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On a composite implicit time integration procedure for nonlinear dynamics

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Abstract

Transient analysis of nonlinear problems in structural and solid mechanics is mainly carried out using direct time integration of the equations of motion. For reliable solutions, a stable and efficient integration algorithm is desirable. Methods that are unconditionally stable in linear analyses appear to be a natural choice for use in nonlinear analyses, but unfortunately may not remain stable for a given time step size in large deformation and long time range response solutions. A composite time integration scheme is proposed and tested in some example solutions against the trapezoidal rule and the Wilson θ-method, and found to be effective where the trapezoidal rule fails to produce a stable solution. These example results are indicative of the merits of the composite scheme.

Keywords: Direct time integration; Nonlinear dynamic analysis; Stability

1. Introduction

Transient analysis of nonlinear problems in solid and structural mechanics requires the stable and accurate solution of the equations

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{F}(\mathbf{U}, \text{time}) = \mathbf{R}(\text{time}) \quad \text{(plus initial conditions)} \tag{1}$$

where **M** is the mass matrix, **C** is the damping matrix, **R** is the vector of externally applied nodal loads, **F** is the vector of nodal forces equivalent to the element stresses, **U** is the vector of nodal displacements (including rotations), and a time derivative is denoted by an overdot. We note that **F** depends on the displacements and time, whereas we assume **M** and **C** to be constant (an assumption that can be removed) [1].

A widely used approach to solve Eq. (1) is direct time integration in which the equilibrium relations are satisfied at discrete time points Δt apart. The solution is stepped forward in time by assuming time variations of displacements, velocities and accelerations within the time interval Δt. The assumptions used result into a specific algorithm and directly affect the stability and accuracy of the procedure.

Direct integration techniques can be either explicit or implicit. Assume that the solutions have been obtained at the discrete time points Δt apart up to time t. Explicit integration techniques use Eq. (1) at the time(s) for which the displacements are known, to obtain the solution at time $t + \Delta t$. This computation is relatively inexpensive to carry out for each time step since no

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solution of coupled linear equations is needed (assuming **M** to be a diagonal matrix, and **C** as well, if present). A widely used explicit technique is the central difference method which, however, is only conditionally stable; that is, the time step size that can be employed without losing the stability of the algorithm must be smaller than, or equal to, the critical time step. This restriction can result in a time step size that can be several orders of magnitude smaller than the step size which should be adequate to accurately resolve the response. In such cases, the use of an implicit integration procedure can be much more effective.

Implicit methods use Eq. (1) at a time for which the solution is not known, to obtain the response at time $t+\Delta t$. The need for the solution of a coupled system of equations makes implicit methods considerably more expensive, computationally, per time step. Hence unconditionally stable implicit schemes are desirable since then the time step size is chosen to satisfy accuracy requirements alone. The use of larger time steps means, of course, that much fewer steps are used than with an explicit, conditionally stable procedure.

Implicit integration schemes like the trapezoidal rule and the Wilson θ-method [2,3] are unconditionally stable in linear analyses, and are also employed in nonlinear analyses. As pointed out long time ago, when using an implicit method in nonlinear analysis, it can be of great importance that equilibrium iterations be carried out at each time step [1,4]. Unfortunately, even with Newton–Raphson iterations carried out to very tight convergence tolerances, a scheme that is unconditionally stable in linear analysis may become unstable in a nonlinear solution. In particular, the trapezoidal rule which is known to be unconditionally stable in linear analysis, may become unstable in nonlinear analysis when a long time response and very large deformations are considered. If present, the instability is clearly seen in that the displacements, velocities and accelerations become unrealistically large.

Much research effort has been directed to improve the stability of integration schemes for nonlinear dynamic analysis of solids and structures. Kuhl and Crisfield [5] have presented a survey of algorithms that have been formulated with this aim. The basic idea followed in the research is to satisfy, either algorithmically or by constraint equations, conservation of momenta and energy, see [5–7] and the references therein. However, high frequency modes which are inaccurately resolved with the time step used may then deteriorate the overall solution accuracy. Also, these methods may result in non-symmetric tangent stiffness matrices and the solution of a scalar variable either at the integration points or over each element in an averaged sense. Hence, these integration schemes are computationally costly.

In this paper we focus on the formulation and study of a single step (but two sub-steps) composite integration procedure. First we present the basic scheme and then we solve various example problems. The calculated results are compared with those obtained using the trapezoidal rule and the Wilson θ-method. The composite procedure is attractive since it only operates on the usual global vectors, only uses the usual symmetric matrices, shows good stability characteristics and is of second-order accuracy.

2. The composite time integration procedure

In general, time integration algorithms formulated using backward difference expressions display some numerical damping and we might use this property to stabilize a time integration scheme. The Houbolt method is such an example which uses a four-point backward difference approximation [1].

A composite, single step, second-order accurate integration scheme for solving first-order equations arising in the simulation of silicon devices and circuits was presented by Bank et al. [8]. This composite scheme is available in the ADINA program for fluid flow structural interaction problems. The first-order fluid flow equations and second-order structural equations are solved fully coupled in time using this procedure [9,10]. Some experience with the algorithm in the solution of structural mechanics problems has been presented in Ref. [11]. In this section we briefly present the formulation of the algorithm, and in Section 5 we give the solutions of some test problems for the evaluation of the scheme. For details on the notation used, see [1].

Assume that the solution is completely known up to time t, and the solution at time $t+\Delta t$ is to be computed. Let $t+\gamma\Delta t$ be an instant in time between times t and $t+\Delta t$, i.e., $\gamma \in (0,1)$. Then using the trapezoidal rule over the time interval $\gamma\Delta t$, we have the following assumptions on velocity and displacement:

$$^{t+\gamma\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \frac{^{t}\ddot{\mathbf{U}} + {}^{t+\gamma\Delta t}\ddot{\mathbf{U}}}{2}\gamma\Delta t \tag{2}$$

and

$$^{t+\gamma\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \frac{^{t}\dot{\mathbf{U}} + {}^{t+\gamma\Delta t}\dot{\mathbf{U}}}{2}\gamma\Delta t \tag{3}$$

or after simplification,

$$^{t+\gamma\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\gamma\Delta t + ({}^{t}\ddot{\mathbf{U}} + {}^{t+\gamma\Delta t}\ddot{\mathbf{U}})\left(\frac{\gamma\Delta t}{2}\right)^2 \tag{4}$$

Solving for $^{t+\gamma\Delta t}\ddot{\mathbf{U}}$ and $^{t+\gamma\Delta t}\dot{\mathbf{U}}$ from the above equations

$$^{t+\gamma\Delta t}\ddot{\mathbf{U}} = ({}^{t+\gamma\Delta t}\mathbf{U} - {}^{t}\mathbf{U} - {}^{t}\dot{\mathbf{U}}\gamma\Delta t)\frac{4}{\gamma^2\Delta t^2} - {}^{t}\ddot{\mathbf{U}} \tag{5}$$

$$^{t+\gamma\Delta t}\dot{\mathbf{U}} = ({}^{t+\gamma\Delta t}\mathbf{U} - {}^{t}\mathbf{U})\frac{2}{\gamma\Delta t} - {}^{t}\dot{\mathbf{U}} \tag{6}$$

The equilibrium equation (1) at time $t+\gamma\Delta t$ is

$$\mathbf{M}\,{}^{t+\gamma\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\gamma\Delta t}\dot{\mathbf{U}} = {}^{t+\gamma\Delta t}\mathbf{R} - {}^{t+\gamma\Delta t}\mathbf{F} \tag{7}$$

Substituting for ${}^{t+\gamma\Delta t}\ddot{\mathbf{U}}$ and ${}^{t+\gamma\Delta t}\dot{\mathbf{U}}$ in the above equation, and linearizing, the following expression is obtained (see [1]),

$$\begin{aligned}&\left({}^{t+\gamma\Delta t}\mathbf{K}^{(i-1)} + \mathbf{M}\frac{4}{\gamma^2\Delta t^2} + \mathbf{C}\frac{2}{\gamma\Delta t}\right)\Delta\mathbf{U}^{(i)} \\ &\quad = {}^{t+\gamma\Delta t}\mathbf{R} - {}^{t+\gamma\Delta t}\mathbf{F}^{(i-1)} \\ &\qquad - \mathbf{M}\left(\frac{4}{\gamma^2\Delta t^2}({}^{t+\gamma\Delta t}\mathbf{U}^{(i-1)} - {}^{t}\mathbf{U}) - \frac{4}{\gamma\Delta t}{}^{t}\dot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}\right) \\ &\qquad - \mathbf{C}\left(\frac{2}{\gamma\Delta t}({}^{t+\gamma\Delta t}\mathbf{U}^{(i-1)} - {}^{t}\mathbf{U}) - {}^{t}\dot{\mathbf{U}}\right)\end{aligned} \tag{8}$$

with ${}^{t+\gamma\Delta t}\mathbf{U}^{(i)} = {}^{t+\gamma\Delta t}\mathbf{U}^{(i-1)} + \Delta\mathbf{U}^{(i)}$, and ${}^{t+\gamma\Delta t}\mathbf{K}^{(i-1)}$ being the consistent tangent stiffness matrix at the configuration corresponding to the displacement ${}^{t+\gamma\Delta t}\mathbf{U}^{(i-1)}$. Once the displacements have been computed, the velocities and accelerations are obtained from the relations given above.

Let the derivative of a function f at time $t+\Delta t$ be written in terms of the function values at times t, $t+\gamma\Delta t$ and $t+\Delta t$ as [12]

$${}^{t+\Delta t}\dot{f} = c_1\,{}^{t}f + c_2\,{}^{t+\gamma\Delta t}f + c_3\,{}^{t+\Delta t}f \tag{9}$$

where

$$c_1 = \frac{(1-\gamma)}{\Delta t\gamma} \tag{10}$$

$$c_2 = \frac{-1}{(1-\gamma)\gamma\Delta t} \tag{11}$$

$$c_3 = \frac{(2-\gamma)}{(1-\gamma)\Delta t} \tag{12}$$

Evaluating velocities in terms of displacements and accelerations in terms of velocities, we have

$${}^{t+\Delta t}\dot{\mathbf{U}} = c_1\,{}^{t}\mathbf{U} + c_2\,{}^{t+\gamma\Delta t}\mathbf{U} + c_3\,{}^{t+\Delta t}\mathbf{U} \tag{13}$$

$${}^{t+\Delta t}\ddot{\mathbf{U}} = c_1\,{}^{t}\dot{\mathbf{U}} + c_2\,{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + c_3\,{}^{t+\Delta t}\dot{\mathbf{U}} \tag{14}$$

Eq. (1) at time $t+\Delta t$ is

$$\mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F} \tag{15}$$

and substituting the above expressions and proceeding as for Eq. (8), we obtain

$$\begin{aligned}&({}^{t+\Delta t}\mathbf{K}^{(i-1)} + c_3c_3\mathbf{M} + c_3\mathbf{C})\Delta\mathbf{U}^{(i)} \\ &\quad = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(i-1)} - \mathbf{M}(c_1\,{}^{t}\dot{\mathbf{U}} + c_2\,{}^{t+\gamma\Delta t}\dot{\mathbf{U}} \\ &\qquad + c_3c_1\,{}^{t}\mathbf{U} + c_3c_2\,{}^{t+\gamma\Delta t}\mathbf{U} + c_3c_3\,{}^{t+\Delta t}\mathbf{U}^{(i-1)}) \\ &\qquad - \mathbf{C}(c_1\,{}^{t}\mathbf{U} + c_2\,{}^{t+\gamma\Delta t}\mathbf{U} + c_3\,{}^{t+\Delta t}\mathbf{U}^{(i-1)})\end{aligned} \tag{16}$$

Note that Newton–Raphson iterations are performed in Eqs. (8) and (16) with $i = 1, 2, 3, \ldots$ and appropriate tight convergence tolerances [1], in order to establish dynamic equilibrium at time $t+\gamma\Delta t$ (using the trapezoidal rule), and at time $t+\Delta t$ (using the three-point backward method), respectively. The solution for ${}^{t+\Delta t}\mathbf{U}$ and the calculation of the velocities and accelerations from the backward difference approximations in Eqs. (13) and (14) gives the complete response at time $t+\Delta t$. In our studies below we use $\gamma = 0.5$.

3. Generalization of the composite scheme

The idea of using sub-steps in a given time step can of course be generalized. For example, for n sub-steps, the trapezoidal rule can be applied $(n-1)$ times, and then the solution can be obtained at the end of the time step by an $(n+1)$-point backward difference scheme. If $n = 3$, the sub-steps being equal in size, solutions at times $t+\Delta t/3$ and $t+2\Delta t/3$ can be obtained by two successive applications of the trapezoidal rule. The solution at time $t+\Delta t$ is then obtained by using the Houbolt method based on the solutions at times t, $t+\Delta t/3$ and

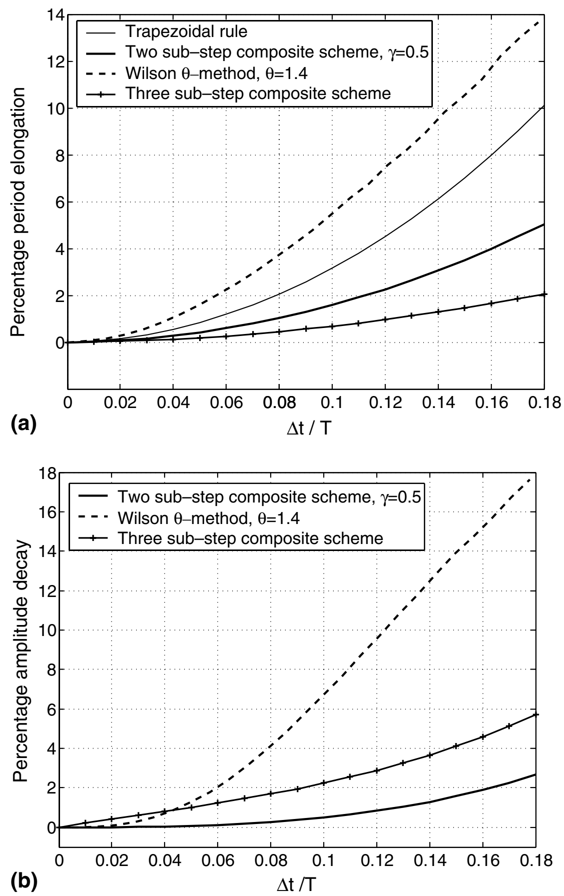

Fig. 1. Percentage period elongation and amplitude decay for the trapezoidal rule, the Wilson θ-method, the two sub-step composite scheme ($\gamma = 0.5$), and the three sub-step composite scheme. (Here, Δt is of course always the total time step size.)

$t + 2\Delta t/3$, see [1]. We call this scheme "the three sub-step composite method".

4. Accuracy of analysis

The scheme presented in Section 2 is in linear analysis unconditionally stable and second-order accurate because so are the trapezoidal rule and the three-point backward difference method [12]. Following the approach in Refs. [1,3], we can evaluate the percentage period elongation and percentage amplitude decay. The evaluations are carried out for a simple spring mass system without any physical damping, and with unit initial displacement and zero initial velocity, as functions of $\Delta t/T$, where Δt is always the complete time step size and T is the natural period of the spring mass system. The curves obtained are given in Fig. 1, along with the curves for the trapezoidal rule and the Wilson θ-method. The composite scheme is seen to perform well when compared to the other methods.

The figures also show the curves calculated for the three sub-step composite method defined in Section 3. This procedure shows considerably more amplitude decay than the two sub-step composite scheme.

5. Numerical examples

In this section we present numerical results for three test problems, obtained using both the proposed composite algorithm (with $\gamma = 0.5$) and the trapezoidal rule. Since the composite scheme retains stability due to the numerical damping introduced by the three-point backward difference method, we also test the Wilson θ-method (which is known to introduce numerical

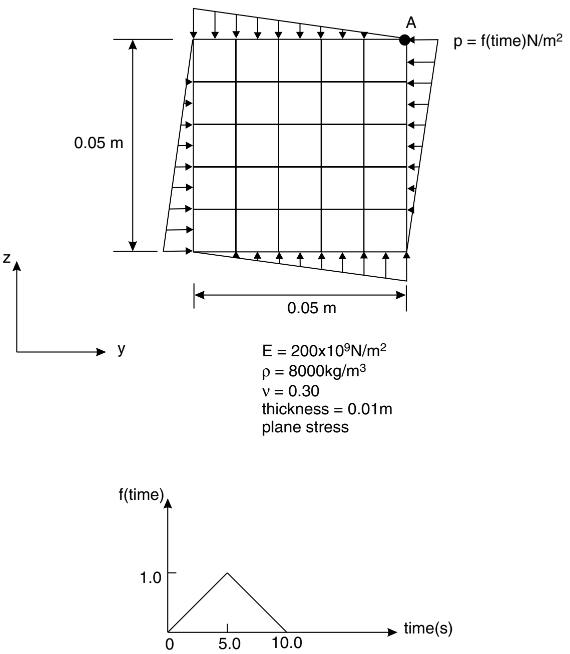

Fig. 2. The rotating plate problem.

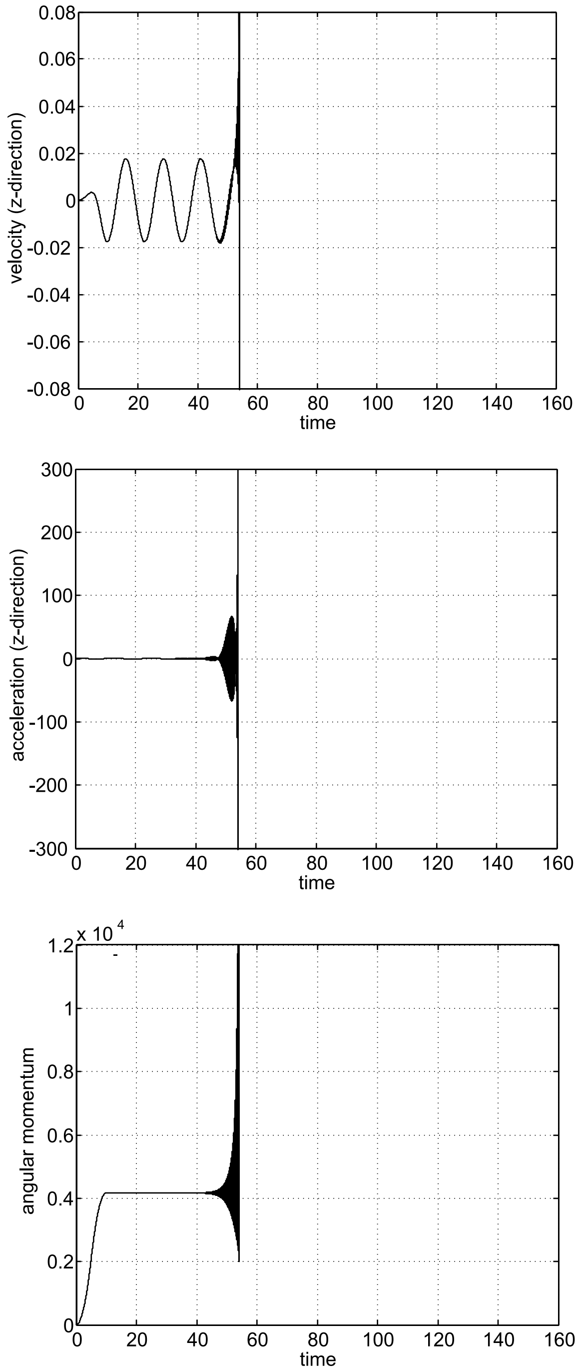

Fig. 3. The rotating plate problem; results using the trapezoidal rule; $\Delta t = 0.02$ s.

damping as well, see Fig. 1) in the solution of the problems, with $\theta = 1.4$. The test problems involve large displacements and rotations and the solutions illustrate the instabilities encountered using the trapezoidal rule. We use linear elastic constitutive relations, therefore the nonlinearities in these problems are only due to large deformations.

5.1. Rotating plate

A plate in plane stress conditions, modeled with four-node elements, is subjected to the loading shown in Fig. 2. The load is applied normal to the plate boundary for 10 s to give the plate a reasonable angular velocity

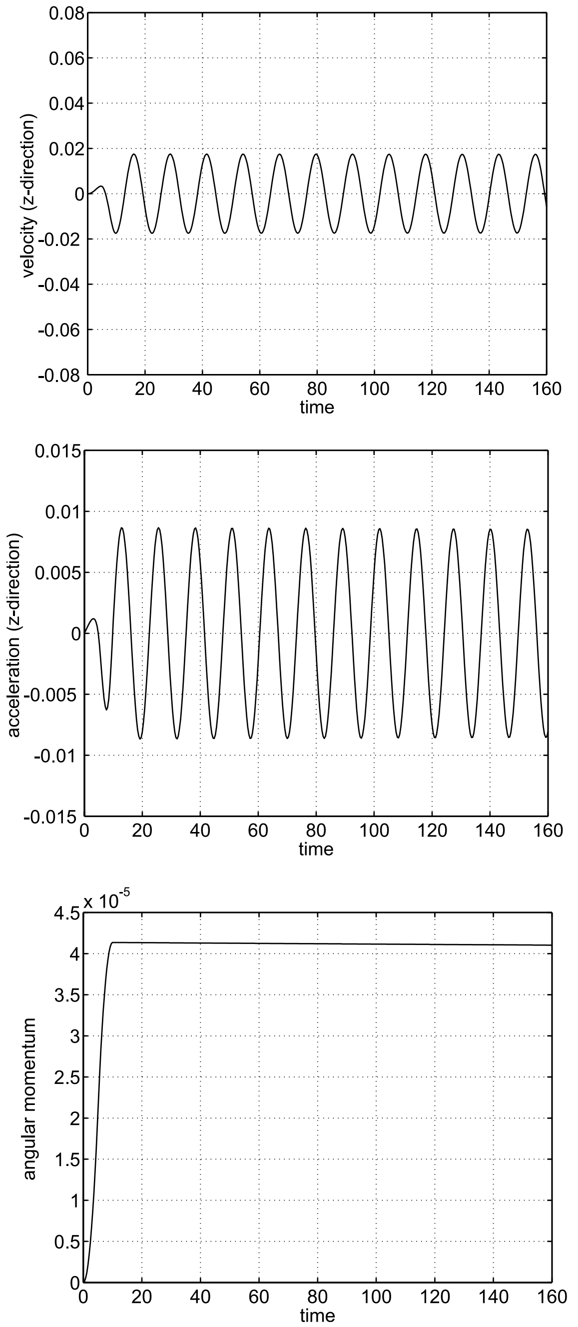

Fig. 4. The rotating plate problem; results using the composite scheme; $\Delta t = 0.4$ s.

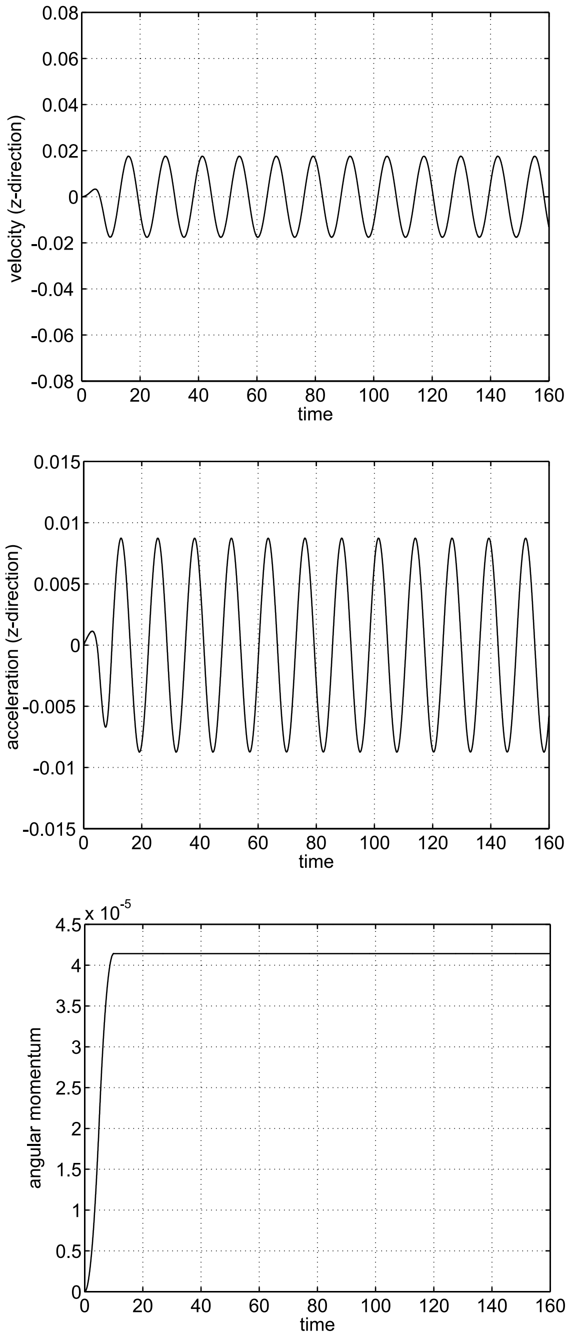

Fig. 5. The rotating plate problem; results using the Wilson θ-method; $\Delta t = 0.02$ s.

and is then taken off to have a conservative system from that instant onwards.

The problem is first solved using the trapezoidal rule with $\Delta t = 0.02$ s. The velocity and acceleration in the z-direction of point A on the plate are plotted along with the angular momentum in Fig. 3. The response is mainly in the rigid body rotational mode. The period of rigid body rotation is about 12.5 s and therefore the time step chosen should be sufficiently small to capture the response very accurately. However, after about three revolutions of the plate, numerical errors start to accumulate significantly, resulting eventually into very large accelerations. Consequently the angular momentum is not conserved, and a point is reached at which the solution cannot proceed any further.

The same problem is next solved using the proposed composite formula with $\Delta t = 0.4$ s (that is, 20 times the time step used with the trapezoidal rule but of course about twice the computational effort per time step). Fig. 4 shows that the quality of response remains excellent. In fact, there is a negligible decay in the angular momentum of the plate. This decay is less than 0.06% per revolution for the time step chosen. This solution illustrates the superior and more robust performance of the composite procedure in this long time duration problem. It is also of interest to test the performance of the Wilson θ-method. Using $\Delta t = 0.02$ s, an accurate solution is also obtained, see Fig. 5. However, the use of a time step size $\Delta t = 0.1$ s resulted in a non-positive definite effective stiffness matrix after only a few time steps, probably because the solution at the discrete time $t + \Delta t$ does not satisfy the dynamic equilibrium accurately (the Newton–Raphson iterations are used to satisfy dynamic equilibrium at time $t + \theta\Delta t$, see [1,2]), and yet this solution is used for the start of the next time step solution.

For comparison purposes, we also present the solution obtained using the three sub-step composite scheme

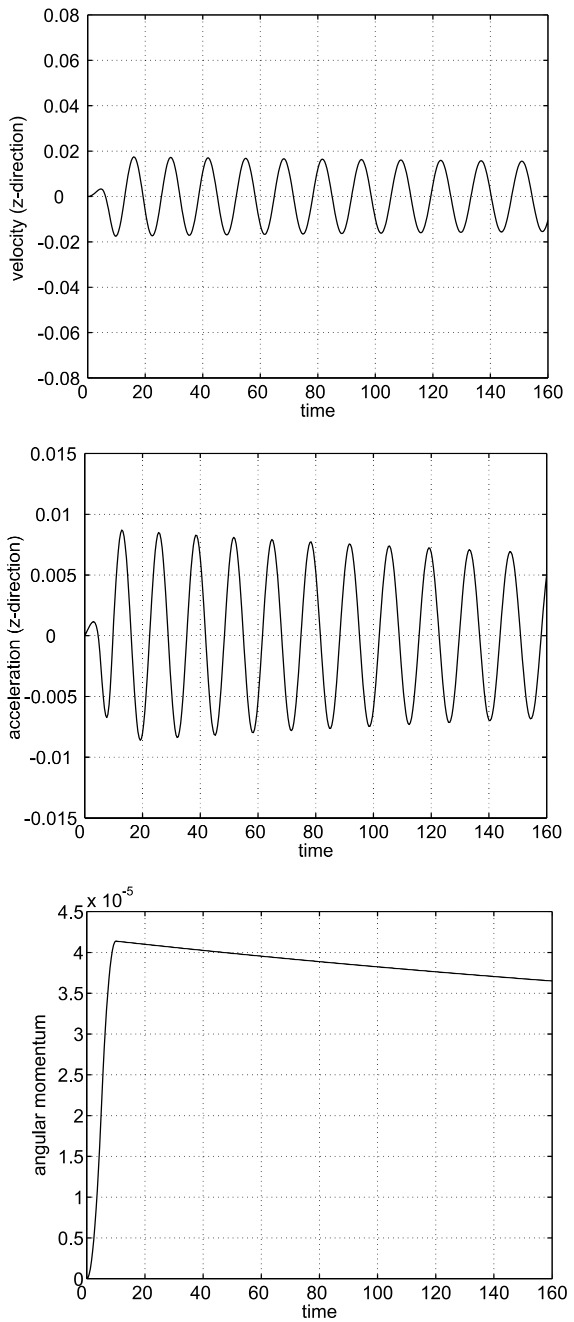

Fig. 6. The rotating plate problem; results using the three sub-step composite scheme; $\Delta t = 0.4$ s.

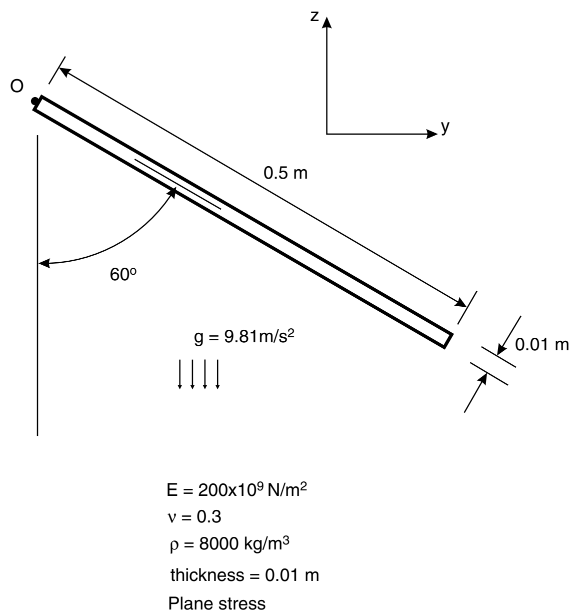

Fig. 7. The compound pendulum in its initial configuration.

of Section 3 with $\Delta t = 0.4$ s, see Fig. 6. The integration remains stable but has considerably more numerical damping (and of course, for a given step size the computational effort is larger than when using the two sub-step composite algorithm).

5.2. Compound pendulum

Fig. 7 shows the compound pendulum considered. The bar is initially at rest and released to swing under the action of the constant gravitational field with a period of about 1.25 s.

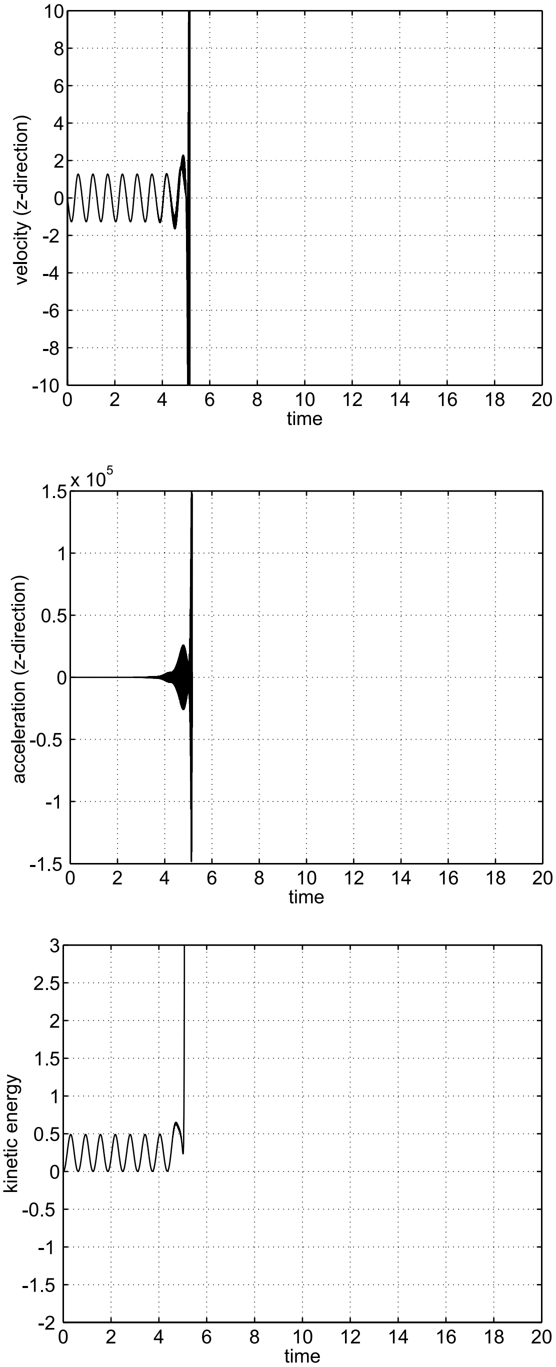

Fig. 8. The compound pendulum; results using the trapezoidal rule; $\Delta t = 0.005$ s.

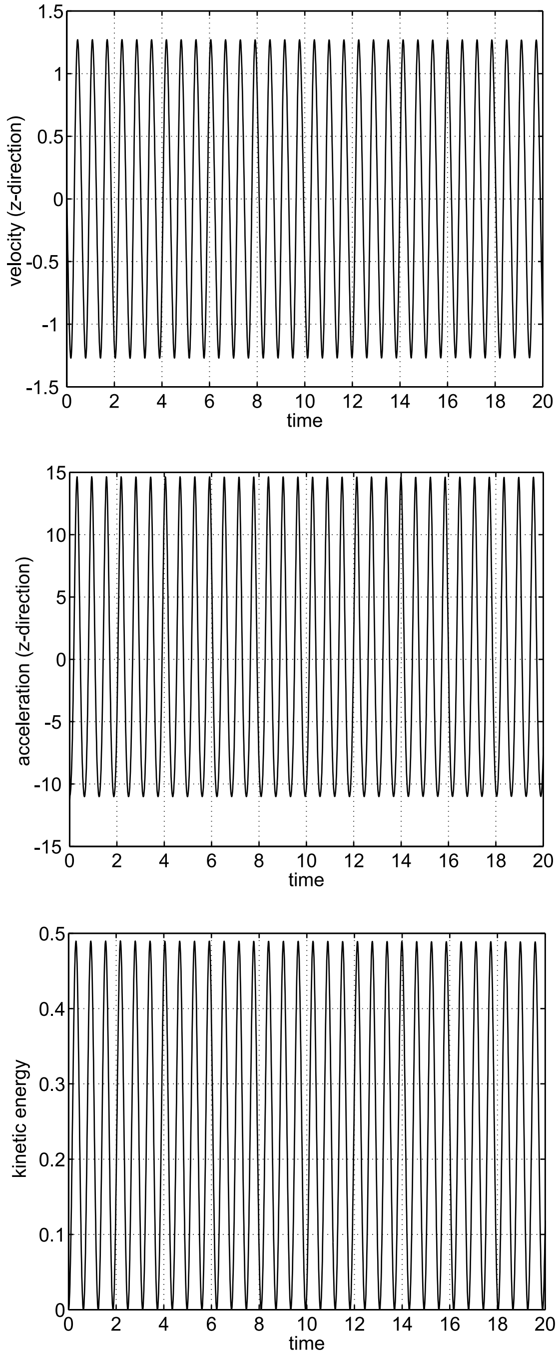

Fig. 9. The compound pendulum; results using the composite scheme; $\Delta t = 0.01$ s.

Forty four-node elements are used to model the pendulum, with 20 elements along the length and 2 in the thickness direction.

The problem is first solved using the trapezoidal rule with a time step $\Delta t = 0.005$ s. This time step size should be small enough to capture the evolution of response accurately. Fig. 8 shows the calculated velocity and acceleration in the z-direction at the tip of the bar, along with the kinetic energy of the system. The trapezoidal rule performs well for a certain length of time after

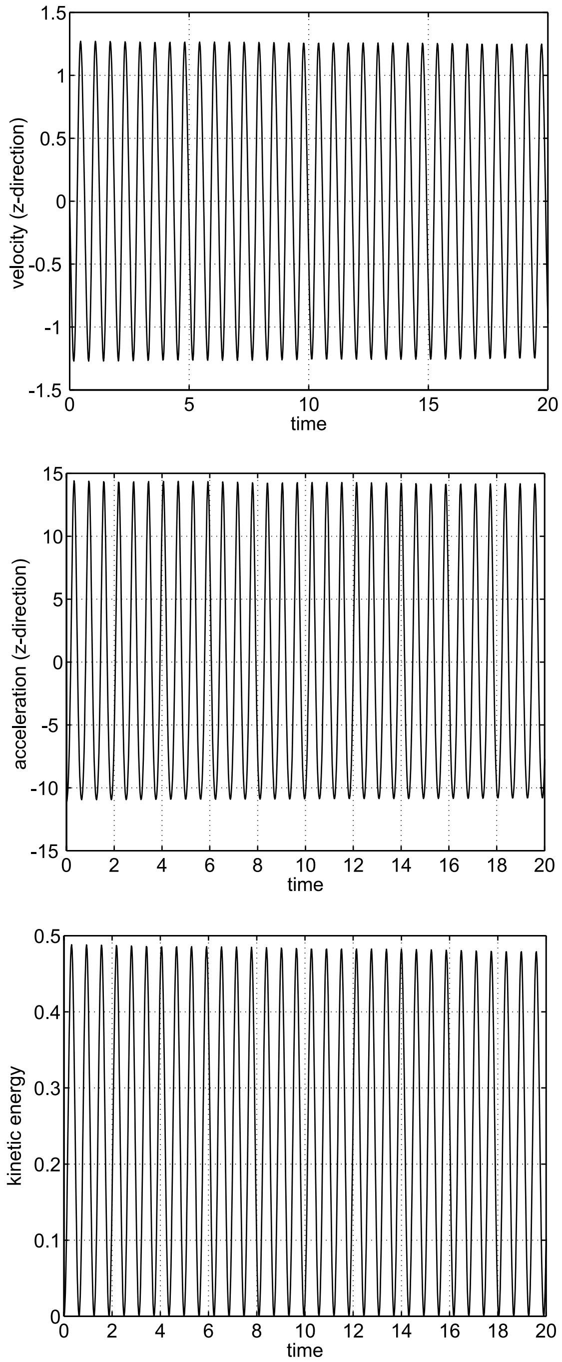

Fig. 10. The compound pendulum; results using the composite scheme; $\Delta t = 0.02$ s.

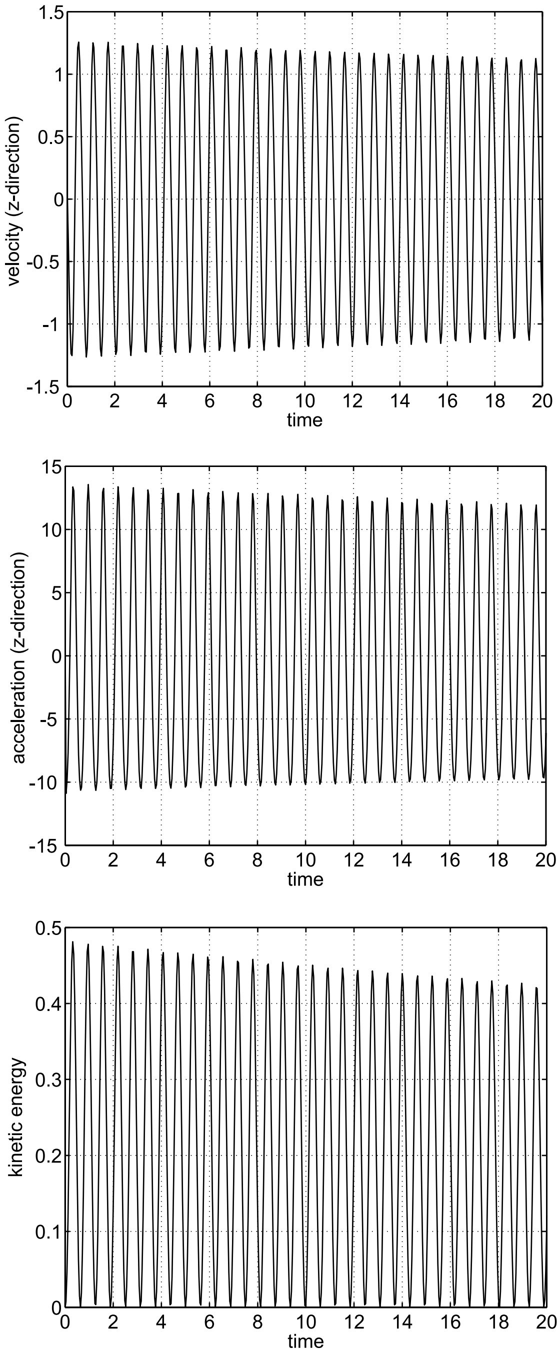

Fig. 11. The compound pendulum; results using the composite scheme; $\Delta t = 0.04$ s.

which the predicted velocity and acceleration response deteriorates noticeably, eventually resulting in very large velocity and acceleration.

The problem is next solved using the composite scheme with $\Delta t = 0.01$ s, which requires about the same solution effort as using the trapezoidal rule with $\Delta t = 0.005$ s. The composite scheme performs well, giving a good velocity and acceleration response, as seen in Fig. 9 which also shows the evolution of kinetic energy of the bar. Although the response for only the first

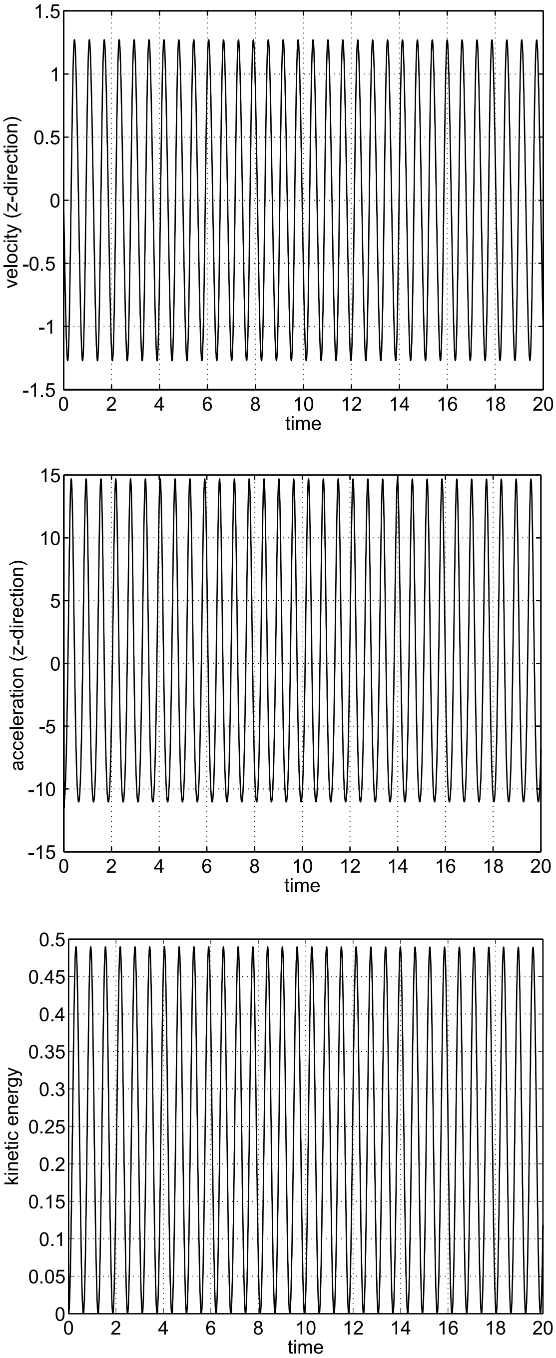

Fig. 12. The compound pendulum; results using the Wilson θ-method; $\Delta t = 0.005$ s.

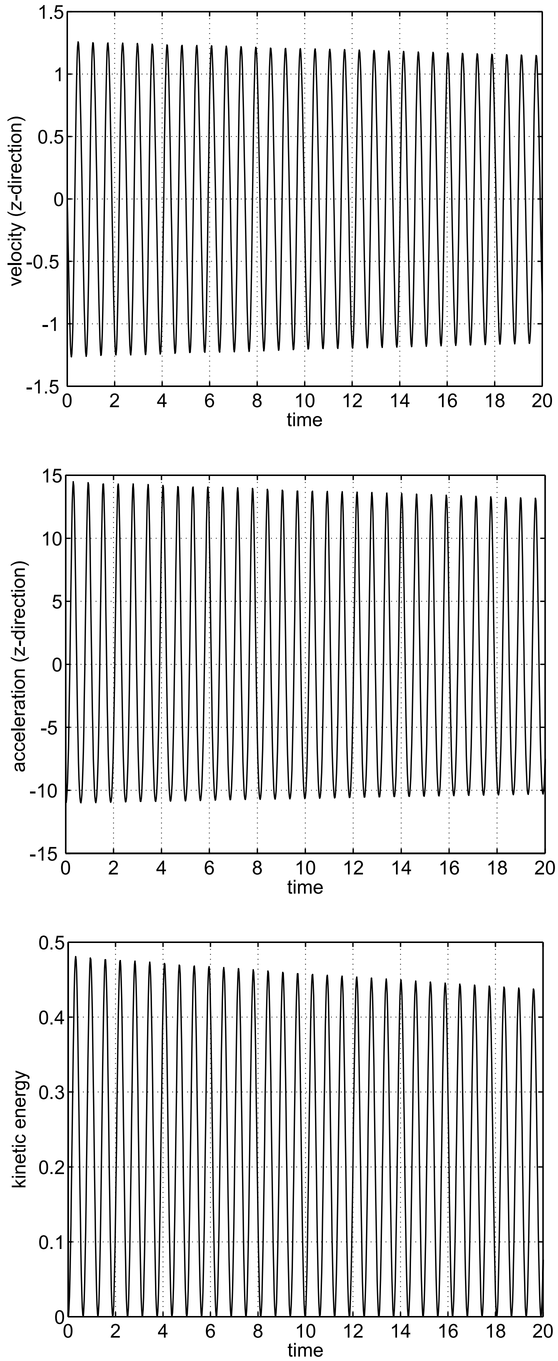

Fig. 13. The compound pendulum; results using the Wilson θ-method; $\Delta t = 0.02$ s.

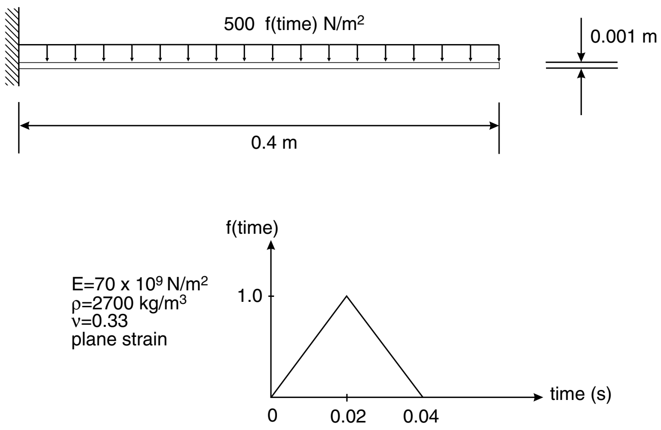

Fig. 14. Cantilever beam modeled using nine-node elements.

20 s is shown, the problem was actually run for a total time of 150 s, and the solution was observed to stay stable and accurate. Also, in our experience, the algorithm remains stable if a larger time step is used, introducing, however, greater numerical damping resulting in reduced accuracy of the solution. This loss of accuracy due to the increase in numerical damping is illustrated in Fig. 10, which shows the results obtained using the composite scheme with $\Delta t = 0.02$ s, and more so in Fig. 11 which shows the results obtained using the composite scheme with $\Delta t = 0.04$ s.

Next we solve the problem using the Wilson θ-method with $\Delta t = 0.005$ s. Fig. 12 shows the solution obtained which is very accurate. Fig. 13 shows the solution calculated using the Wilson θ-method with $\Delta t = 0.02$ s and this figure shows that the solution accuracy is similar to when using the composite scheme with $\Delta t = 0.04$ s. Since the composite scheme uses two solutions per time step, the solution effort is about the same in these two cases. However, the use of a larger time step, e.g., $\Delta t = 0.03$ s, with the Wilson θ-method resulted in a non-positive definite effective stiffness matrix after only a few steps (an instability we also encountered in the previous example).

5.3. Cantilever beam

The response solved for in the two previous test problems involved large rigid body motions over long time intervals. Here we consider the cantilever beam shown in Fig. 14, modeled with a 400×1 mesh of nine-node elements and subjected to pressure loading. The beam is supported to prevent rigid body motion but undergoes large displacements.

Figs. 15–17 show the calculated response of the beam at its tip using the trapezoidal rule, the composite scheme and the Wilson θ-method. As in the solution of the previous problems considered, the solution provided by the trapezoidal rule is not acceptable, whereas the composite scheme and the Wilson θ-method perform well.

6. Conclusions

We focused on a composite single step direct time integration scheme. The procedure uses two sub-steps per time step Δt: in the first sub-step the usual trapezoidal rule is used and in the second sub-step a three-point backward difference approximation is used. There are no special calculations to start the time integration. The composite scheme is available in the ADINA program, in particular for the solution of fluid flow structural interaction problems, where first- and second-order system equations are fully coupled.

In this paper we presented the scheme for nonlinear dynamic analysis of structures and demonstrated its performance relative to the trapezoidal rule and the Wilson θ-method—representative of other widely used time integration methods—by solving three test problems that are useful to test time integration methods in nonlinear dynamics. For a given time step size the composite scheme is about twice as expensive computationally as the usual trapezoidal rule and the Wilson θ-method, and hence we are primarily interested in the scheme when the other techniques are not stable. The numerical examples solved show the algorithm to remain stable for large time step sizes; but when the time step is too large the numerical damping can be appreciable.

When comparing the performance of the methods, the composite scheme can be significantly more effective than the trapezoidal rule when large deformations over long time ranges need be calculated. The Wilson θ-method also provides quite effective solutions for the elastic analysis problems considered herein. However, for inelastic and contact problems, integration methods that establish equilibrium at the actual physical times, and use these equilibrium states to march forward are preferable.

The basic approach used in the construction of the composite scheme is to use two second-order accurate (and in linear analysis, unconditionally stable) schemes, one of which introduces a small amount of numerical

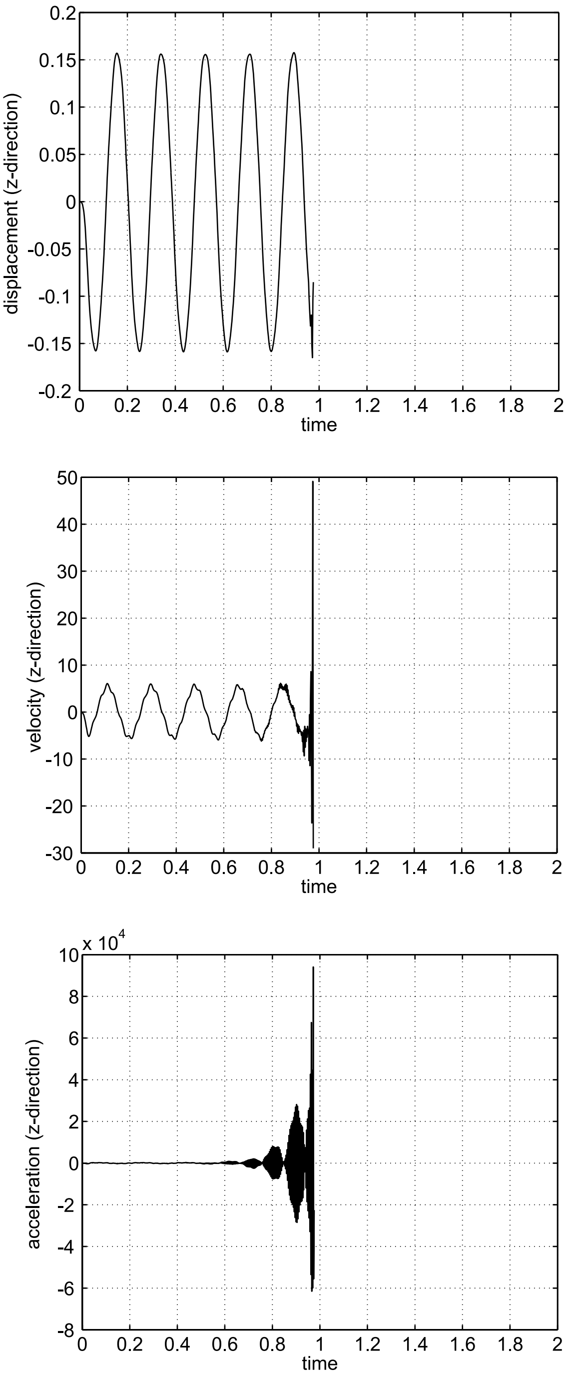

Fig. 15. Cantilever beam; results using the trapezoidal rule; $\Delta t = 0.002$ s.

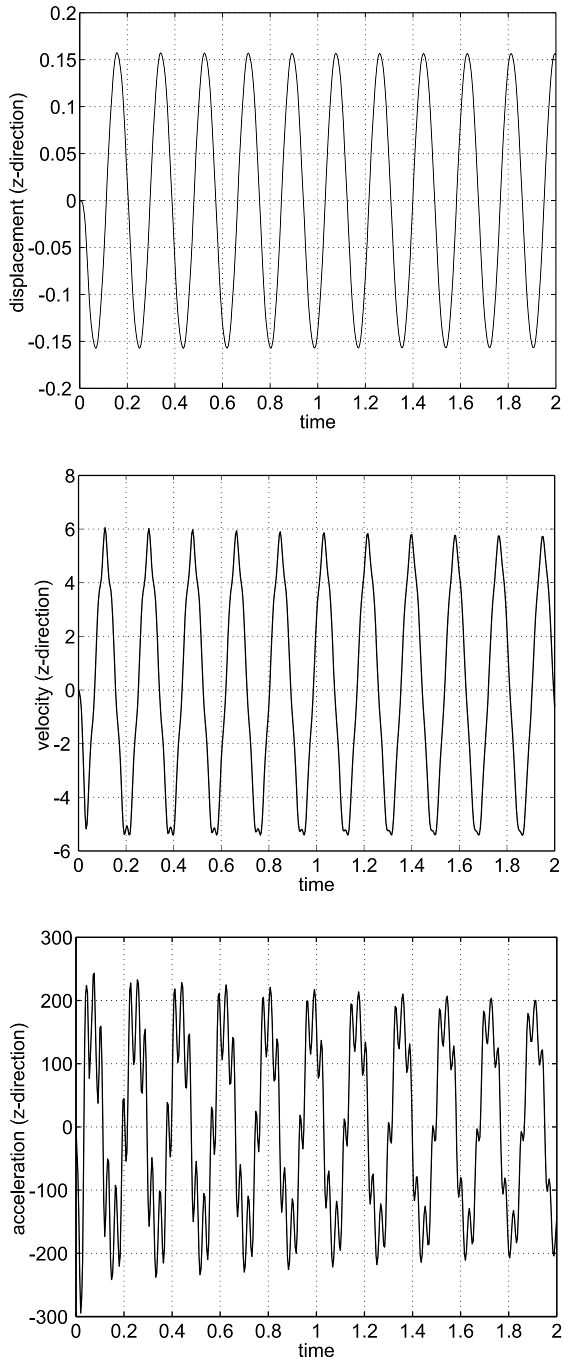

Fig. 16. Cantilever beam; results using the composite scheme; $\Delta t = 0.004$ s.

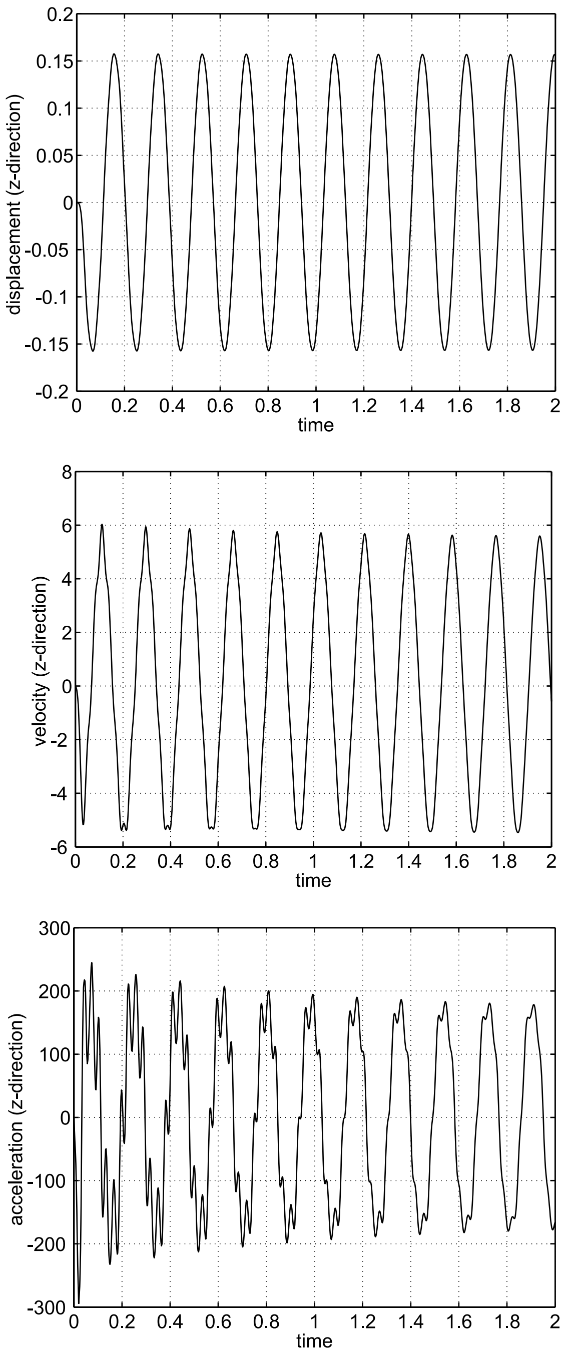

Fig. 17. Cantilever beam; results using the Wilson θ-method; $\Delta t = 0.002$ s.

damping. This approach is quite different from the approach of constraining the energy and momenta of the finite element system. Of course, a small amount of damping can also be introduced in other ways, but the basic aim then needs to be to preserve stability *and* second-order accuracy.

The composite scheme is attractive because only the usual symmetric stiffness, mass and damping matrices are used, and no additional unknown variables (i.e., Lagrange multipliers) need to be solved for. Indeed, the implementation is as straightforward as the implementation of the trapezoidal rule. Therefore the composite scheme is of value in analyses where the trapezoidal rule and other techniques do not give sufficiently accurate solutions.

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