Throughout, [S] denotes Serre’s book \textit{Finite groups: an introduction}.

\textbf{September 15}

1. Chapter 1, exercise 1 from [S].

2. Chapter 1, exercise 7 from [S].

3. Prove the third isomorphism theorem for groups: if $H$ is a normal subgroup of $G$ and $K$ is a normal subgroup of $H$, then

\[
(G/K)/(H/K) \cong G/H.
\]

4. Prove that if

\[1 \to A \to B \to C \to 1\]

is a short exact sequence of groups, then $A$ is isomorphic to a normal subgroup of $B$. Use this to show that if $B$ is simple, then $C$ is isomorphic to either $B$ or 1.

\textbf{September 20}

1. Prove that a finite group $G$ is a $p$-group if and only if every element of $G$ is order a power of $p$.

2. If $H$ is a Sylow $p$-subgroup of $G$, prove that $H$ is the only Sylow $p$-subgroup of its normalizer.

3. If $p > q$ and $G$ is a group of order $pq$, show

\begin{enumerate}
  \item $G$ has only one Sylow $p$-subgroup, and
  \item if $q$ does not divide $p - 1$, then $G$ is cyclic.
\end{enumerate}

4. (a) If $G$ has exactly one Sylow $p$-subgroup $P$ and one Sylow $q$-subgroup $Q$, show that they are normal, and that $P$ and $Q$ commute;

(b) If all Sylow subgroups are normal, show $G$ is the product of its Sylow subgroups.
September 22

1. Let $S$ be a Sylow $p$-subgroup of $G$, and let $x$ and $y$ be elements of $U$ a subgroup of $G$, and $g \in G$ such that $g^{-1}Ug = U$ and $g^{-1}xg = y$, and let $n$ be the order of $g$. Show that

$$V := \left\{ \prod_i g^{a_i} \alpha_i \right\} \text{ such that } n \mid \sum_i a_i$$

is a subgroup of $U$.

2. Let $H$ be a subgroup of $G$. Show that $[G, G] \subseteq H$ if and only if $H$ is normal and $G/H$ is abelian.

3. Show two of the identities (1.1), (1.3), (1.4), and (1.5) on page 29 of [S].

September 27

1. Chapter 3, exercise 2 from [S].

2. Chapter 3, exercise 9 from [S].

3. Chapter 3, exercise 12 from [S].

4. Show that the commutator $[-,-]$ defined in class for $gr(G)$ satisfies the Jacobi identity.

October 6

1. Show that the dihedral group $D_{2n}$ is nilpotent if and only if $n$ is a power of 2.

2. Chapter 3, exercise 5, parts (a) and (b) from [S].

3. Chapter 3, exercise 14 part (a) from [S].

4. Chapter 3, exercise 14 part (b) from [S].

October 13

1. Let $M$ be a $\mathbb{Z}/m\mathbb{Z}$-module, $C$ a cyclic subgroup of $M$ of order $n$. Show there exists $N$ a subgroup of $C$ such that $M \cong C \oplus N$.

2. Show that $\Phi(G)$ is a characteristic subgroup, i.e. it is fixed by every automorphism of $G$.

3. Given a field $\mathbb{k}$ and $n > 0$, define the projective special linear group $PSL_{2n}(\mathbb{k})$ to be the quotient group $SL_{2n}(\mathbb{k})/\{\pm I\}$, where $I$ is the identity matrix.
(a) Show that $|\text{PSL}_2(\mathbb{F}_5)| = |A_5|.$
(b) Deduce that $\text{PSL}_2(\mathbb{F}_5) \cong A_5,$ using the fact that $A_5$ is the unique simple nonabelian group of smallest order, and $\text{PSL}_2(\mathbb{F}_5)$ is simple.

4. Half of Chapter 3, exercise 19 in [S]: Show that if $q$ is not one of the values listed on the bottom of page 45 in Section 3.10 of [S], then $\text{PSL}_2(\mathbb{F}_q)$ is not minimal simple.

October 18

1. Chapter 4, exercise 1 in [S].
2. Chapter 4, exercise 2 in [S].
3. Describe all extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$.
4. Let $A$ be a $G$-module and let $\tau$ be an element of the center of $G$. Let the map $\phi : Z^1(G, A) \to C^1(G, A)$ be given by $\phi(f) = \tau f - f$ for all $f \in Z^1(G, A)$. Prove that $\phi(H^1(G, A)) = 0$ (or, equivalently, that $\phi(Z^1(G, A)) \subseteq B^1(G, A))$.

October 20

1. Show that the splittings of a split extension are conjugate to each other if and only if $H^1(G, A) = 0$.
2. Chapter 4, exercise 9 in [S].
3. Chapter 4, exercise 13 in [S].