considered as a trivial } G\text{-module, and let } M \text{ be a Noetherian } C\text{-module on which } G \text{ acts by } C\text{-linear transformations. Assume } n \text{ is not a multiple of } p. \text{ Then } H^*(G, M) \text{ is a Noetherian module over } \bigotimes_{i=1}^r S^*((g_n^{(r)})^\#(2^i-1)) \otimes C. \n
\text{In particular, } H^*(G, k) \text{ is a finite module over } \bigotimes_{i=1}^r S^*((g_n^{(r)})^\#(2^i-1)) \text{ and hence is finitely generated.} \]

We have now completed part (1) of our sketch above. Part (2) consists of proving the following theorem [2, Theorem 1.5].

**Theorem 1.0.4.** Let \( G \subset (\text{GL}_{n,k})_{(r)} \) be an infinitesimal group scheme over \( k \) of height \( r \). Let \( C \) be a commutative \( k\)-algebra considered as a trivial \( G\)-module, and let \( M \) be a Noetherian \( C\)-module on which \( G \) acts by \( C\)-linear transformations. Assume \( n \) is not a multiple of \( p \). Then \( H^*(G, M) \) is a Noetherian module over \( \bigotimes_{i=1}^r S^*((g_n^{(r)})^\#(2^i-1)) \otimes C. \n
In particular, \( H^*(G, k) \) is a finite module over \( \bigotimes_{i=1}^r S^*((g_n^{(r)})^\#(2^i-1)) \) and hence is finitely generated.

The proof of Theorem 1.0.4 is a spectral sequence argument. We first prove a fundamental general technical lemma, which will be used several times to deduce finite generation properties from spectral sequences. This appears as [2, Lemma 1.6].

**Lemma 1.0.5.** Let \( E^{p,q}_2 \Rightarrow E^{p,q}_\infty \) be a first quadrant graded commutative ring spectral sequence (e.g. the sum \( \bigoplus_{p,q} E^{p,q}_2 \) is a doubly graded ring and the differentials are derivations, that is, \( d(ab) = d(a)b + (-1)^{\deg a}ad(b) \)). Let further \( \tilde{E}^{p,q}_2 \Rightarrow E^{p,q}_\infty \) be a first quadrant graded spectral sequence which is a module over \( E \). Let \( A \) and \( B \) be commutative, graded rings concentrated in even degree with graded homomorphisms \( A \to E_2, B \to E_2^{*,0} \). Using the natural graded ring homomorphisms

\[ E_2^{*,0} \to E_\infty^{*,0} \to E_\infty \]

and

\[ E_\infty \to E_\infty^{0,*} \to E_2^{0,*} \]

we produce graded ring homomorphisms

\[ A \otimes B \to E_\infty, \quad A \otimes B \to E_2^{*,*}. \]

Assume \( g \) makes \( \tilde{E}_\infty^{*,*} \) into a Noetherian module over \( A \otimes B \). Then \( f \) makes \( \tilde{E}_\infty \) into a Noetherian module over \( A \otimes B \).

We will skip the proof of this general lemma.

We begin the proof of Theorem 1.0.4 for infinitesimal group schemes of height 1. Note that this was already proven by Friedlander-Parshall [1], but we need to know the details of their proof.
For $G$ height 1, $k[G]^\#$ is canonically isomorphic to $V(g)$ the restricted universal enveloping algebra of $g$ the Lie algebra of $G$. Recall what this means: $g$ is defined as the left-invariant derivations $D : k[G] \to k[G]$ (that is, derivations $D$ which satisfy $D\lambda x = \lambda x D$ for each $x \in G(k)$, where $\lambda : k[G] \to k[G]$ is induced by left multiplication by $x$. Then one can check that in characteristic $p$, raising to the $p$th power preserves the left invariant derivations, so it gives a map $g \to g$. Denote this map by $x \mapsto x^p$ (the notation chosen to emphasize that this version of “raising $x$ to the $p$th power” is distinct from raising $x$ to the $p$th power in $U(g)$, that is, $x^p$). Then the restricted enveloping algebra $V(g)$ (also often denoted $U^p(g)$) is defined as $U(g)/((x^p - x)[p])$. See e.g. [4, Section 11.h] or [3, Section I.7.10]. For $g = gl_n$, the $p$ power map $x \mapsto x^p$ is just raising to the $p$th power, as a matrix.

The proof of finite generation for infinitesimal group schemes of height 1 will be an application of our spectral sequence lemma Lemma 1.0.5, applied to the May spectral sequence, as in [3, Lemma I.9.16]: the spectral sequence, which converges to $H^{s+t}(G, M)$, defined as the spectral sequence corresponding to the double complex

$$
\Lambda^i(g^\#) \otimes M \longrightarrow 0 \longrightarrow (g^{(1)})^\# \otimes \Lambda^i(g^\#) \otimes M \longrightarrow \cdots \longrightarrow S^j((g^{(1)})^\#) \otimes \Lambda^i(g^\#) \otimes M
$$

In other words, the 0 page is given by the formula

$$
E_0^{s,t}(M) = \begin{cases} 
0 & s \text{ is odd} \\
S^n((g^{(1)})^\#) \otimes \Lambda^i(g^\#) \otimes M & s = 2n. 
\end{cases}
$$

Clearly, the horizontal differentials are 0. The vertical differentials are the same as those which are used to define the Lie algebra cohomology of $g$ (see [3, Section I.9.17]). These vertical differentials are given by

$$
M \to g^\# \otimes M
$$

$$
m \mapsto \sum \phi_j \otimes m_j \text{ where } xm = \sum \phi_j(x)m_j \forall x \in g
$$

and

$$
g^\# \otimes M \to \Lambda^2(g^\#) \otimes M
$$

$$
\psi \otimes m \mapsto \sum (\phi_j \land \psi) \otimes m_j + d'_1(\psi) \otimes m
$$
where \( m_j \) are \( \phi_j \) are as above, and \( d'_i \) is the map \( g^\# \to \Lambda^2(g^\#) \cong (\Lambda^2 g)^* \) which is the transpose of

\[
\Lambda^2 g \to g
\]

\[
x \land y \mapsto -[x, y].
\]

This map is then used iteratively to define \( d^i_{i+j}(\phi \land \psi) = d'_i(\phi) \land \psi + (-1)^i \phi \land d'_j(\psi) \)

for \( \phi \in \Lambda^i g^\# \) and \( \psi \in \Lambda^j g^\# \).

Since the differentials of the columns are the same differentials used to define the Lie algebra cohomology of \( g \), it follows that

\[
E_2^{st}(M) = E_1^{st}(M) = \begin{cases} 0 & s \text{ is odd} \\ S^n((g^{(1)})^\#) \otimes H^i(g, M) & s = 2n. \end{cases}
\]

We now have a ring spectral sequence \( E(C) \) and a module over it \( E(M) \), for \( E \) and \( M \) as in Theorem 1.0.4 (recall: \( C \) is a commutative \( \mathbb{k} \)-algebra considered as a trivial \( G \)-module, \( M \) is a Noetherian \( C \)-module, \( G \) acts on \( M \) by \( C \)-linear transformations). We will apply Lemma 1.0.5 with \( A := C \) and \( B := S^*((g^{(1)})^\#) \). We need maps \( A = C \to E_\infty = H^*(G, C) = H^*(G, \mathbb{k}) \otimes C \) and \( B \to E_2 \), these are straightforward. By Lemma 1.0.5, \( E_\infty = H^*(G, M) \) is finitely generated over \( A \otimes B = C \otimes S^*((g^{(1)})^\#) \). To complete the proof, we need to verify that the map \( e_1|_G : (g^{(1)}_n)^\# \to H^2(G, \mathbb{k}) \) (the claimed map which by which we obtain finite generation) coincides with the one obtained from Lemma 1.0.5: this is the map

\[
(g^{(1)}_n)^\# \to (g^{(1)})^\# \to H^2(G, \mathbb{k}),
\]

the first map coming from the dual map \( g \to gl_n \), and the second map from the closed embedding \( G \to GL_{n, \mathbb{k}} \). Using the naturality of the May spectral sequence, this follows from the more general result, given [2, Lemma 1.7].

**Lemma 1.0.6.** Assume \( n \not\equiv 0 \pmod{p} \). Let \( (g^{(r)}_n)^\# \to H^{2p-1}((GL_n, \mathbb{k}), \mathbb{k}) \) be a nonzero \( GL_n, \mathbb{k} \)-equivariant homomorphism vanishing on \( Tr^{(r)} \in (g^{(r)}_n)^\# \). Then this homomorphism coincides (up to nonzero scalar factor) with

\[
(g^{(r)}_n)^\# \to S^{p-1} ((g^{(1)}_n)^\#) \to H^{2p-1}((GL_n, \mathbb{k}), \mathbb{k})
\]

where the first arrow is raising to the \( p-1 \) power, and the second arrow is the edge homomorphism in the May spectral sequence.

Why is this relevant in our case? Because each \( e_{i-1}(r-1)|_G : (g^{(r)}_n)^\# \to H^{2p-1}(G, \mathbb{k}) \) vanishes on \( Tr^{(r)} \), by the commutativity of

\[
\begin{array}{ccc}
H^{2p-1}(GL_n, \mathbb{k}, g^{(r)}_n) & \longrightarrow & H^{2p-1}(G, g^{(r)}_n) \cong \text{Hom}_\mathbb{k}((g^{(r)}_n)^\#, H^{2p-1}(G, \mathbb{k})) \\
\downarrow & & \downarrow \circ Tr^{(r)} \\
H^{2p-1}(GL_n, \mathbb{k}, \mathbb{k}) & \longrightarrow & H^{2p-1}(G, \mathbb{k})
\end{array}
\]

and the fact that \( H^*(GL_n, \mathbb{k}) = 0 \) for \( * > 0 \): this is a corollary of Kempf vanishing, see [3, Corollary II.4.11].
Proof. From the May spectral sequence, we get an exact sequence:

\[ 0 \to H^1((GL_{n,k})_{(1)}, k) \to E^{0,1}_2 \to E^{2,0}_2 \to H^2((GL_{n,k})_{(1)}, k), \]

in other words

\[ 0 \to H^1((GL_{n,k})_{(1)}, k) \to H^1(gl_n, k) \to (gl_n^{(1)})^\# \to H^2((GL_{n,k})_{(1)}, k). \]

One can compute directly that \( H^1(gl_n, k) \cong k \)–this gives a cocycle in \( gl_n^{(1)} \). Such an element \( \phi \) corresponds to an extension of \( gl_n \)-modules, via \( M(\phi) = k \oplus k \), and \( \lambda_1, \lambda_2 = (x\lambda_1 + \lambda_2\phi(x), 0) = (\lambda_2\phi(x), 0) \). Specifically, in our case, \( \phi \) is the trace map. The extension will be an extension of the restricted universal enveloping algebra if and only if \( x^p \) acts by the same formula as \( x \) for all \( x \in gl_n \), and in our case, it does not. For example, the matrix

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

acts by

\[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \]

and so clearly while \( A^p = A \) acts by the same matrix, \( A^p \) acts by 0. The conclusion is that \( H^1((GL_{n,k})_{(1)}, k) = 0 \).

Thus, the exact sequence above becomes

\[ 0 \to H^1(gl_n, k) \cong k \to (gl_n^{(1)})^\# \to H^2((GL_{n,k})_{(1)}, k). \]

Since \( Tr^{(1)} \) is in the image of \( k \to (gl_n^{(1)})^\# \), it is in the kernel of the following map, which therefore factors

\[ H^1(gl_n, k) \cong k \to (gl_n^{(1)})^\# \to H^2((GL_{n,k})_{(1)}, k) \]

By general representation theory of \( GL_{n,k} \), since we are assuming \( n \neq 0 \) modulo \( p \), \( sl_n^{(1)} \) is an irreducible \( GL_{n,k} \)-representation of highest weight \( \epsilon_n - \epsilon_1 \), and it follows that \( (sl_n^{(1)})^{(r)} \) is an irreducible \( GL_{n,k} \)-representation of highest weight \( p^r(\epsilon_n - \epsilon_1) \). Now, examine the May spectral sequence. In degree \( 2p^{r-1} \), one can check that the only term which has this highest weight is the

\[ E^{2p^{r-1},0}_2 = S^{p^{r-1}}((gl_n^{(1)})^\#), \]

and it appears with multiplicity 1, so there is a unique map

\[ (sl_n^{(1)})^{(r)} \to H^{2p^{r-1}}((GL_{n,k})_{(1)}, k). \]

Specifically, in our case, this means that \( e_1|_G \) coincides with the edge homomorphism coming from the May spectral sequence, and the proof of finite generation for infinitesimal group schemes of height 1 follows now immediately.

To complete the proof of finite generation for arbitrary group schemes, we now have two steps:

(1) Upgrade from infinitesimal group schemes of height 1 to arbitrary infinitesimal group schemes.
(2) Upgrade from infinitesimal group schemes and finite groups to arbitrary group schemes.

Both are accomplished by spectral sequence arguments. For (1), given an infinitesimal group scheme of height \( r \), taking \( G \hookrightarrow \text{GL}_{n,k} \), let \( G' \subset (\text{GL}_{n,k})^{(r-1)} \) be the image of \( G \) under the restriction of the Frobenius map \( \text{GL}_{n,k} \to \text{GL}_{n,k} \). Then \( G' \) is height \( r-1 \), and the kernel \( H \) of the map \( G \to G' \) is height 1. We now have an extension of infinitesimal group schemes:

\[
1 \to H \to G \to G' \to 1.
\]

Similarly, for (2), an arbitrary group scheme \( G \) may be written as \( \pi \ltimes G_0 \) where \( \pi \) is the finite group of \( k \)-points in \( G \), and \( G_0 \) is the infinitesimal group scheme which is the connected component of \( G \).

In both cases, we want to relate the cohomology of the extension to the cohomology of its composition factors. Both are argued by versions of the Lyndon-Hochschild-Serre spectral sequence, see [3, Proposition I.6.6]. If \( N \) is a normal flat exact subgroup scheme in \( G \), this is a spectral sequence

\[
E_2^{n,m} = H^n(G/N, H^m(N,V)) \Rightarrow H^{n+m}(G,V).
\]

References