LOCAL LANGLANDS SEMINAR: BASICS OF WEIL-DELIGNE REPRESENTATIONS

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Abstract. We will cover the basic definitions involved in the statement for the Local Langlands Correspondence for GL_n in characteristic zero.

Fix a prime number p, a finite extension $F/\mathbb{Q}_p$, and an algebraic closure $\overline{F}$ of F throughout the talk.

1. Smooth representations of $\mathrm{GL}_n(F)$

Let $\mathbf{C}$ be an algebraically closed field of characteristic zero.

First of all, topologize the group $\mathrm{GL}_n(F)$ by making it inherit the product topology from $M_n(F) = F^{n^2}$. We will define a *smooth* representation of $\mathrm{GL}_n(F)$ over $\mathbf{C}$:

Proposition-Definition 1. Let (V, π) be a representation of $\mathrm{GL}_n(F)$ over $\mathbf{C}$, i.e., V is a $\mathbf{C}$-vector space and $\pi\colon \mathrm{GL}_n(F) \to \mathrm{GL}(V)$ is a group homomorphism. The representation (V, π) is *smooth* if one of the following equivalent conditions hold:

(i) For any compact open subgroup $K \subset \mathrm{GL}_n(F)$, let $V^K := \{v \in V : \pi(K)v = v\}$. Then

$$V = \bigcup_K V^K.$$

(ii) For any vector $v \in V$ there is a compact open subgroup $K \subset \mathrm{GL}_n(F)$ such that $\pi(K)v = v$.

(iii) For any vector $v \in V$ the stabilizer of v

$$\mathrm{Stab}(v) := \{g \in \mathrm{GL}_n(F) : \pi(g)v = v\}$$

is open in $\mathrm{GL}_n(F)$.

(iv) The action map $\mathrm{GL}_n(F) \times V \to V$ is continuous, where V is given the discrete topology.

Proof. The equivalence of (i) and (ii) is clear. (ii) clearly implies (iii). Moreover (iii) implies (ii) since $\mathrm{GL}_n(F)$ is a locally compact group (i.e., has an open basis consisting of compact open subgroups). Now note that (iv) is equivalent to: for any $v, w \in V$,

$$(\mathrm{GL}_n(F) \times \{v\}) \cap a^{-1}(w) = \{g \in \mathrm{GL}_n(F) : \pi(g)v = w\}$$

is open in $\mathrm{GL}_n(F)$. Thus (iv) implies (iii) by letting $v = w$. Conversely, clearly the set $(\mathrm{GL}_n(F) \times \{v\}) \cap a^{-1}(w)$ is open if it is empty. If it is non-empty, say it contains $g_0 \in \mathrm{GL}_n(F)$, then

$$(\mathrm{GL}_n(F) \times \{v\}) \cap a^{-1}(w) = g_0 \,\mathrm{Stab}(v)$$

is open in $\mathrm{GL}_n(F)$. □

Definition 1.0.1. A smooth representation (V, π) of $\mathrm{GL}_n(F)$ is *admissible* if for any compact open subgroup $K \subset \mathrm{GL}_n(F)$ the subspace $V^K \subset V$ is finite-dimensional.

2. Weil-Deligne representation

Definition 2.0.1. Let W_F be the Weil group of F. A *Weil-Deligne representation* of W_F on a finite-dimensional L-vector space V is a pair (r, N) where $r\colon W_F \to \mathrm{GL}(V)$ is a continuous semisimple representation, and $N\colon V \to V$ is an endomorphism, such that for all $\sigma \in W_F$,

$$r(\sigma)Nr(\sigma)^{-1} = q_F^{-v_F(\sigma)}N.$$

A Weil-Deligne representation is *bounded* if for all $\sigma \in W_F$ the operator $r(\sigma)$ is bounded, i.e., the determinant is in $\mathfrak{o}_L^\times$ and the characteristic polynomial is in $\mathfrak{o}_L[X]$ (equivalently, all of the eigenvalues are in $\mathfrak{o}_{\overline{L}}^\times$).

2.1. Grothendieck's monodromy theorem.

Let $\ell \neq p$ be two primes. We will consider ℓ-adic representations of G_F, i.e., a representation into a finite-dimensional L-vector space, where $L/\mathbb{Q}_\ell$ is algebraic.

Now recall Grothendieck's monodromy theorem ([Gee22, Prop 2.18], [BH06, Thm 32.5], [ST68]):

Proposition 2.1.1. *Suppose $\ell \neq p$, let $F/\mathbb{Q}_\ell$ be a finite extension, let $L/\mathbb{Q}_p$ be an algebraic extension, and let V be a finite-dimensional L-vector space. Fix:*

- *φ, a lift of Fr_F; and*
- *a compatible system $(\zeta_m)_{(m,\ell)=1}$ of primitive roots of unity.*

Then for any continuous representation $\rho\colon G_F \to \mathrm{GL}(V)$ there is a finite extension F'/F and a uniquely determined nilpotent endomorphism $N\colon V \to V$ such that for all $\sigma \in I_{F'}$,

$$\rho(\sigma) = \exp(Nt_{\zeta,p}(\sigma)),$$

where for all $\sigma \in W_F$, we have $\rho(\sigma)N\rho(\sigma)^{-1} = q_F^{-v_F(\sigma)}N$, where t_ζ is an isomorphism $I_F/P_F \simeq \prod_{p\neq\ell}\mathbb{Z}_p$.

Moreover, there is an equivalence of categories:

$$\left\{\begin{matrix}\textit{continuous representations of } G_F \textit{ on}\\ \textit{finite-dimensional } L\textit{-vector spaces}\end{matrix}\right\} \simeq \left\{\begin{matrix}\textit{bounded Weil-Deligne representations}\\ \textit{on finite dimensional } L\textit{-vector spaces}\end{matrix}\right\}$$
$$\rho \mapsto (V, r, N),$$

where $r(\tau) := \rho(\tau)\exp(-t_{\zeta,p}(\varphi^{-v_F(\tau)}\tau)N)$.

Then the *Local Langlands Correspondence for* GL_n, proved by [HT01], [Hen00], and [Sch13] is a bijection:[1]

(2.1.2)

$$\mathrm{Irr}\,\mathrm{GL}_n(F) := \left\{\begin{matrix}\text{irreducible smooth represen-}\\ \text{tations of } \mathrm{GL}_n(F) \text{ over } \mathbb{C}\end{matrix}\right\} \simeq \Phi_n(W_F) := \left\{\begin{matrix}n\text{-dimensional representations } \rho \text{ of}\\ WD_F \text{ such that } \rho|_{W_F} \text{ is semisimple}\\ \text{and } \rho|_{\mathbb{C}} \text{ is algebraic}\end{matrix}\right\}$$

satisfying compatibility with parabolic induction, central characters, etc. Denote the correspondence as $\pi \mapsto \rho_\pi$.

Example 2.1.3. Consider the vector space V of smooth functions on $\mathbf{P}^1_F$. Then since $\mathrm{GL}_2(F)$ acts on $\mathbf{P}^1_F$, it becomes a smooth representation of $\mathrm{GL}_2(F)$. It has a subspace $\mathbf{1}$ consisting of constant functions, and the quotient $V/\mathbf{1}$ is irreducible, called the *Steinberg representation.* The L-parameter

[1] For a proof when $n = 2$, see [BH06].

is given by:

$$\begin{aligned} W_F \rtimes \mathbb{C} &\to \mathrm{GL}_2(\mathbb{C}) \\ (w,0) &\mapsto \begin{pmatrix} \|w\|^{1/2} & \\ & \|w\|^{-1/2} \end{pmatrix} \\ (1,x) &\mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}. \end{aligned}$$

The Local Langlands Correspondence translates properties of representations of $\mathrm{GL}_n(F)$ to properties of representations of WD_F in the following way [Bor79]:

Definition 2.1.4. Let π be a smooth irreducible representation of $\mathrm{GL}_n(F)$. Then

(1) π is *tempered* if the central character ω_π is unitary, and for any matrix coefficient f of π and any $\epsilon > 0$,

$$\int_{\mathrm{GL}_n(F)/F^\times} |f(g)|^{2+\epsilon} dg < \infty.$$

(2) π is *square-integrable* if the central character ω_π is unitary, and for any matrix coefficient f of π,

$$\int_{\mathrm{GL}_n(F)/F^\times} |f(g)|^{2} dg < \infty.$$

π is *essentially square-integrable* or *discrete series* if there exists a character χ of $F^\times$ such that $\pi \otimes \chi$ is square-integrable.

(3) π is *supercuspidal* if all matrix coefficients of π have compact support modulo $F^\times$.

Remark 2.1.5. Supercuspidal implies square-integrable implies tempered.

Proposition 2.1.6. *Let π be a smooth irreducible representation of $\mathrm{GL}_n(F)$, and let $\rho_\pi \colon WD_F \to \mathrm{GL}_n(\mathbb{C})$ be its Langlands parameter. Then the following hold:*

(1) π is tempered if and only if the image of $\rho_\pi|_{W_F}$ is bounded
(2) π is square-integrable modulo center if and only if ρ_π is indecomposable
(3) π is supercuspidal if and only if $\rho_\pi|_{W_F}$ is irreducible.

2.2. **Satake correspondence.** For unramified representations of $\mathrm{GL}_n(F)$, the correspondence is particularly simple:

Definition 2.2.1. An irreducible smooth representation π of $\mathrm{GL}_n(F)$ is *unramified* if π has a $\mathrm{GL}_n(\mathfrak{o})$-invariant vector.

The *Satake correspondence* is a bijection between subsets of $\mathrm{Irr}\,\mathrm{GL}_n(F)$ and $\Phi_n(W_F)$, compatible with the Local Langlands Correspondence (and much easier!):

$$\left\{\begin{matrix}\text{unramified representations of} \\ \mathrm{GL}_n(F) \text{ over } \mathbb{C}\end{matrix}\right\} \simeq S_n \backslash (\mathbb{C}^\times)^n. \tag{2.2.2}$$

Here, $(z_1, \dots, z_n) \in S_n\backslash(\mathbb{C}^\times)^n$ is viewed as a n-dimensional representations of WD_F by

$$\begin{aligned} WD_F &\to \mathrm{GL}_n(\mathbb{C}) \\ (w,x) &\mapsto \mathrm{diag}(z_1^{\mathrm{val}_F(w)}, \dots, z_n^{\mathrm{val}_F(w)}). \end{aligned}$$

Thus, (2.2.2) can be re-written to resemble the general Local Langlands Correspondence:

$$\left\{\begin{matrix}\text{unramified representations of} \\ \mathrm{GL}_n(F) \text{ over } \mathbb{C}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{representations of } WD_F \text{ triv-} \\ \text{ial on } \mathbb{C} \text{ and the inertia } I_F\end{matrix}\right\}. \tag{2.2.3}$$

Given a n-tuple $(z_1, \dots, z_n) \in S_n\backslash(\mathbb{C}^\times)^n$, let

$$B_n = \{(x_{ij})_{i,j=1}^n \in \mathrm{GL}_n(F) : x_{ij} = 0 \text{ for } i > j\}$$

be the *Borel subgroup* of $\mathrm{GL}_n(F)$, the subgroup of upper triangular $n\times n$-matrices. Then $(z_1,\dots,z_n)$ defines a character $\chi\colon B_n \to \mathbb{C}^\times$ by:

$$\chi(x_{ij})_{i,j=1}^n := z_1^{\mathrm{val}_F(x_{11})}\cdots z_n^{\mathrm{val}_F(x_{nn})},$$

where $\mathrm{val}_F\colon F^\times \to \mathbb{Z}$ is the *valuation*. Then the *normalized parabolic induction* of χ, denoted $\mathrm{i}_B^G\chi$, is such that $(\mathrm{i}_B^G\chi)^{\mathrm{GL}_n(\mathfrak{o}_F)}$ is 1-dimensional, hence has a unique unramified subquotient. The subquotient is moreover independent of the order of the z_i's, hence realizes the map from the right to left in (2.2.2). To go from the left to the right, we take eigenvalues of certain Hecke operators on the representation, and we denote the correspondence by $\pi_v \mapsto t_{\pi_v}$.

3. L-factors and ϵ-factors

Definition 3.0.1. Let (ρ, V) be a finite-dimensional semisimple representation of W_F. Then let

$$L(\rho,s) := \det(1-\rho(\mathrm{Fr})q^{-s}; V^{I_F})^{-1}.$$

To define the local constant, we must discuss the functional equation for the group $\mathrm{GL}_1(F) = F^\times$: The local constant ($\epsilon$-factor) is more convoluted. For $\Phi \in C_c^\infty(F)$, define the *Fourier transform* $\hat{\Phi}$ of Φ to be:

$$\hat{\Phi}(x) = \int_F \Phi(y)\psi(xy)d\mu(y).$$

Now set

$$z_m := \int_{\varpi^m \mathfrak{o}_F^\times} \Phi(x)\chi(x)d\mu^\times(x)$$

and let

$$\zeta(\Phi,\chi,s) := \sum_{m\in\mathbb{Z}} z_m q^{-ms}$$

Then there is a unique rational function $\gamma(\chi,s,\psi)\in\mathbb{C}(q^{-s})$ such that

$$\zeta(\hat{\Phi},\chi^{-1},1-s) = \gamma(\chi,s,\psi)\zeta(\Phi,\chi,s).$$

Now let

$$\epsilon(\chi,s,\psi) := \gamma(\chi,s,\psi)\frac{L(\chi,s)}{L(\chi^{-1},1-s)}.$$

Fix a non-trivial continuous character $\psi\colon F\to\mathbb{C}^\times$, and for any finite extension E/F let $\psi_E = \psi\circ\mathrm{tr}_{E/F}$.

Theorem 3.0.2. *There is a unique family of $\epsilon(\rho,s,\psi_E)\in\mathbb{C}[q^{\pm s}]^\times = \mathbb{C}^\times(q^s)^{\mathbb{Z}}$ for all finite extensions E/F inside $\overline{F}$ and semisimple representation ρ of W_E such that:*

(1) If χ is a character of $E^\times$ then

$$\epsilon(\chi\circ\mathrm{Art}_E,s,\psi_E) = \epsilon(\chi,s,\psi_E),$$

where $\mathrm{Art}_E\colon W_E^{ab}\to E^\times$ is the Artin reciprocity map

(2) If ρ_1 and ρ_2 are semisimple then

$$\epsilon(\rho_1\oplus\rho_2,s,\psi_E) = \epsilon(\rho_1,s,\psi_E)\epsilon(\rho_2,s,\psi_E).$$

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