18.708: SIMPLICIAL COMMUTATIVE RINGS

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Abstract.

Contents

1. Motivation and overview

The goal of this course is to introduce the formalism of *simplicial commutative rings* and the *cotangent complex*.

1.1. Why simplicial commutative rings? Algebraic geometry is about a theory of geometry whose basic building block is the notion of *scheme*. This is a space built out of local pieces called *affine schemes* (much like how manifolds are locally built out of pieces looking like Euclidean space), which are governed by commutative rings. Thus, classical commutative ring theory can be seen as the local study of algebraic geometry.

In a completely analogous manner, *simplicial commutative rings* are the basic building blocks of an enhanced theory of algebraic geometry called *derived algebraic geometry*. Recently, derived algebraic geometry has found various interesting applications (which we will comment on shortly). However, it is technically quite formidable; for example, the definition of a "simplicial commutative ring" is quite involved, and for this reason will be delayed until later. For concreteness, in this mini-course we will focus not on derived algebraic geometry but on applications of the simplicial theory to *classical algebraic geometry*.

1.2. What are simplicial commutative rings? For now, we will ease into things by using analogies and intuitions. First of all, the notion of simplicial commutative ring is meant to be a model for a "topological commutative ring". The adjective "simplicial" has technical advantages over "topological"; for example, we will also see a notion of "simplicial set" which is a technical replacement for "topological space" in the spirit of CW complex, avoiding pathologies that can occur with the latter notion.

In particular, a simplicial commutative ring $\mathcal{R}$ has associated (abelian) homotopy groups $\pi_0(\mathcal{R}), \pi_1(\mathcal{R}), \pi_2(\mathcal{R})$. These should be thought of as the homotopy groups of the underlying "topological space" of $\mathcal{R}$, but moreover the commutative ring structure on $\mathcal{R}$ equips $\pi_*(\mathcal{R})$ with the structure of a graded commutative ring. Understanding this graded commutative ring is a first approximation to understanding $\mathcal{R}$, in the same sense that the homotopy groups of a space are an approximation to understanding the space.

1.3. **Geometric perspective.** Any commutative ring R can be viewed as a simplicial commutative ring $\underline{R}$, intuitively by "equipping it with the discrete topology". Then $\pi_0(\underline{R}) = R$, while $\pi_i(\underline{R}) = 0$ for $i > 0$. This induces a fully faithful embedding of the category of commutative rings into the category of simplicial commutative rings, and in particular justifies the perspective that simplicial commutative rings form an "enlargement" of commutative rings. (There is a more refined meaning of fully faithful, in which the Hom spaces are equipped with a simplicial set structure, for which this is till true.)

On the other hand, for any simplicial commutative ring $\mathcal{R}$, the zero-th homotopy group $\pi_0(\mathcal{R})$ has a natural ring structure. These functors fit into an adjunction

$$\{\text{commutative rings}\} \underset{R \mapsto \underline{R}}{\overset{\pi_0(\mathcal{R}) \leftarrow\!\shortmid \mathcal{R}}{\rightleftarrows}} \{\text{simplicial commutative rings}\} \tag{1.1}$$

In particular, given a "derived affine scheme" X, which for now means an object in the opposite category to the category of simplicial commutative rings, it has an "underlying classical scheme" X_{cl} equipped with a map $X_{\mathrm{cl}} \to X$. How should one think about this? At a formal level, the relation in (2.1) is quite analogous to the adjunction

$$\{\text{reduced commutative rings}\} \underset{R \mapsto R}{\overset{R/\mathrm{Nil} \leftarrow\!\shortmid R}{\rightleftarrows}} \{\text{commutative rings}\} \tag{1.2}$$

The relationship between derived algebraic geometry and algebraic geometry is quite analogous to the relationship between algebraic geometry and "reduced algebraic geometry" (that is to say, the theory of schemes built out of *reduced* commutative rings). In particular, the map $X_{\mathrm{cl}} \to X$ for a "derived affine scheme X" is analogous to the map $X_{\mathrm{red}} \to X$ for a classical (classical, possibly non-reduced affine scheme) X. Thus, geometrically one can think of a "derived affine scheme" as a type of "infinitesimal thickening" of its underlying classical scheme. This analogy goes beyond just the adjunctions above; it also applies (for example) when reasoning about the étale topology.

1.4. **Examples of derived schemes.** Let X be a proper variety over a field k and $\mathcal{E}$ a vector bundle on X. Consider the scheme representing the functor $S \in \mathrm{Sch}_{/k}$ to global sections of $\mathcal{E}$ on X_S, this is represented by $\underline{H^0(X, \mathcal{E})} = H^0(X, \mathcal{E}) \otimes_k \mathbf{A}^1_k$, the k-vector space $H^0(X, \mathcal{E})$ regarded as an affine space over k. It can be described as $\mathrm{Spec}\,(\mathrm{Sym}\, H^0(X, \mathcal{E})^\vee)$.

A perspective one learns in homological algebra is to view $H^0(X, \mathcal{E})$ as the zeroth cohomology group of a complex $R\Gamma(X, \mathcal{E})$, whose higher cohomology groups are $H^i(X, \mathcal{E})$ for $i \geq 1$. There will be a derived affine scheme

$$\underline{R\Gamma(X, \mathcal{E})} = \text{“}\operatorname{Spec}(\operatorname{Sym} R\Gamma(X, \mathcal{E})^\vee)\text{”},$$

whose classical truncation is $\underline{H^0(X, \mathcal{E})}$.

Just as $R\Gamma$ and H^i, for $i \geq 1$, can be viewed as "derived functors of H^0", many of the constructions that we will encounter can be viewed as "derived functors" of constructions on commutative rings. In particular, the *cotangent complex* can be viewed as the output of "deriving" the formation of Kähler differentials. However the theory of "derived functors" that you learn in homological algebra applies for *abelian* categories; generalizing this is a reason we need the simplicial theory.

Why consider a space like $\underline{R\Gamma(X, \mathcal{E})}$? One thing we know about $R\Gamma(X, \mathcal{E})$ is that it behaves well in families. For example, suppose X is a smooth projective curve. Then there is a (well-behaved) moduli stack of rank n vector bundles on X, called Bun_G. There is also a stack $\mathcal{M}$ parametrizing families of rank n vector bundles $\mathcal{E}/X$ and a section of $\mathcal{E}$. The fiber of $\mathcal{M} \to \operatorname{Bun}_n$ over $\mathcal{E}$ is $\underline{H^0(X, \mathcal{E})}$. Since the dimension of $H^0(X, \mathcal{E})$ varies discontinuously in $\mathcal{E}$, this morphism has horrible smoothness properties. However, the Euler characteristic of $R\Gamma(X, \mathcal{E})$ is locally constant in $\mathcal{E}$ – that is a first sign that the derived version is "well-behaved". This has recently been used in joint work with Yun and Zhang, in enumerative geometry problems related to automorphic forms.

1.5. **What is the cotangent complex?** Given a map of commutative rings $f \colon A \to B$, we have the module of Kähler differentials $\Omega_f := \Omega_{B/A}$. If f is smooth, then we may think of this as the module of relative 1-forms.

The *cotangent complex* of f will be a cochain complex $\mathbf{L}_f$, concentrated in non-positive degrees, such that $H^0(\mathbf{L}_f) = \Omega_f$. If f is smooth then $H^i(\mathbf{L}_f) = 0$ for $i \neq 0$, and if f is of finite presentation and A, B are noetherian then there is a converse: if $\operatorname{Hom}_B(\mathbf{L}_f, -)$ vanishes in degrees ≥ 1, then f is smooth. The cotangent complex can be viewed as a "derived" functor of formation of Kähler differentials. For example, given a sequence of maps

$$A \xrightarrow{f} B \xrightarrow{g} C, \qquad h = g \circ f \colon A \to C$$

we get an exact sequence

$$\Omega_f \otimes_B C \to \Omega_h \to \Omega_g \to 0.$$

This will extend to an exact *triangle* of cotangent complexes

$$\mathbf{L}_f \overset{\mathbf{L}}{\otimes}_B C \to \mathbf{L}_h \to \mathbf{L}_g \xrightarrow{+1}$$

which in particular gives a long exact sequence in cohomology,

$$\ldots \to H^{-1}(\mathbf{L}_h) \to H^{-1}(\mathbf{L}_g) \to \Omega_f \otimes_B C \to \Omega_h \to \Omega_g \to 0.$$

The definition of $\mathbf{L}_f$ will use simplicial commutative rings (even though it is already interesting for classical rings)! That is because simplicial commutative rings are needed to create the "resolutions" used to derive the functor of Kähler differentials.

The objective of this mini-course will be to define the cotangent complex and explain some of its applications to *classical* algebraic geometry. For example, we will see how it can be used to characterize certain geometric properties of morphisms. One example, indicated above, is a characterization of smoothness for $f\colon A \to B$ in terms of $\mathbf{L}_f$. We will now give another example. Recall that a *regular sequence* $r_1, \ldots, r_m$ in a commutative ring R is a sequence of elements, not generating the unit ideal, such that r_i is not a zero-divisor on $R/(r_1, \ldots, r_{i-1})$. Geometrically, this means we are cutting down by a sequence of equations that is never redundant on a component.

Let $f\colon A \to B$ be a finite type morphism of noetherian rings. We say that f is a *local complete intersection* (LCI) if it is Zariski-locally on A the composition of a map of the form $A \to A[x_1, \ldots, x_n]$ followed by a quotient by a regular sequence.

Theorem 1.1. *Let $f\colon A \to B$ be a finite type homomorphism of Noetherian rings. Then f is LCI if and only if $\mathbf{L}_f$ is represented by a complex with cohomology concentrated in degrees $[-1, 0]$.*

Example 1.2. Important recent applications of Theorem 1.1 have to do with generalizing the notion of LCI homomorphisms to non-noetherian contexts. This has arisen for example in perfectoid geometry, where one is always dealing with very large rings (e.g., in characteristic p they must be perfect) which are almost never noetherian.

1.6. **Application to deformation theory.** The cotangent complex was originally invented for applications to deformation theory.

Example 1.3 (Deformation theory for schemes)**.** Let $X \to S$ be a scheme and $S \hookrightarrow S'$ a square-zero thickening. We consider deformations of X to $X' \to S'$, as in the diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ S & \hookrightarrow & S' \end{array}$$

Then this is a matter of constructing an extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \end{array}$$

where $\mathcal{J}$ may be regarded as a sheaf on $\mathcal{O}_X$ because of the square-zero property, and as such it is isomorphic to $f^*\mathcal{I}$. The "obstruction group" for such a diagram lies in $\operatorname{Ext}^2(\mathbf{L}_f, \mathcal{J})$. If it vanishes, then the set of deformations is a torsor for $\operatorname{Ext}^1_X(\mathbf{L}_f, \mathcal{J})$, and the automorphisms are isomorphic to $\operatorname{Ext}^0_X(\mathbf{L}_f, \mathcal{J}) \cong \operatorname{Hom}_X(\Omega_X, \mathcal{J})$. More generally, this story generalizes when just given a map $f^*\mathcal{I} \to \mathcal{J}$.

2. Simplicial sets and simplicial commutative rings

2.1. **Why "simplicial"?** We are going to introduce a notion of "simplicial widgets", where widgets are objects of a category $\mathcal{C}$, which is a combinatorial model for the notion of "topological widget". For example, simplicial sets will form a model for "well-behaved" topological spaces (e.g., those having the homotopy type of a simplicial complex).

2.2. **The simplex category.** The *simplex category* Δ has as its objects the sets $[n] := \{0, 1, \ldots, n\}$ for $n \geq 0$, and $\mathrm{Hom}_\Delta([n], [m])$ consists of all maps of sets $f \colon [n] \to [m]$ that do not reverse order, i.e., if $i \leq j$ then $f(i) \leq f(j)$.

We will name specific generator morphisms between $[n]$ and $[n+1]$. For $0 \leq i \leq n+1$, the *coface map*

$$\delta_i \colon [n] \to [n+1]$$

is the morphism "skipping over i", or more formally

$$\delta_i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

For $0 \leq i \leq n$, the *codegeneracy map*

$$\sigma_i \colon [n+1] \to [n]$$

be the morphism "doubling up on i", or more formally

$$\delta_i(j) = \begin{cases} j & j \leq i \\ j-1 & j \geq i+1 \end{cases}$$

Exercise 2.1. Show that all non-identity morphisms in Δ can be written in the form

$$\delta_{i_1} \circ \ldots \circ \delta_{i_m} \circ \sigma_{j_1} \circ \ldots \circ \sigma_{j_n}.$$

2.3. **Simplicial sets.** A *simplicial set* is a functor $\underline{X} \colon \Delta^{\mathrm{op}} \to \mathsf{Sets}$. Concretely, it is specified by a collection of sets $X_n := \underline{X}([n])$, for $n \geq 0$, with maps between them indexed by the morphisms in Δ^{op}. In particular, we have *face maps*

$$d_i := \underline{X}(\delta_i) \colon X_{n+1} \to X_n \quad \text{for } 0 \leq i \leq n+1,$$

and *degeneracy maps*

$$s_i := \underline{X}(\sigma_i) \colon X_n \to X_{n+1} \quad \text{for } 0 \leq i \leq n$$

Because of Exercise 2.1, to specify a simplicial set it it suffices to specify the data of the $\{X_n\}_{n \geq 0}$ and the d_i and s_i for each n, satisfying the "simplicial identities".

Exercise 2.2. Show that all relations between the face and degeneracy maps are generated by the following:

- $d_i d_j = d_{j-1} d_i$ for $i < j$.
- $d_i d_i = d_i d_{i+1}$.
- $d_j s_j = d_{j+1} s_j = \mathrm{Id}$.
- $d_i s_j = s_{j-1} d_i$ for $i < j$.
- $d_i s_j = s_j d_{i-1}$ for $i > j+1$.
- $s_i s_j = s_{j+1} s_i$ for $i \leq j$.

Example 2.3. Each element $[n] \in \Delta$ induces a representable functor on Δ^{op}, namely $\mathrm{Hom}_\Delta(-, [n])$, and so induces a simplicial set that we shall call $\Delta[n]$. Intuitively, we think of it as corresponding to the topological space of the standard n-simplex, denoted $|\Delta[n]$.

Example 2.4. The definition of simplicial set is an axiomatization of the structure obtained from a topological space X by setting $\underline{X}[n]$ to be the set of continuous maps from the n-simplex $|\Delta[n]|$ to X, i.e. X_n is the set of "n-simplices of X". The face maps d_i are the usual boundary maps. The degeneracy maps account for ways to view lower-dimensional simplices as "degenerate" instances of higher-dimensional simplices; these do not play an important role in the topological theory, but are convenient in the simplicial theory. This defines the "singular simplices" functor

$$\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{sSets}.$$

Therefore, for a general simplicial set $\underline{X}$ we will refer to $\underline{X}[n]$ as the "n-simplices of $\underline{X}$".

2.3.1. *Geometric realization.* Given a simplicial set $\underline{X}$, we can build a topological space $X := |\underline{X}|$ from $\underline{X}$ by viewing $\underline{X}$ as a recipe for assembling a simplicial complex. In formulas,

$$|\underline{X}| := \frac{\coprod_n \underline{X}[n] \times \Delta^n}{(d_i x, u) \sim (x, \delta_i u), (s_i x, u) \sim (x, \sigma_i u)}.$$

An alternative, concise way to express this is as follows: we declare $|\Delta[n]|$ to be the standard n-simplex $|\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \colon 0 \leq x_i \leq 1, \sum x_i = 1\}$ and then we define

$$|\underline{X}| = \varinjlim_{\Delta[n] \to \underline{X}} |\Delta^n|$$

where the indexing category has as its objects the maps $\Delta[n] \to \underline{X}$ for varying n, and as its morphisms the maps $\Delta[m] \to \Delta[n]$ respecting the given maps to $\underline{X}$.

Exercise 2.5. Show that geometric realization is left adjoint to the singular simplices functor:

$$\mathrm{Hom}_{\mathrm{Top}}(|\underline{X}|, Y) \cong \mathrm{Hom}_{\mathrm{sSet}}(\underline{X}, \mathrm{Sing}\, Y).$$

Remark 2.6. If we restrict ourselves to the subcategory of suitably "nice" topological spaces, then these adjoint functors define an "equivalence" in a suitable sense (which we are not yet equipped to define precisely).

2.3.2. *Homotopy groups of a simplicial set.* A simplicial set $\underline{X}$ has *homotopy groups* $\pi_i(\underline{X})$, which could be defined as the homotopy groups (in the usual sense) of the geometric realization $|\underline{X}|$. They could also be defined directly within the category of simplicial sets, basically by using a combinatorial model for the sphere as the simplicial set Δ^n modulo its boundary, but this involves the subtlety of replacing $\underline{X}$ by an "equivalent" *Kan complex.* This is a first indication of the principle that in "homotopy-theoretic" categories (such as the category of simplicial sets), objects may not have "enough" maps in or out of them and so must be replaced by "equivalent" objects that do. This will be discussed a bit more in the next lecture.

2.3.3. *Internal Hom.* Given two simplicial sets $\underline{X}$ and $\underline{Y}$, we make a simplicial set $\underline{\mathrm{Map}}(\underline{X}, \underline{Y})$ whose set of 0-simplices is identified with $\mathrm{Hom}_\Delta(\underline{X}, \underline{Y})$. Intuitively, we are trying to put a topological structure on the set of maps from $\underline{X}$ to $\underline{Y}$. This will make the category of simplicial sets into a *simplicial category* – a category enriched over simplicial sets.

We define

$$\underline{\mathrm{Map}}(\underline{X}, \underline{Y})([n]) := \underline{\mathrm{Map}}_\Delta(\underline{X} \times \Delta[n], \underline{Y}).$$

Note that the product of presheaves is formed level-wise.

Remark 2.7. A simplicial category is a possible model for the notion of ∞-category. Intuitively, this is a type of category where objects have mapping *spaces.*

2.4. **Simplicial widgets.** More generally, if $\mathcal{C}$ is any category, then we say that a *simplicial object of* $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. If $\mathcal{C}$ is the category of "widgets", then the functor category $\mathrm{Hom}(\Delta, \mathcal{C})$ will be called the category of "simplicial widgets".

Example 2.8. The following examples will be of interest to us.

- A *simplicial abelian group* is a functor from Δ^{op} to the category of abelian groups.
- Given a commutative ring R (for us this always means commutative and with unit), a *simplicial R-module* is a functor from Δ^{op} to the category of R-modules.
- A *simplicial commutative ring* is a functor from Δ^{op} to the category of commutative rings.
- Given a commutative ring R, a *simplicial R-algebra* is a functor from Δ^{op} to the category of R-algebras.

The categories of such objects is, in each case, enriched over sSet. First we observe that for a simplicial set $\underline{S}$ and a simplicial widget $\underline{X}$, we have a simplicial widget

$$(\underline{X}^{\otimes \underline{S}})[n] = \coprod_{S_n} X_n$$

with the coproduct formed in the category of widgets. Then we define

$$\underline{\mathrm{Hom}}_{\mathsf{s}\mathcal{C}}(\underline{X}, \underline{Y})[n] := \mathrm{Hom}_{\mathsf{s}\mathcal{C}}(\underline{X}^{\Delta[n]}, \underline{Y}).$$

Example 2.9. Let $\underline{X}$ be a simplicial set. Then there is a simplicial abelian group $\mathbf{Z}\langle \underline{X} \rangle$, obtained by forming the free abelian group level-wise: define $\mathbf{Z}\langle \underline{X} \rangle([n]) := \mathbf{Z}\langle \underline{X}([n]) \rangle$ and let the face and degeneracy maps be induced by those of $\underline{X}$.

There is also a simplicial $\mathbf{Z}$-algebra $\mathbf{Z}[\underline{X}]$ obtained by forming the free (polynomial) $\mathbf{Z}$-algebra level-wise.

Remark 2.10. It is a fact that simplicial groups are *automatically* Kan complexes. Therefore, the homotopy groups of simplicial groups can be calculated as maps from simplicial spheres, in contrast to the situation for general simplicial sets as cautioned in §2.3.2. Because of how the model structures are defined, the same applies for simplicial abelian groups, simplicial commutative rings, etc.

2.5. **The Dold-Kan Theorem.** Recall that if $\mathcal{C}$ is an abelian category, then we can form the category of chain complexes of elements in $\mathcal{C}$. This is essential for example in homological algebra.

Theorem 2.11 (Dold-Kan)**.** *The category of simplicial R-modules is equivalent to the category of non-negatively graded chain complexes of R-modules.*

We will indicate the functors used to define the equivalence, starting with the functor from simplicial R-modules to chain complexes of R-modules. Let $\underline{M}$ be a simplicial R-module. First we explain an auxiliary construction called the *Moore complex* of $\underline{M}$. From $\underline{M}$ we form a chain complex M_* by taking $M_n := \underline{M}[n]$, and taking the differential $d_n \colon M_n \to M_{n-1}$ to be the alternating sum of the face maps,

$$\partial := \sum_{i=0}^{n} (-1)^i d_i.$$

Example 2.12. The singular chain complex of a topological space X is by definition the Moore complex of the simplicial abelian group $\mathbf{Z}\langle \mathrm{Sing}\, X \rangle$.

Remark 2.13. The homology groups of the Moore complex M_* coincide with the homotopy groups of the underlying simplicial set $\underline{M}$.

Let us analyze the structure on the Moore complex. There are degeneracy maps $s_i \colon M_n \to M_{n+1}$ for $0 \leq i \leq n$. Topological intuition suggests that "removing" the degenerate simplices should not affect the homology. Define $DM_* \subset M_*$ so that DM_{n+1} is the span of the image of $s_0, \ldots, s_n$.

Exercise 2.14. Check that DM_* is a complex of M_*, i.e. is preserved by the differential ∂. Then show that the map $M_* \to (M/DM)_*$ is a quasi-isomorphism.

Basically we want to instead consider the chain complex $(M/DM)_*$. However, it is convenient to use a different normalization of this, which we will call the *normalized Moore complex.* Define NM_n to be the kernel of *all* the face maps d_i for $i < n$ (but not $i = n$). Then $(-1)^n d_n$ defines a differential $NM_n \to NM_{n-1}$.

Exercise 2.15. Show that NM_* is a chain complex.

Exercise 2.16. Show that the sum map

$$NM_n \oplus DM_n \to M_n$$

is an isomorphism. Deduce that NM_* maps isomorphically to $(M/DM)_*$.

The functor from simplicial R-modules to chain complexes of R-modules is $\underline{M} \mapsto NM_*$. A key step in the proof of the Dold-Kan equivalence is to show that

$$\underline{M}[n] \cong \bigoplus_{[n] \twoheadrightarrow [k]} NM_k$$

functorially in $\underline{M}$. This tells us how to define the inverse functor: given a chain complex M_*, we will define a simplicial R-module by

$$\underline{M}[n] := \bigoplus_{[n] \twoheadrightarrow [k]} M_k.$$

2.6. **Simplicial commutative rings.** A *simplicial commutative ring* is a functor $\mathcal{R}$ from Δ^{op} to the category of commutative rings. Because we shall work with this a lot, we adopt a slightly more economical notation. Simplical commutative rings will be denoted using calligraphic letters, and we abbreviate $\mathcal{R}_n := \mathcal{R}[n]$. Classical commutative rings will be denoted using Roman letters such as R. So, a simplicial commutative can be specified concretely by a collection of commutative rings $\mathcal{R}_n$, for $n \geq 0$, with maps between them indexed by the morphisms in Δ^{op}. This is an axiomatization of the structure that exists on the singular simplices of a topological commutative ring. The category of commutative rings is denoted CR and the category of simplicial commutative rings is denoted SCR.

Definition 2.17. If $\mathcal{R}$ is a simplicial commutative ring, you can form its homotopy groups $\pi_*(\mathcal{R})$. As groups they are the homotopy groups of the underlying simplicial set/abelian group, but the ring structure on $\mathcal{R}$ equips $\pi_*(\mathcal{R})$ with the additional structure of a graded *algebra.* Note that this graded-commutative ring structure is what one would expect to see on a "nice" topological commutative ring.

Example 2.18. Let R be a commutative ring. Then may define a simplicial commutative ring $\underline{R}$ such that $\underline{R}[n] = R$, and all face and degeneracy maps are the identity map. Intuitively, $\underline{R}$ corresponds to the topological commutative ring which is R equipped with the discrete topology.

2.6.1. *Homotopy groups.* The homotopy groups of a simplicial commutative ring $\mathcal{R}$ can be computed using the underlying simplicial abelian group, and therefore by the Moore complex

$$\ldots \to R_n \xrightarrow{\partial_n} R_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_0} R_0.$$

However, this does not make transparent the *graded-commutative ring structure* on

$$\pi_*(\mathcal{R}) := \bigoplus \pi_i(\mathcal{R}).$$

Example 2.19. For the constant simplicial commutative ring $\underline{R}$, the Moore complex reads

$$\ldots \xrightarrow{\mathrm{Id}} R \xrightarrow{0} R \xrightarrow{\mathrm{Id}} R \xrightarrow{0} R$$

Hence we see $\pi_0(\underline{R}) = R$ and $\pi_i(\underline{R}) = 0$ for $i > 0$, in accordance with Example 2.18.

Remark 2.20. It turns out that the augmentation ideal $\pi_{>0}(\mathcal{R})$ always has a *divided power structure* in $\pi_*(\mathcal{R})$. We will not explain what this means or why it exists, except that for a $\mathbf{Q}$-algebra a divided power structure always exists and is unique, whereas in positive characteristic it is a rather special piece of structure.

Recall that last time we claimed there was an adjunction

$$\{\text{commutative rings}\} \underset{R \mapsto \underline{R}}{\overset{\pi_0(\mathcal{R}) \leftarrow\!\shortmid \mathcal{R}}{\rightleftarrows}} \{\text{simplicial commutative rings}\} \tag{2.1}$$

To justify this, we need to argue that

$$\mathrm{Hom}_{\mathsf{SCR}}(\mathcal{R}, \underline{R}) \cong \mathrm{Hom}_{\mathsf{CR}}(\pi_0(\mathcal{R}), R).$$

A morphism $f \in \mathrm{Hom}_{\mathsf{SCR}}(\mathcal{R}, \underline{R})$ amounts to a system of maps $f_n \colon \mathcal{R}_n \to R$ compatible with the face and degeneracy maps. Since all maps on $\underline{R}$ are constant, it must therefore be the case that $f_n(r)$ can be calculated by composing with any of the $n+1$ distinct maps $\mathcal{R}_n \to \mathcal{R}_0$ and then applying $f_0 \colon \mathcal{R}_0 \to R$. For this to be well-defined, we see that f_0 must vanish on the image of $d_1 - d_0$, and thus factor through $\pi_0(\mathcal{R}) \to R$. It remains to show that any such f_0 does induce a well-defined map of simplicial commutative rings, which is completed by the following exercise.

Exercise 2.21. Show that all of the $n+1$ maps $\mathcal{R}_n \to \mathcal{R}_0$ induced by the $n+1$ maps $[0] \to [n]$ have the same composition with $\mathcal{R}_0 \twoheadrightarrow \pi_0(\mathcal{R})$.

2.6.2. *Enrichment over simplicial sets.* Because of its importance, let us explicate the enrichment of simplicial commutative rings over simplicial sets. Given a set S and a ring R, we can form $R^{\otimes S} := \coprod_S R$. Then for a simplicial set $\underline{S}$ and a simplicial commutative $\mathcal{R}$, we define $\mathcal{R}^{\underline{S}}$ by $\mathcal{R}^{\underline{S}}[n] := \mathcal{R}[n]^{\otimes \underline{S}[n]}$. Finally,

$$\underline{\mathrm{Hom}}_{\mathsf{SCR}}(\mathcal{R}, \mathcal{R}')[n] := \mathrm{Hom}_{\mathsf{SCR}}(\mathcal{R}^{\otimes \Delta[n]}, \mathcal{R}').$$

Exercise 2.22. For commutative rings R and R', calculate the homotopy groups of $\underline{\mathrm{Hom}}_{\mathsf{SCR}}(\underline{R}, \underline{R}')$. (Answer: $\pi_0 = \mathrm{Hom}_{\mathrm{CR}}(\underline{R}, \underline{R}')$, and π_i vanish for $i > 0$.) This justifies thinking of CR as being embedded fully faithfully, in the sense of *simplicial categories*, in SCR.

3. Simplicial resolutions

3.1. **Derived functors.** Let us recall the paradigm arising from homological algebra, which you are probably already familiar with. If F is a right-exact functor on an abelian category $\mathcal{C}$, then we define the "higher derived functors" of F on $M \in \mathcal{C}$ by finding a "free" resolution

$$\dots P_n \to P_{n-1} \to \dots \to P_0 \to M \to 0$$

and then applying F instead to the complex $\dots P_n \to P_{n-1} \to \dots \to P_0$, which we view as a "replacement" for M. We are being deliberately vague about what the adjective "free" should mean.

Example 3.1. Let $\mathcal{C}$ be the category of R-modules and N an R-module, $F = N \otimes_R -$. The higher derived functors of F are $\operatorname{Tor}_i^R(N, -)$.

Example 3.2. Let $\mathcal{C}$ be the category of R-modules and N an R-module, $F = \operatorname{Hom}_R(N, -)$. The higher derived functors of F are $\operatorname{Ext}^i_R(N, -)$.

Simplicial commutative rings will play the role for commutative rings that chain complexes of R-modules play for R-modules. In particular, we will see that replacing a commutative ring by a type of "free resolution" will allow us to build a theory of "derived functors" on the category of simplicial commutative rings. We will illustrate this in the particular example of the cotangent complex.

Recall that we said that for a smooth morphism $f\colon A \to B$, we have the B-module of Kähler differentials Ω^1_f, which is "well-behaved" if f is smooth. For general f, we will "resolve" B by a "smooth" simplicial A-algebra in order to define the cotangent complex $\mathbf{L}_f$. For abelian categories this meant replacing B by a complex of objects which level-wise had good properties (e.g., "free"), but that does not apply to the category of A-algebras, which is far from abelian. Instead, the this theory of "homological algebra" will be replaced by a theory of "homotopical algebra" developed by Quillen. The correct framework for this is that of Quillen's *model categories*, but this would involve a significant digression to explain completely, so we will adopt an ad hoc presentation and then hint at the general theory later, in §3.3.

Definition 3.3. A morphism of simplicial commutative rings $f\colon \mathcal{R} \to \mathcal{R}'$ is a *weak equivalence* if it induces an isomorphism of homotopy groups $\pi_*(f)\colon \pi_*(\mathcal{R}) \xrightarrow{\sim} \pi_*(\mathcal{R}')$.

This notion is an analogue for simplicial commutative rings of the notion of *quasi-isomorphism* for chain complexes. Indeed, we could make an analogous definition for simplicial R-modules, which would transport to precisely the notion of quasi-isomorphism under the Dold-Kan Theorem.

3.2. **Free simplicial algebras.** Let $\mathcal{A}$ be a simplicial commutative ring. A *free simplicial $\mathcal{A}$-algebra* $\mathcal{R}$ is a simplicial $\mathcal{A}$-algebra of the following form:

- There is a system of sets X_n such that $\mathcal{R}_n = \mathcal{A}[X_n]$.
- The degeneracy maps send $s_j(X_n) \subset X_{n+1}$. (Note that there is *no* condition on face maps!)

The following exercise illustrates that a free simplicial $\mathcal{A}$-algebra has good "mapping out" properties, similar to those of free resolutions of modules.

Exercise 3.4. Consider a commutative diagram of simplicial algebras

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & \nearrow & \sim\downarrow \Phi \\ \mathcal{R} & \longrightarrow & \mathcal{C} \end{array}$$

were Φ is level-wise surjective and a weak equivalence, and $\mathcal{R}$ is a free simplicial $\mathcal{A}$-algebra. Show that there exists a lift $\mathcal{R} \to \mathcal{B}$ making the diagram commutative.

Proposition 3.5 (Existence of resolutions)**.** *Let $f\colon A \to B$ be a map of (classical) commutative rings. Then there exists a free simplicial A-algebra $\mathcal{B}$ and a diagram*

$$\begin{array}{ccc} & & \mathcal{B} \\ & \nearrow & \downarrow\sim \\ A & \xrightarrow[f]{} & B \end{array}$$

Furthermore, we can even arrange that the formation of this diagram is functorial in B.

Proof sketch. We will give a "canonical" resolution, which is functorial in B. Note that for any A-algebra R, we have a map

$$A[R] \to R \tag{3.1}$$

sending $[r]$ to r. Now, from the map $A[B] \to B$, we get two maps $A[A[B]] \to A[B]$:

- Apply (3.1) with $R = B$ to get a map $A[B] \to B$, and then take $A[-]$. For example, this map sends $[a[b]] \mapsto [ab]$.
- Apply (3.1) with $R = A[B]$. For example, this maps sends $[a[b]] \mapsto a[b]$.

We also have a map $A[B] \to A[A[B]]$ sending $[b]$ to $[[b]]$, which is a section of both maps above.

Contemplating the combinatorics of the situation, one gets a simplicial object

$$\ldots \to A[A[B]] \rightrightarrows A[B] \tag{3.2}$$

where the maps defined previously are the first face and degeneracy maps. Clearly this resolution is level-wise a polynomial algebra over A, and one can check it is free. Why is it a resolution of B?

This comes from an "extra degeneracy" argument, which is part of a more general pattern. If we have a simplicial set $\underline{X}$

$$\ldots X_1 \rightrightarrows X_0$$

with an augmentation $d_0\colon X_0 \to X_{-1}$ and "extra degeneracies" $s_{-1}\colon X_{n-1} \to X_n$ satisfying the extensions of the relations (2.2), then the map $\underline{X} \to \underline{X_{-1}}$ is a weak equivalence. Intuitively, $s_{-1}\colon X_{-1} \to X_0$ specifies a basepoint in each connectd component, and for $n \geq 1$ the extra degeneracy $s_{-1}\colon X_{n-1} \to X_n$ specifies the homotopy from an $(n-1)$-simplex to the n-simplex spanned by it and the basepoint of the connected component.

Going back to (3.2), there are extra degeneracies $B \to A[B]$ sending $b \mapsto [b]$ and $s_{-1}\colon A[B] \mapsto A[A[B]]$ sending $a[b] \mapsto [a[b]]$, etc.

∎

Remark 3.6. The construction of the proof is a special case of a more general one, and we give the general statement because it actually elucidates the situation. Let T be a monad on a category $\mathcal{C}$, meaning the data of an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ plus natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \to T$ and $\mu\colon T \circ T \to T$ satisfying the coherence conditions. The example of interest is: $\mathcal{C}$ is the category of sets, and T is the functor taking a set S to the underlying set of $A[S]$. Then the category of algebras over T is equivalent to the category of A-algebras.

An algebra over the monad T consists of the data of $x \in \mathcal{C}$ plus a morphism $a\colon Tx \to x$ satisfying coherence conditions. Given an algebra X over T, one can form a simplicial set called the *bar construction* $B(T,X)_\bullet$, which has $B(T,X)_n = T^{\circ(n+1)}X$. By formal combinatorial analysis, the bar construction is equipped with extra degeneracies.

Exercise 3.7. Complete the above proof sketch by defining the simplicial structure on

$$\ldots A[A[B]] \rightrightarrows A[B]$$

and showing that it is weakly equivalent to B. You may find it clarifying to work in the generality of Remark 3.6.

Let us discuss "how unique" these resolutions are. In the classical situation of homological algebra, for example, projective resolutions are unique up to homotopy. There is a sense in which any free simplicial resolution is unique up to homotopy. The best language for formulating such statements is model category theory, but we can say what it specializes to in this situation. There is a product morphism $\mu\colon A[X] \otimes_A A[X] \to A[X]$, where $A[X] \otimes_A A[X]$ is formed by taking tensor products levelwise. Let us call $A[X,X] := A[X] \otimes_A A[X]$. This represents the "homotopy coproduct"; in particular, given $f, g\colon A[X] \to B$ we get $f \otimes g\colon A[X,X] \to B$.

Let $A[X,X,Y]$ be a simplicial resolution of $A[X]$ over $A[X,X]$. Then we say that f and g are *homotopic* if $f \otimes g$ extends to a map

$$\begin{array}{ccc} A[X,X] & \longrightarrow & A[X,X,Y] \\ & \searrow^{f\otimes g} & \downarrow \\ & & B \end{array}$$

Exercise 3.4 may be used to show that the lifts (as in the Exercise) are unique up to homotopy, and also that any two simplicial resolutions are homotopic.

3.3. **A vista of model categories.**

Let us hint at the more general framework underlying these procedures, which is Quillen's theory of *model categories*. The purpose of a model category is to incorporate homotopy theory into category theory.

A model category is a category equipped with several distinguished classes of morphisms, satisfying various properties. The most important are the *weak equivalences*, which are not necessarily isomorphisms but are meant to be "homotopic". The weak equivalences are morphisms that we want to "invert". Familiar examples are:

- In the category of chain complexes of R-modules, weak equivalences should be the quasi-isomorphisms. Inverting them leads to the *derived category of R-modules.*
- In the category of simplicial commutative rings, weak equivalences should be as defined in Definition 3.3.

In particular, model categories allow to construct the localization of a category at the weak equivalence, or informally speaking, the category obtained by "inverting weak equivalences". The other data in a model category are classes of morphisms called the *cofibrations* and *fibrations*, which are roughly the generalizations of projective and injective resolutions. There are various axioms placed on these classes of morphisms: for example, they are preserved by compositions and retracts. An object is called *fibrant* if its map to the terminal object is a fibration, and an object is called *cofibrant* if its map from the initial object is a cofibration. The axioms of a model category imply that every object can be "resolved" by a cofibrant object or a fibrant object. As we know, derived functors on abelian categories are defined using such resolutions. A model category structure is more general since it makes sense for not necessarily abelian categories, and allows to construct the "non-abelian derived functor" of F by providing an appropriate notion of "resolution".

Our particular constructions with simplicial commutative rings are examples of general constructions with model categories. For more, we highly recommend Dwyer-Spalinski.

3.4. **The cotangent complex.** We may now define the *cotangent complex* of a morphism $f\colon A \to B$. Let $\mathcal{B} \xrightarrow{\sim} B$ be a simplicial resolution of B as a free simplicial A-algebra. Then let $\mathbf{L}_{\mathcal{B}/A}$ be the simplicial $\mathcal{B}$-module obtained by forming Kähler differentials level-wise: $\mathbf{L}_{\mathcal{B}/A}[n] := \Omega^1_{\mathcal{B}_n/A}$. Finally we set the *cotangent complex*

$$\mathbf{L}_f := \mathbf{L}_{B/A} := \mathbf{L}_{\mathcal{B}/A} \otimes_{\mathcal{B}} B.$$

Exercise 3.8. Show that if $\mathcal{B}$ and $\mathcal{B}'$ are two free simplicial resolutions of B, then $\mathbf{L}_{\mathcal{B}/A} \otimes_{\mathcal{B}} B$ is homotopic to $\mathbf{L}_{\mathcal{B}'/A} \otimes_{\mathcal{B}} B$. Therefore, $\mathbf{L}_{B/A}$ is well-defined up to homotopy. In particular, it gives a well-defined object of the derived category of B-modules.

We expect $\mathbf{L}_{B/A}$ to have reasonable finiteness properties when the morphism f has good finiteness properties. However, the canonical resolutions used to build $\mathcal{B}$ in the proof of Proposition 3.5 are extremely large. Therefore, we would like to know that there are "smaller" resolutions. We will examine this question next.

3.5. **Economical resolutions.** We will show:

Proposition 3.9. *Let A be a noetherian commutative ring and $f\colon A \to B$ a finite type morphism. Then there exists a simplicial A-algebra resolution of B by a free simplicial A-algebra $A[X]$ with each X_n finite.*

Corollary 3.10. *Let A be a noetherian commutative ring and $f\colon A \to B$ a finite type morphism. Then there exists a representative of $\mathbf{L}_{B/A}$ by a complex of finite free A-modules.*

Remark 3.11. Typically $\mathbf{L}_{B/A}$ cannot be represented by a perfect complex; that is, it will have homology groups in infinitely many degrees. For example, if A is a field and $\mathbf{L}_{B/A}$ is perfect, then $\mathbf{L}_{B/A}$ has amplitude at most 2, and B is LCI.

As motivation, we recall how to construct "efficient" resolutions of a finite type A-module M by a complex of free A-modules. We can build a sequence of complexes of finite free A-modules $\{F_i\}$ whose homology approximates M in degrees up to i. For F_0, we pick any surjection from a free module $A^n \twoheadrightarrow M$. Then we pick generators for $I := \ker(F_0 \twoheadrightarrow M)$, which induces a map $A^m \xrightarrow{\partial} A^n$ with image I, whose map to M induces an isomorphism in H_0. So we may take $F_0 = [A^m \xrightarrow{\partial} A^n]$. We then inductive build F_i from F_{i-1} by picking representatives in F_{i-1} for $H_i(F_{i-1})$, and adding a free summand in degree $i+2$ whose boundary is these generators.

To perform an analogous construction for simplicial commutative rings, what we want is a way to "kill cycles" in a simplicial commutative ring. This is accomplished by the following Lemma.

Lemma 3.12. *Given a simplicial commutative ring $\mathcal{A}$, and a homotopy class $[z] \in \pi_i(\mathcal{A})$, there exists a free simplicial $\mathcal{A}$-algebra $\mathcal{A}'$ such that:*

- *$\mathcal{A}'_n$ is a free $\mathcal{A}_n$-algebra on finitely many generators for each n, and $\mathcal{A}'_n = \mathcal{A}_n$ for $n \leq i$.*
- *The map $\mathcal{A} \to \mathcal{A}'$ induce an isomorphism on π_n for $n < i$, and for $n = i$ an exact sequence*

$$0 \to \mathcal{A}_i([z]) \to \pi_i(\mathcal{A}) \to \pi_i(\mathcal{A}') \to 0$$

Indeed, supposing Lemma 3.12 is true, we can make a resolution as in Proposition 3.9 by picking a surjection $\underline{A[x_1, \ldots, x_d]} \twoheadrightarrow B$. Then, by repeatedly applying Lemma 3.12, we may build a sequence of free simplicial A-algebras with the "correct" finiteness properties and homotopy groups in all finite degrees. Taking their colimit gives the desired resolution.

Now to prove Lemma 3.12, we want to "adjoin" a variable whose boundary is $[z]$. However, because of the simplicial identities, the process of "adjoining" variables is necessarily quite complicated. For this we will imitate what happens to the singular simplices when attaching cells to a topological space X to kill a homology class in degree $i-1$. If we attach a cell in degree i, then this creates additional degenerate simplices in all higher dimensions. For this reason, the construction is considerably more complicated.

Proof of Lemma 3.12. We define

$$X_n := \{x_t \mid t \colon [n] \twoheadrightarrow [i+1] \in \Delta\}$$

and we take $\mathcal{A}'_n = \mathcal{A}_n[X_n]$, with faces and degeneracies defined as follows.

- We have $s_j(x_t) = x_{t \circ s^j}$.
- Note that $X_{i+1} = \{x_{\mathrm{Id}}\}$. We set $d_0(x_{\mathrm{Id}}) = z$ and $d_j(x_{\mathrm{Id}}) = 0$ for $j > 0$. For $n > i+1$, we define $d_j(x_t) = x_{t \circ d^j}$ if $t \circ d^j$ is surjective, and otherwise it factors through a face map $d_{j'} \colon [i] \to [i+1]$ so we define it to make the following diagram commute

$$\begin{array}{ccc} [n] & \xrightarrow{t} & [i+1] \\ {\scriptstyle d^j}\uparrow & & \uparrow{\scriptstyle d^{j'}} \\ [n-1] & \longrightarrow & [i] \end{array}$$

Exercise 3.13. Check that this $\mathcal{A}'_n$ is a simplicial commutative ring satisfying the conclusions of Lemma 3.12.

□

Remark 3.14. A more highbrow way to phrase this construction is as follows. For a simplicial A-algebra $\mathcal{A}$, a class in $\pi_i(\mathcal{A})$ is represented by a map of simplicial sets $\partial\Delta[i+1] \to \mathcal{A}$ (here we are using Remark 2.10). This induces a map of simplicial commutative rings $A[\partial\Delta[i+1]] \to \mathcal{A}$, where $A[\partial\Delta[i+1]]$ is the free simplicial A-algebra on the simplicial set $\partial\Delta[i+1]$. We may then form

$$\mathcal{A}' := \mathcal{A} \otimes_{A[\partial\Delta[i+1]]} A[\Delta_{i+1}]$$

and we claim that it satisfies the conclusions of Lemma 3.12. The first bullet point is satisfied by inspection. To compute the effect on homotopy groups, we note that $\pi_*(A) \xrightarrow{\sim} \pi_*(A[\Delta_{i+1}])$ because Δ^{i+1} is contractible, and the lowest positive-degree homotopy group of $A[\partial\Delta[i+1]]$ is in degree i and maps to $[z]$ in $\pi_i(A)$ by construction. Then use the Tor spectral sequence ♠♠♠ TONY: [qu70 (5.2)]

$$\mathrm{Tor}_p^{\pi_*(A[\partial\Delta[i+1]])}(\pi_*(\mathcal{A}), \pi_*(A[\Delta_{i+1}]))_q \implies \pi_{p+q}(\mathcal{A} \otimes_{A[\partial\Delta[i+1]]} A[\Delta_{i+1}]).$$

4. The cotangent complex

4.1. **Derived tensor products.** Let $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to \mathcal{B}'$ be two maps of simplicial commutative rings. Let $\mathcal{A} \hookrightarrow \mathcal{P} \xrightarrow{\sim} \mathcal{B}$ be a free resolution of $\mathcal{B}$ as an $\mathcal{A}$-algebra and $\mathcal{A} \hookrightarrow \mathcal{P}' \xrightarrow{\sim} \mathcal{B}'$ be a free resolution of $\mathcal{B}'$ as an $\mathcal{A}$-algebra. The *derived tensor product* "$\mathcal{B} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{B}'$" is the simplicial commutative ring $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}'$, with

$$(\mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}')_n = \mathcal{P}_n \otimes_{\mathcal{A}_n} \mathcal{P}'_n.$$

Actually, it suffices to resolve only one of the terms, e.g., $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{B}'$. Indeed, there is an evident map

$$\mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}' \to \mathcal{P} \otimes_{\mathcal{A}} \mathcal{B}'.$$

We claim that these are quasi-isomorphisms of free $\mathcal{B}'$-algebras. This can be checked on the underlying simplicial $\mathcal{A}$-modules, where it follows from the familiar fact that the derived tensor product of modules can be computed by resolving only one factor. If A is classical and $\mathcal{P}$ is a free simplicial A-algebra, then the underlying simplicial A-module of $\mathcal{P}$ associates under the Dold-Kan correspondence to a complex of free A-modules. In particular, this takes free simplicial resolutions to the familiar notion of free resolutions of chain complexes. In particular, we see that if $\mathcal{A} \cong A$, $\mathcal{B} \cong B$, and $\mathcal{B}' \cong B'$ are all classical, then we have

$$\pi_i(\mathcal{B} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{B}') \cong \mathrm{Tor}_i^A(B, B').$$

Lemma 4.1. *Let $A \to B$ and $A \to B'$ be maps of classical commutative rings, such that $\mathrm{Tor}_A^i(B, B') = 0$ for all $i > 0$. Then $B \overset{\mathbf{L}}{\otimes}_A B' \to \pi_0(B \overset{\mathbf{L}}{\otimes}_A B') \cong B \otimes_A B'$ is a weak equivalence.*

4.2. **Properties of the cotangent complex.** Now we establish some properties of the cotangent complex which are familiar for the module of Kähler differentials, at least in the smooth case.

Proposition 4.2. *If B, B' are A-algebras with $\mathrm{Tor}_A^i(B, B') = 0$ for $i > 0$, then*

(i) $\mathbf{L}_{B \otimes_A B'/B'} \cong \mathbf{L}_{B/A} \otimes_A B'$.
(ii) $\mathbf{L}_{B \otimes_A B'/A} \cong (\mathbf{L}_{B/A} \otimes_A B') \oplus (B \otimes_A \mathbf{L}_{B'/A})$.

Remark 4.3. It is true in general that

(i) $\mathbf{L}_{\mathcal{B} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{B}'/\mathcal{B}'} \cong \mathbf{L}_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B}'$.
(ii) $\mathbf{L}_{\mathcal{B} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{B}'/\mathcal{A}} \cong (\mathbf{L}_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B}') \oplus (\mathcal{B} \otimes_{\mathcal{A}} \mathbf{L}_{\mathcal{B}'/\mathcal{A}})$.

The Tor-vanishing assumption is used to ensure that $B \overset{\mathbf{L}}{\otimes}_A B' \xrightarrow{\sim} B \otimes_A B'$.

Proof. Both arguments are completely formal from the properties of the Kähler differentials, using the observation that:

- If $\mathcal{P}$ is a free simplicial $\mathcal{A}$-algebra, and $\mathcal{B}$ is any simplicial $\mathcal{A}$-algebra, then $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{B}$ is a free simplicial $\mathcal{B}$-algebra.

(1) Let $A \to \mathcal{B} \xrightarrow{\sim} B$ be a free resolution of B as a simplicial A-algebra. Then $B' \to \mathcal{B} \otimes_A B' \xrightarrow{\sim} B \otimes_A B'$ is a free resolution of $B \otimes_A B'$ as a simplicial B'-algebra. (Really $\mathcal{B} \otimes_A B'$ is the derived tensor product, so here we are using the vanishing of $\mathrm{Tor}^i_A(B, B')$ for $i > 0$ to see that $B \otimes_A B'$ is weakly equivalent to the derived tensor product.) Furthermore, for each n we have

$$\Omega_{\mathcal{B}'_n/B'} \cong \Omega_{\mathcal{B}_n \times_A B'/B'} \cong \Omega_{\mathcal{B}_n/A} \otimes_A B'.$$

Hence $\mathbf{L}_{\mathcal{B} \otimes_A B'/B'} \cong \mathbf{L}_{\mathcal{B}/A} \otimes_A B'$, and then the result follows from tensoring with B over $\mathcal{B}$.

(2) Let $A \to \mathcal{B}' \xrightarrow{\sim} B'$ be a free resolution of B' as a simplicial A-algebra. Then $\mathcal{B} \otimes_A \mathcal{B}'$ is a free simplicial A-algebra resolution of $B \otimes_A B'$ (again we are using the Tor-vanishing assumption here), and we have

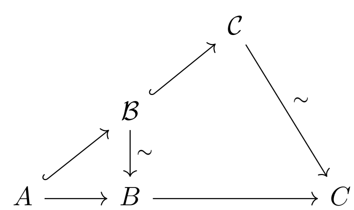
$$\Omega_{\mathcal{B}'_n/A} \cong (\Omega_{\mathcal{B}_n/A} \otimes_A B') \oplus (\Omega_{\mathcal{B}'_n/A} \otimes_A B).$$

Then the isomorphism follows level-wise, as in part (1). □

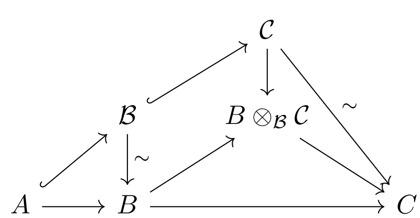
Proposition 4.4. *If $A \to B \to C$ is a composition of morphisms, then there is an exact triangle in the derived category of C-modules:*

$$\mathbf{L}_{B/A} \otimes_B C \to \mathbf{L}_{C/A} \to \mathbf{L}_{C/B}$$

Proof. Choose a free resolution $A \hookrightarrow \mathcal{B} \xrightarrow{\sim} B$. This gives C the structure of a $\mathcal{B}$-algebra, so we may then choose a free resolution $\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{\sim} C$. This gives a diagram



Then we form the tensor product



We have an exact triangle

$$\mathbf{L}_{\mathcal{B}/A} \otimes_{\mathcal{B}} \mathcal{C} \to \mathbf{L}_{\mathcal{C}/A} \to \mathbf{L}_{\mathcal{C}/\mathcal{B}}.$$

Applying $\otimes_{\mathcal{C}} C$, we get an exact triangle. Let us verify that the terms are as claimed.

- $\mathbf{L}_{\mathcal{B}/A} \otimes_{\mathcal{B}} \mathcal{C} \otimes_{\mathcal{C}} C \cong \mathbf{L}_{\mathcal{B}/A} \otimes_{\mathcal{B}} C \cong \mathbf{L}_{B/A} \otimes_B C$.
- $\mathbf{L}_{\mathcal{C}/A} \otimes_{\mathcal{C}} C \cong \mathbf{L}_{C/A}$.

- Let's abbreviate $C' := B \otimes_{\mathcal{B}} \mathcal{C}$. We have $\mathbf{L}_{\mathcal{C}/\mathcal{B}} \otimes_{\mathcal{B}} B \cong \mathbf{L}_{B \otimes_{\mathcal{B}} \mathcal{C}/B}$. Since this a free resolution of C as a B-algebra, tensoring with $\mathcal{C}$ gives $\mathbf{L}_{C/A}$.

□

Corollary 4.5 (Cotangent complex of localizations)**.**

(i) If S is a multiplicative system in A, then $\mathbf{L}_{S^{-1}A/A} \cong 0$.
(ii) If S is a multiplicative system in A and T is a multiplicative system in B, then

$$\mathbf{L}_{T^{-1}B/S^{-1}A} \cong \mathbf{L}_{B/A} \otimes_B T^{-1}B.$$

Proof. (i) We have $S^{-1}A \otimes_A S^{-1}A \cong S^{-1}A$. Putting this into Proposition 4.2(ii), we get

$$\mathbf{L}_{S^{-1}A/A} \cong \mathbf{L}_{S^{-1}A \otimes_A S^{-1}A/A} \cong \mathbf{L}_{S^{-1}A/A} \oplus \mathbf{L}_{S^{-1}A/A}.$$

(ii) By Proposition 4.2(i) We have $\mathbf{L}_{S^{-1}B/S^{-1}A} \cong \mathbf{L}_{B/A} \otimes_B S^{-1}B$. Apply Proposition 4.4 to get an exact triangle

$$\mathbf{L}_{S^{-1}B/S^{-1}A} \otimes_{S^{-1}B} T^{-1}B \to \mathbf{L}_{T^{-1}B/S^{-1}A} \to \mathbf{L}_{T^{-1}B/S^{-1}B}.$$

The rightmost term vanishes according to (i), from which the result follows.

□

4.3. **Special cases.** Now we will identify the cotangent complex in some special cases. Simplicial resolutions are almost always too unwieldy to compute with by hand, so we will instead need to make clever use of the formal properties explained above.

Proposition 4.6. *Let A be a noetherian commutative ring and $f \colon A \to B$ a finite type morphism. Then*

(i) f is étale if and only if $\mathbf{L}_f \cong 0$.
(ii) f is smooth if and only if $\mathbf{L}_f \cong \Omega^1_f$.

Proof. We will only prove the forward directions for now. The converses will be established after we have developed the connection between $\mathbf{L}_f$ and deformation theory.

Recall that $f \colon A \to B$ is weakly étale if and only if B is flat over A and $\Delta \colon \operatorname{Spec} B \to \operatorname{Spec} B \otimes_A B$ is an open immersion. If f is a finite type morphism of noetherian commutative rings, as in our situation, then weakly étale is equivalent to étale.

Suppose f is étale. Then $B \to B \otimes_A B$ is a localization, so Corollary 4.5 implies that $\mathbf{L}_{B \otimes_A B/B} \cong 0$. By Proposition 4.2 we have $\mathbf{L}_{B \otimes_A B} \cong \mathbf{L}_{B/A} \otimes_A B$. Since we only need to check a condition locally on $\operatorname{Spec} B$, we may as well assume that f is faithfully flat. The the preceding vanishing implies that $\mathbf{L}_{B/A} = 0$.

Next suppose f is smooth. The local structure theorem for smooth maps says that locally on $\operatorname{Spec} B$, f can be factored as an étale map over an affine space:

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \mathbf{A}^n_A \\ & \searrow^{f} & \downarrow \\ & & \operatorname{Spec} A \end{array}$$

At the level rings, this means we have a factorization $A \to P \to B$ where P is a polynomial ring over A. Then the morphism of constant simplicial A-algebras $A \to P$ is already free, so $\mathbf{L}_{P/A} \cong \Omega_{P/A}$. From Proposition 4.4 we have an exact triangle $\mathbf{L}_{P/A} \otimes_P B \to \mathbf{L}_{B/A} \to \mathbf{L}_{B/P}$. We know that $\mathbf{L}_{P/A} \otimes_P B \cong \Omega_{P/A}$ and $\mathbf{L}_{B/P} = 0$. Also we have

$$0 \to \Omega_{P/A} \otimes_P B \to \Omega_{B/A} \to \Omega_{B/P} \to 0.$$

□

Recall that an ideal $I \subset A$ is *regular* if it has a sequence of generators $r_1, \ldots, r_n$ such that each r_i is a non-zerodivisor modulo $A/(r_1, \ldots, r_{i-1})$.

Proposition 4.7. *If $B = A/I$ and I is regular then*

$$\mathbf{L}_{B/A} \xrightarrow{\sim} I/I^2[1].$$

Proof. Let us first treat the case $A = \mathbf{Z}[x]$ and $B = \mathbf{Z}$, $I = (x)$. In this case we have a section $\mathbf{Z} \to \mathbf{Z}[x]$. Apply Proposition 4.4 to the sequence $\mathbf{Z} \to \mathbf{Z}[x] \to \mathbf{Z}$ to get an exact triangle

$$\mathbf{L}_{\mathbf{Z}[x]/Z} \otimes_{\mathbf{Z}[x]} \mathbf{Z} \to \mathbf{L}_{\mathbf{Z}/\mathbf{Z}} \to \mathbf{L}_{\mathbf{Z}/\mathbf{Z}[x]}.$$

This shows that $\mathbf{L}_{\mathbf{Z}/\mathbf{Z}[x]} \cong \mathbf{L}_{\mathbf{Z}[x]/\mathbf{Z}}[1] \otimes_{\mathbf{Z}[x]} \mathbf{Z} \cong (x)/(x^2)$.

Now we tackle the general case. By induction, it suffices to handle the case where $I = (r)$ is principal. Note that a choice of r induces a map $\mathbf{Z}[x] \to A$ sending $x \mapsto r$, and this map induces $A/(r) \cong \mathbf{Z} \otimes_{\mathbf{Z}[x]} R$. We claim that r is a non zerodivisor if and only if $\mathrm{Tor}^i_{\mathbf{Z}[x]}(\mathbf{Z}, A) = 0$ for all $i > 0$, or in other words if and only if $\mathbf{Z} \overset{\mathbf{L}}{\otimes}_{\mathbf{Z}[x]} A \xrightarrow{\sim} A/(r)$. The claim is an elementary computation in homological algebra, using the resolution $\mathbf{Z}[x] \xrightarrow{x} \mathbf{Z}[x]$ of $\mathbf{Z}$ over $\mathbf{Z}[x]$. Then by Proposition 4.2, we have

$$\mathbf{L}_{B/A} \cong \mathbf{L}_{\mathbf{Z}/\mathbf{Z}[x]} \otimes_{\mathbf{Z}[x]} A \cong (x)/(x^2) \otimes_{\mathbf{Z}[x]} A \cong (r)/(r^2).$$

□