TRANSLATION OF DELIGNE *ÉQUATIONS DIFFÉRENTIELLES À POINTS SINGULIERS RÉGULIERS*

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Abstract. We translate §I.3 and §I.4 of [Del70].

3. Translation in terms of first order partial differential equations

3.1. Let X be a complex analytic variety. If $\mathcal{V} = \underline{V_0}$ is the constant vector bundle on X, it carries a canonical connection ${}_0\nabla$. Then any connection ∇ on $\mathcal{V}$ is of the form

$$\nabla = {}_0\nabla + \Gamma$$

where $\gamma \in \Omega^1_X \otimes End(\mathcal{V})$. In other words, given a section v of $\mathcal{V}$ viewed as a function $X \to V_0$,

$$\nabla v = dv + \Gamma \cdot v.$$

Once we choose a basis (e_α) of V_0, Γ represents a matrix (ω^α_β). Then

$$(\nabla v)^\alpha = dv^\alpha + \sum_\beta \omega^\alpha_\beta v^\beta. \tag{3.1}$$

Now let $\mathcal{V}$ be *any* holomorphic vector bundle on X. Given a trivialization $e\colon \underline{\mathbb{C}}^n \simeq \mathcal{V}$ of $\mathcal{V}$, by the above we can view a connection as a $n \times n$ matrix ω_e of differential forms on X. If $f\colon \underline{\mathbb{C}}^n \simeq \mathcal{V}$ is another trivialization, then they differ by a coordinate change $A = e^{-1}f \in \mathrm{GL}_n(\mathcal{O}_X)$, and a computation shows

$$\omega_f = A^{-1}dA + A^{-1}\omega_e A.$$

If in addition we have a system of local coordinates (x^i) on X, they define a basis dx^i of Ω^1_x, and we can write

$$\omega^\alpha_\beta = \sum_i \Gamma^\alpha_{\beta i} dx^i,$$

where $\Gamma^\alpha_{\beta i}$ are the <u>coefficients of the connection</u>. Then (3.1) can be re-written as

$$(\nabla_i v)^\alpha = \partial_i v^\alpha + \sum_\beta \Gamma \beta i^\alpha v^\beta.$$

Now a section v is horizontal, i.e., $\nabla v = 0$ if and only if

$$\partial_i v^\alpha = -\sum_\beta \Gamma^\alpha_{\beta i} v^\beta. \tag{3.2}$$

3.2. Now using Einstein notation,

$$\begin{aligned}
\nabla^2 v &= \nabla((dv^\alpha + \omega^\alpha_\beta v^\beta)e_\alpha) \\
&= d(dv^\alpha + \omega^\alpha_\beta v^\beta)e_\alpha - (dv^\alpha + \omega^\alpha_\beta v^\beta) \wedge \omega^\gamma_\alpha \cdot e_\gamma \\
&= (d\omega^\gamma_\beta - \omega^\alpha_\beta \wedge \omega^\gamma_\alpha)v^\beta e_\gamma.
\end{aligned}$$

Thus the Riemann curvature tensor is

$$R^\alpha_\beta = d\omega^\alpha_\beta + \sum_\gamma \omega^\alpha_\gamma \wedge \omega^\gamma_\beta, \tag{3.3}$$

which can equivalently be written as

$$R = d\omega + \omega \wedge \omega.$$

Now (3.3) gives:

$$R^\alpha_{\beta,i,j} = (\partial_i \Gamma^\alpha_{\beta j} - \partial_j \Gamma^\alpha_{\beta i}) + (\Gamma^\alpha_{\gamma,i}\Gamma^\gamma_{\beta,j} - \Gamma^\gamma_{\gamma,j}\Gamma^\gamma_{\beta,i})$$
$$R^\alpha_\beta = \sum_{i<j} R^\alpha_{\beta,i,j} dx^i \wedge dx^j.$$

The condition $R^\alpha_{\beta,i,j} = 0$ is equivalent to (3.2) being *integrable* in the classical sense.

4. Differential equations of order n

4.1. Solving the differential equation

$$\frac{d^n}{dx^n} y = \sum_{i=1}^n a_i(x) \frac{d^{n-i}}{dx^{n-i}} y$$

is equivalent to solving the n equations

$$\frac{d}{dx} y_i = y_{i+1} (1 \leq i < n)$$
$$\frac{d}{dx} y_n = \sum_{i=1}^n a_i(x) y_{n+1-i}.$$

By §3, this is equivalent to a finding the horizontal sections in a rank n vector bundle with connection.

4.2. Let X be a complex analytic variety of dimension 1, and let X_n be the n-th infinitesimal neighborhood of the diagonal of $X \times X$, and let p_1 and p_2 be the projections of X_n to X. Let $\pi_{k,\ell}$ denote the injection $X_\ell \hookrightarrow X_k$ for $\ell \leq k$.

Letting $\mathcal{I}$ be the sheaf of ideals defining $\Delta X \subset X \times X$, we have canonical isomorphisms

$$\mathcal{I}/\mathcal{I}^2 \simeq \Omega^1_X$$
$$\mathcal{I}^n/\mathcal{I}^{n+1} \simeq (\Omega^1_X)^{\otimes n}.$$

For an invertible sheaf $\mathcal{L}$ on X, let $P^n(\mathcal{L})$ be the vector bundle of jets of order n,

$$P^n(\mathcal{L}) := p_{1*} p_2^* \mathcal{L}.$$

The $\mathcal{I}$-adic filtration on $p_2^*\mathcal{L}$ defines a filtration on $P^n(\mathcal{L})$ given by

$$\mathbf{gr} P^n(\mathcal{L}) = \mathbf{gr} P^n(\mathcal{O}) \otimes \mathcal{L}$$
$$\mathbf{gr}^i P^n(\mathcal{L}) = \Omega^{\otimes i} \otimes \mathcal{L} (0 \leq i \leq n). \tag{4.1}$$

Recall that we inductively defined a differential operator of degree $\leq n$ to be a $\mathbb{C}$-linear homomorphism of $\mathcal{O}_X$-modules $A \colon \mathcal{M} \to \mathcal{N}$ such that:

- when $n = 0$: A is $\mathcal{O}_X$-linar: and
- when $n = m + 1$: for all local sections f of $\mathcal{O}$, the commutator $[A, f]$ is of order $\leq m$.

For all local sections s of $\mathcal{L}$, the pullback p_2^* defines a local section $D^n(s)$ of $P^n(\mathcal{L})$. The $\mathbb{C}$-linear morphism $D^n \colon \mathcal{L} \to P^n(\mathcal{L})$ is the universal differential operator of degree $\leq n$.

Definition 4.3. (1) A homogeneous linear differential equation of order n on $\mathcal{L}$ is a $\mathcal{O}_X$-linear section of the inclusion $\Omega^{\otimes n} \otimes \mathcal{L} \hookrightarrow P^n(\mathcal{L})$.
(2) A local section s of $\mathcal{L}$ is a *solution* of a differential equation E if $E^n(s) = 0$.

4.4. Let $\mathcal{L} = \mathcal{O}_X$ and let x be a local coordinate on X. A choice of x identifies $P^k(\mathcal{O})$ with $\mathcal{O}^{[0,k]}$, and the homomorphism D^k becomes

$$D^k \colon \mathcal{O} \to P^k(\mathcal{O}) \simeq \mathcal{O}^{[0,k]}$$
$$f \mapsto (\partial_x^i f)_{0 \le i \le k}.$$

A choice of x identifies Ω^1 with $\mathcal{O}$, so a differential equation of order n is a morphism $E \in \mathrm{Hom}(\mathcal{O}^{[0,n]}, \mathcal{O})$, and so has coordinates $(b_i)_{0 \le i \le n}$ where $b_n = 1$. The solutions to E are exactly holomorphic functions f such that

$$\sum_{i=0}^{n} b_i(x) \partial_x^i f = 0 (b_n = 1).$$

Cauchy's theorem on the existence and uniqueness of solutions to differential equations can be re-phrased as:

Theorem 4.5 (Cauchy)**.** *Let X and $\mathcal{L}$ be as in 4.2 and let E be a differential equation of order n on $\mathcal{L}$. Then*

(1) The subsheaf of $\mathcal{L}$ of solutions to E is a rank n local system $\mathcal{L}^E$ on X; and
(2) the canonical morphism $D^{n-1} \colon \mathcal{L}^E \to P^{n-1}(\mathcal{L})$ induces an isomorphism

$$\mathcal{O} \otimes_{\mathbb{C}} \mathcal{L}^E \simeq P^{n-1}(\mathcal{L}).$$

The result (2) and 2.17 shows that E defines a canonical connection on $P^{n-1}(\mathcal{L})$, and horizontal sections are the images under D of the solutions to E.

4.6. For a differential equation E on $\mathcal{L}$, we thus associated:

(1) a holomorphic vector bundle $\mathcal{V} = P^{n-1}(\mathcal{L})$ with a (integrable) connection; and
(2) a surjective homomorphism $\lambda \colon \mathcal{V} \to \mathcal{L}$, i.e., the $i = 0$ case of (4.1).

Moreover, the solutions to E are images under λ of horizontal sections of $\mathcal{V}$. This is simply another way of phrasing the equivalence in (4.1).

4.7. Let $\mathcal{V}$ be a vector bundle of rank n with connection ∇. Let v be a local section of $\mathcal{V}$, and w a vector field[1] on X which does not die on any point. We will say that v is cyclic if the local sections $(\nabla_w)^i v$ of $\mathcal{V}$ $(0 \le i \le n)$ form a basis. The condition does not depend on the choice of w, and if f is an invertible holomorphic function, then v is cyclic if and only if fv is cyclic. In fact one can show that $(\nabla_{gw})^i fv$ is in the sub-module of $\mathcal{V}$ generated by $(\nabla_w)^j(v)$ $(0 \le j \le i)$.

If $\mathcal{L}$ is an invertible module, we will say a section v of $\mathcal{V} \otimes \mathcal{L}$ is cyclic if for all local isomorphisms of $\mathcal{L}$ and $\mathcal{O}$, the corresponding section of $\mathcal{V}$ is cyclic. This in particular applies to any section v of $Hom(\mathcal{V}, \mathcal{L}) = \mathcal{V}^\vee \otimes \mathcal{L}$.

Lemma 4.8. *With the notation and hypothesis from 4.6, λ is a cyclic section of $Hom(\mathcal{V}, \mathcal{L})$.*

The problem is local on X, so we reduce to the case $\mathcal{L} = \mathcal{O}$ and there exists a local coordinate x. A section (f^i) of $P^{n-1}(\mathcal{O}) \simeq \mathcal{O}^{[0,n-1]}$ is horizontal if and only if it satisfies

$$\partial_x f^i = f^{i+1} (0 \le i \le n-2)$$
$$\partial_x f^{n-1} = -\sum_{i=0}^{n-1} b_i f^i.$$

[1] A section of the tangent bundle of X.

This provides us with a connection

$$\nabla = d + \begin{pmatrix} 0 & -1 & & & \\ & 0 & -1 & & \\ & & \ddots & & \\ & & & 0 & -1 \\ b^0 & b^1 & \cdots & b^{n-2} & b^{n-1} \end{pmatrix} dx.$$

In our coordinate system $\lambda = e^0$, and we calculate

$$\nabla_x^i \lambda = e^i (0 \leq i \leq n-1),$$

which proves Lemma 4.8.

Proposition 4.9. *The construction in 4.6 gives an equivalence of the following categories*[2] *where we take the morphsims to be isomorphisms:*

- *the category of invertible sheaves on X, with a differential equation of order n; and*
- *the category of a vector bundle $\mathcal{V}$ of rank n equipped with a connection and an invertible sheaf $\mathcal{L}$, together with a cyclic homomorphism $\lambda \colon \mathcal{V} \to \mathcal{L}$.*

We construct a quasi-inverse to the functor of 4.6. Let $\mathcal{V}$ be a vector bundle with connection and let λ be a homomorphism from $\mathcal{V}$ to an invertible sheaf $\mathcal{L}$. Let V be the local system of horizontal sections of $\mathcal{V}$. For any $\mathcal{O}$-module $\mathcal{M}$, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{V}, \mathcal{M}) \simeq \mathrm{Hom}_{\mathbb{C}}(V, \mathcal{M}).$$

In particular, we define a homomorphism $\gamma^k \colon \mathcal{V} \to P^k(\mathcal{L})$ by setting, for a horizontal section v of $\mathcal{V}$,

$$\gamma^k(v) := D^k(\lambda(v)). \tag{4.2}$$

Lemma. *The homomorphism λ is cyclic if and only if γ^{n-1} is an isomorphism.*

The question is local on X, so let $\mathcal{L} = \mathcal{O}$ and let x be a local coordinate. Then in the notation of 4.4, we have $\partial_x^i \lambda = \nabla_x^i \lambda$ for $0 \leq i \leq k$. For $k = n-1$, they form a basis of $\mathrm{Hom}(\mathcal{V}, \mathcal{O})$ if and only if γ^{n-1} is an isomorphism.

For $k \geq \ell$, the diagram

$$\begin{array}{ccc} & \mathcal{V} & \\ {}^{\gamma^k}\swarrow & & \searrow^{\gamma^\ell} \\ P^k(\mathcal{L}) & \xrightarrow{\pi_{\ell,k}} & P^\ell(\mathcal{L}) \end{array} \tag{4.3}$$

is commutative; if λ is cyclic, we deduce from the lemma that $\gamma^n(\mathcal{V})$ is a direct summand of codimension 1 in $P^n(\mathcal{L})$, and $\Omega^{\otimes n} \otimes \mathcal{L} \simeq \ker(\pi_{n-1,n})$. Therefore there exists a unique equation of order n on $\mathcal{L}$

$$E \colon P^n(\mathcal{L}) \to \Omega^{\otimes n} \otimes \mathcal{L}$$

such that $E \circ \gamma^n = 0$. By (4.2), if v is a horizontal section of $\mathcal{V}$, then $ED^n \lambda v = E\gamma^n v = 0$, so that λv is a solution of E. Let us equip $P^{n-1}(\mathcal{L})$ with the connection from 4.6 defined by E. If v is a horizontal section of $\mathcal{V}$, then $\gamma^{n-1}(v) = D^{n-1}\lambda v$ where λv is a solution to E, and $\gamma^{n-1} v$ is

[2] actually groupoids

therefore horizontal. We deduce that γ^{n-1} is comptaible with the connections. Now a special case $(k = n-1, \ell = 0)$ of (4.3) shows that the diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\gamma^{n-1}} & P^{n-1}(\mathcal{L}) \\ \lambda\downarrow & & \downarrow\pi_{0,n-1} \\ \mathcal{L} & \xrightarrow{=} & \mathcal{L} \end{array}$$

is commutative. The functor

$$(\mathcal{V}, \mathcal{L}, \lambda) \mapsto (\mathcal{L}, E)$$

is a quasi-inverse to the functor in 4.6.

4.10. Let's summarize the relation between the triple $(\mathcal{V}, \mathcal{L}, \lambda)$ and $(\mathcal{L}, E)$ under the category equivalence in 4.6 and Proposition 4.9. We have a homomorphism $\gamma^k \colon \mathcal{V} \to P^k(\mathcal{L})$, such that:

- for v horizontal, $\gamma^k(v) = D^k \lambda v$.
- we have $\gamma^0 = \lambda$ and $\pi_{\ell,k}\gamma^k = \gamma^\ell$.
- γ^{n-1} is an isomorphism (λ is cyclic).
- $E\gamma^n = 0$.
- λ induces an isomorphism between the local system V of horizontal sections in $\mathcal{V}$ and the local system $\mathcal{L}^E$ of solutions of E.

References

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