18.708 PROBLEM SET 3

Problem 1. Let $E = \langle g_1, g_2 : g_1 g_2 = g_2 g_1, g_i^2 = 1\rangle$ denote the Klein four-group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

(a) Let $(a, b) \in \Bbbk^2 \backslash \{0, 0\}$ and let $M(a, b)$ denote the 2-dimensional E-module where g_1 and g_2 act by

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

respectively. Compute directly (i.e. without appealing to the Avrunin-Scott Theorem) the cohomological support variety $V(M(a, b))$ of $M(a, b)$ (i.e., show that $V(M(a, b))$ corresponds to the rank variety $V^r(M(a, b))$ that was computed in class).

(b) (optional). Classify the remaining indecomposable modules for the Klein four-group E, and compute their support varieties.

Problem 2. Let E be an elementary abelian group $(\mathbb{Z}/p\mathbb{Z})^{\oplus c}$ and $\Bbbk$ a field of characteristic p. Complete the proof sketched in class that $M \otimes N$ is free if and only if $M \otimes' N$ is free, where the action of $\Bbbk E$ on $M \otimes' N$ is given by the diagonal action with respect to a linearly independent collection $\{u_{a(1)}, ..., u_{a(c)}\}$ of units $u_{a(i)} = 1 + \sum_{j=1}^{c} a(i)_j (g_j - 1)$ for some collection of points $a(i) \in \Bbbk^c$. Namely, show that $M \otimes N$ (and, by a similar argument, $M \otimes' N$) is free if and only if

$$\dim \mathrm{Hom}_E(M, N) = \frac{1}{|E|}(\dim M)(\dim N).$$

Problem 3. The following problem shows that the direct analogue of Dade's lemma does not hold for infinite-dimensional modules. Let $E := (\mathbb{Z}/2\mathbb{Z})^r$ for some $r > 1$ with standard generators $\{g_i\}$, and $\Bbbk$ be a field of characteristic 2. Set $\mathbb{K} := \Bbbk(t_1, ..., t_r)$ a transcendental extension of $\Bbbk$ of transcendence degree r, and let $M := \mathbb{K} \oplus \mathbb{K}$ as a $\Bbbk$-vector space, equipped with E-action given by

$$g_i \mapsto \begin{pmatrix} \mathrm{Id} & 0 \\ t_i \,\mathrm{Id} & \mathrm{Id} \end{pmatrix}.$$

Show that M is not a free $\Bbbk E$-module, but its restriction to every cyclic shifted subgroup of $\Bbbk E$ is free.

Problem 4. This problem deals with the stable category $\mathsf{stmod}(G)$ of a finite group scheme G.

(a) Recall $\Sigma(M)$ is defined as the cokernel of an injective envelope of M. Show that while Σ does **not** induce a functorial auto-equivalence of $\mathsf{mod}(G)$, it **does** induce an equivalence of $\mathsf{stmod}(G)$. Give an explicit characterization of the inverse functor.

(b) It was claimed in class that $\mathsf{stmod}(G)$ is a triangulated category, where distinguished triangles are those triangles isomorphic to those arising from short exact sequences in $\mathsf{mod}(G)$. Prove that axiom (TR2) holds for $\mathsf{stmod}(G)$: the triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$$

is distinguished.

(c) (optional). Verify that $\mathsf{stmod}(G)$ satisfies the remaining axioms of triangulated categories.