Homework, Math 18.704, Fall 2022

Throughout, [S] denotes Serre's book *Finite groups: an introduction*.

September 15

1. Chapter 1, exercise 1 from [S].
2. Chapter 1, exercise 7 from [S].
3. Prove the third isomorphism theorem for groups: if H is a normal subgroup of G and K is a normal subgroup of H, then
$$(G/K)/(H/K) \cong G/H.$$
4. Prove that if
$$1 \to A \to B \to C \to 1$$
is a short exact sequence of groups, then A is isomorphic to a normal subgroup of B. Use this to show that if B is simple, then C is isomorphic to either B or 1.

September 20

1. Prove that a finite group G is a p-group if and only if every element of G is order a power of p.
2. If H is a Sylow p-subgroup of G, prove that H is the only Sylow p-subgroup of its normalizer.
3. If $p > q$ and G is a group of order pq, show
   (a) G has only one Sylow p-subgroup, and
   (b) if q does not divide $p-1$, then G is cyclic.
4. (a) If G has exactly one Sylow p-subgroup P and one Sylow q-subgroup Q, show that they are normal, and that P and Q commute;
   (b) If all Sylow subgroups are normal, show G is the product of its Sylow subgroups.

September 22

1. Let S be a Sylow p-subgroup of G, and let x and y be elements of U a subgroup of G, and $g \in G$ such that $g^{-1}Ug = U$ and $g^{-1}xg = y$, and let n be the order of g. Show that
$$V := \left\{ \prod_i g^{a_i} x^{b_i} \text{ such that } n | \sum_i a_i \right\}$$
is a subgroup of U.

2. Let H be a subgroup of G. Show that $[G, G] \subseteq H$ if and only if H is normal and G/H is abelian.

3. Show two of the identities (1.1), (1.3), (1.4), and (1.5) on page 29 of [S].

September 27

1. Chapter 3, exercise 2 from [S].

2. Chapter 3, exercise 9 from [S].

3. Chapter 3, exercise 12 from [S].

4. Show that the commutator [-,-] defined in class for $\mathrm{gr}(G)$ satisfies the Jacobi identity.

October 6

1. Show that the dihedral group D_{2n} is nilpotent if and only if n is a power of 2.

2. Chapter 3, exercise 5, parts (a) and (b) from [S].

3. Chapter 3, exercise 14 part (a) from [S].

4. Chapter 3, exercise 14 part (b) from [S].

October 13

1. Let M be a $\mathbb{Z}/m\mathbb{Z}$-module, C a cyclic subgroup of M of order n. Show there exists N a subgroup of C such that $M \cong C \oplus N$.

2. Show that $\Phi(G)$ is a characteristic subgroup, i.e. it is fixed by every automorphism of G.

3. Given a field $\Bbbk$ and $n > 0$, define the *projective special linear group* $\mathrm{PSL}_{2n}(\Bbbk)$ to be the quotient group $\mathrm{SL}_{2n}(\Bbbk)/\{\pm I\}$, where I is the identity matrix.

(a) Show that $|\operatorname{PSL}_2(\mathbb{F}_5)| = |A_5|$.

(b) Deduce that $\operatorname{PSL}_2(\mathbb{F}_5) \cong A_5$, using the fact that A_5 is the unique simple nonabelian group of smallest order, and $\operatorname{PSL}_2(\mathbb{F}_5)$ is simple.

4. Half of Chapter 3, exercise 19 in [S]: Show that if q is not one of the values listed on the bottom of page 45 in Section 3.10 of [S], then $\operatorname{PSL}_2(\mathbb{F}_q)$ is not minimal simple.

October 18

1. Chapter 4, exercise 1 in [S].

2. Chapter 4, exercise 2 in [S].

3. Describe all extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$.

4. Let A be a G-module and let τ be an element of the center of G. Let the map $\phi : Z^1(G, A) \to C^1(G, A)$ be given by $\phi(f) = \tau f - f$ for all $f \in Z^1(G, A)$. Prove that $\phi(H^1(G, A)) = 0$ (or, equivalently, that $\phi(Z^1(G, A)) \subseteq B^1(G, A)$).

October 20

1. Show that the splittings of a split extension are conjugate to each other if and only if $H^1(G, A) = 0$.

2. Chapter 4, exercise 9 in [S].

3. Chapter 4, exercise 13 in [S].

October 27

1. Chapter 4, exercise 15 in [S].

2. Chapter 4, exercise 16 in [S].

3. Assuming all pullbacks exist, show that $(A \times_C B) \times_B D = A \times_C D$. (You can do this explicitly in the category of groups if you want).

November 1

1. Show that any 1-dimensional representation of G must be constant over its conjugacy classes.

2. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation. Define ϕ by $\phi(g) = \overline{\rho(g)}$ (that is, entry-wise complex conjugation). Show that $\phi : G \to \mathrm{GL}_n(\mathbb{C})$ is also a representation.

3. Let G be a simple group, and $\rho : G \to \mathrm{GL}(V)$ be a representation where $V = \mathbb{C}^n$. Prove that if there exists $g \neq 1$ such that $\chi_V(g) = \chi_V(1)$ then ρ is the trivial representation (i.e. $\rho(g)$ is the identity matrix for all $g \in G$).

4. Let h be a positive-definite Hermitian form on $V = \mathbb{C}^n$, and let r and g be its real and imaginary parts respectively, defined by $h(x, y) = r(x, y) + g(x, y)i$. Prove that when V is made into a real vector space by restricting scalars to $\mathbb{R}$, r is a positive definite symmetric form and g is a skew-symmetric form.

November 8

1. Find all irreducible representations of C_5. (Hint: they are all contained in its regular representation.)

2. Suppose $G \neq q$. Show that equivalence of the two properties:

   (a) G is simple.

   (b) $\chi(g) \neq \chi(1)$ for every irreducible character $\chi \neq 1$ and every $g \neq 1$.

3. Determine all 1-dimensional representations of S_n.

4. Find character tables for S_3 or A_4.

November 10

1. Is there a representation of S_3 such that its character χ satisfies:

   (a) $(\chi(1), \chi((12)), \chi((123))) = (3, 3, 0)$,

   (b) $(\chi(1), \chi((12)), \chi((123))) = (5, -1, 0)$,

   (c) $(\chi(1), \chi((12)), \chi((123))) = (8, 2, 2)$?

2. With terminology as in the proof of [Proposition 8.29, S], show that $V_\chi = W_\chi \oplus \cdots \oplus W_\chi$.

3. Find all positive integers n such that $\frac{i\sqrt{n}-1}{2}$ is an algebraic integer. *Hint: if a complex number is the root of a polynomial with real coefficients, so is its conjugate.*

4. Show that for a representation ρ of a finite group G, the eigenvalues of $\rho(g)$ are roots of unity for all $g \in G$. *Hint: In a finite group, all elements have finite order.*