

Quantum Physics II (8.05) Fall 2002

Assignment 10

Nov 13, 2002

Due Nov 20, 2002

Readings

As given on Assignment 9. Note that the supplementary notes on solving the radial wave equation for the Coulomb potential using operator methods are now posted on the web page. Problems 1 and 2 refer to these notes.

Problem Set 10

1. Analyzing a Pair of Potentials Related by Supersymmetry (12 points)

Consider a particle moving in one-dimension confined to the interval $-\pi/2 < x < \pi/2$. That is, $\psi(-\pi/2) = \psi(\pi/2) = 0$. You will investigate the energy eigenvalues and eigenstates for a particle moving within this interval in the presence of two different potentials.

Throughout this problem, assume that units have been scaled in such away that $\hbar$ and $2m$ have vanished from the Schrödinger equation.

Consider the superpotential

$$\mathcal{W}(x) = \tan x . \tag{1}$$

- (a) Use equation (6) from the supplementary notes to derive two Hamiltonians $H^{(1)}$ and $H^{(2)}$ from this superpotential, whose eigenvalues and eigenstates are related as described in the notes.
- (b) One of these two Hamiltonians should be very familiar, from 8.04. Use your 8.04 experience to write down its eigenvalues and eigenstates.
- (c) The other Hamiltonian has quite an unfamiliar looking potential. Use the general results proved in the supplementary notes, and your answer to part (b), to determine all the energy eigenvalues and eigenstates of the unfamiliar looking Hamiltonian.
- (d) Draw a sketch of the two potentials, side by side. Label the lowest few energy eigenvalues of each. (Lets say that by “few” I mean 3 for the Hamiltonian with a zero eigenvalue, and 2 for the other.) Sketch the eigenstates for each Hamiltonian associated with these eigenvalues.

2. Analyzing a Set of Potentials Related by Supersymmetry (18 points)

If you are not familiar with the hyperbolic functions, eg with relations like $\cosh^2 x - \sinh^2 x = 1$ and $\tanh^2 x = 1 - \operatorname{sech}^2 x$, I would suggest that you work on this problem collaboratively with one of your fellow students who is.

As in the previous problem, we will be studying one dimensional quantum mechanics problems in which we have scaled x and energy to make everything in the Schrödinger equation dimensionless, and to set $\hbar = 2m = 1$.

The Pöschl-Teller Hamiltonians are defined to be

$$H_n^{\text{PT}} = -\frac{d^2}{dx^2} - n(n+1)\operatorname{sech}^2 x \quad (2)$$

where $n = 0, 1, 2, 3, \dots$ can be any nonnegative integer.

Your task in this problem is to find energies and eigenstates for all of the bound states for all of these infinitely many Hamiltonians.

- (a) Consider the infinite set of superpotentials

$$\mathcal{W}_n = n \tanh x \quad (3)$$

where $n = 0, 1, 2, 3, \dots$. Using the methods from the supplementary notes, construct $\mathcal{A}_n$, $\mathcal{A}_n^\dagger$, $H_n^{(1)}$ and $H_n^{(2)}$ from the superpotentials (3). Show that

$$H_n^{(1)} = H_{n+1}^{(2)} - (2n+1) \quad (4)$$

Relate $H_n^{(1)}$ and $H_n^{(2)}$ to the Pöschl-Teller Hamiltonians.

- (b) Show that $H_n^{(1)}$ has a ground state with energy eigenvalue $E = 0$. Find the ground state wave function for each n . [You need not normalize the wave functions. That is, find the form of the wave functions, up to an overall normalization constant.]
- (c) You have shown that $H_1^{(1)}$ has a bound state with energy $E = 0$. Use this fact, the relation (4), and results proved in general in the supplementary notes to conclude that $H_2^{(1)}$ must have a bound state with a particular positive energy. Find this energy eigenvalue. (You have now found two bound states of $H_2^{(1)}$: the ground state that you found in part (b) and the excited state that you have found here.)
- (d) Suppose that I tell you that $H_1^{(1)}$ has *only one bound state*, namely the ground state that you found in part (b). Assuming this to be the case, show that $H_n^{(1)}$ has precisely n bound states. Find the energy eigenvalues for all n bound states of $H_n^{(1)}$. Write an expression for the wave function associated with each eigenstate in the form of differential operator(s) acting on wave functions you have found explicitly in part (b).
- (e) Consider the Hamiltonian $H_0^{(1)}$. What is the lowest energy eigenvalue of this Hamiltonian? Use this fact to demonstrate that $H_1^{(1)}$ cannot have a second bound state with energy eigenvalue between 0 and 1. [This proves that $H_1^{(1)}$ has only one bound state, its ground state with $E = 0$, as I had you assume above.]

- (f) What are all the bound state energies of the Pöschl-Teller Hamiltonians H_n^{PT} ?
- (g) **This last part is optional and will not be graded, although a solution will be provided.** The Pöschl-Teller potentials also have unbound “scattering” states with positive energy. [Unlike the harmonic oscillator, and unlike the potentials you investigated in Problem 1, the potential $-n(n+1)\text{sech}^2x$ goes to zero as $x \rightarrow \pm\infty$ and therefore allows for scattering.] You know what the scattering states are for H_0^{PT} : the wave functions are $\exp(ikx)$ with energy k^2 . Use the methods developed earlier in this problem to find the wave function for the scattering state with energy k^2 in the potentials $-n(n+1)\text{sech}^2x$ with $n = 1$ and $n = 2$. Calculate the reflection and transmission coefficients $R_n(k)$ and $T_n(k)$, and show that $R_n = 0$ and $|T_n| = 1$, meaning that these potentials are “reflectionless”. [Do the calculation only for $n = 1, 2$. But, the potentials are in fact reflectionless for all n .]

3. Ground State of the Hydrogen Atom (6 points)

Write out the properly normalized wavefunction of the ground state of the hydrogen atom with all numerical factors included. [You can take this from Problem Set 9 or from a book.]

- (a) Find $\langle r \rangle$, $\langle r^2 \rangle$ and $\langle 1/r \rangle$ for the ground state of the hydrogen atom. Express your answers in terms of the Bohr radius $a_0 = \hbar^2/me^2$.
- (b) The virial theorem says $\langle \frac{p^2}{2m} \rangle = \frac{1}{2} \langle \frac{e^2}{r} \rangle$ for a Coulomb potential. Use the result of part (a) plus the fact that $E = \langle \frac{p^2}{2m} - \frac{e^2}{r} \rangle$ to check the virial theorem for the ground state.
- (c) Calculate the expectation value of p^2/m^2c^2 for the hydrogen ground state. Call your answer α^2 . Evaluate α numerically and show that it is dimensionless and equals approximately 1/137.

This result shows how relativistic is the motion of the electron in hydrogen. Note also that α is a dimensionless parameter measuring the strength of electromagnetic forces, since it involves only e and the fundamental constants $\hbar$ and c .

4. Beta Decay of Tritium (6 points)

An electron is in the ground state of tritium. Tritium is just like hydrogen, except that its nucleus consists of one proton and two neutrons. The tritium nucleus is unstable. After the decay (which can be thought of as instantaneous for our purposes) the nucleus is a ^3He nucleus made of two protons and one neutron. Thus, Z has changed from 1 to 2. [The reduced mass μ has changed by a tiny amount, but you can ignore this.] The statement that the decay is “instantaneous” means that immediately after the decay, the wavefunction of the electron in the atom is unchanged from what it was immediately before the decay. The electron is, however, no longer in an energy eigenstate. Calculate the probability that the electron is in the ground state of ^3He . What is the probability that it is in some state with $\ell = 0$? [No calculation should be required to answer the latter question.]

[Note that the nuclear decay produces an electron and an antineutrino. These exit the system quickly enough that we can neglect them. This is (part of) what I mean when I say that the decay is instantaneous.]

5. Making Contact with the Bohr Atom (6 points)

We have now learned how to treat the hydrogen atom in quantum mechanics. In 8.04, you studied a partially successful attempt by Bohr. He mixed classical concepts (electrons moving along classical circular orbits) and quantum mechanical concepts (quantization of angular momentum) and concluded that only orbits with certain radii are allowed and managed to get the energy spectrum right. Now that we've learned how to do things properly, and have confirmed Bohr's energy spectrum, we should go back and see whether there is some hint of Bohr's allowed radii in the wavefunctions we have found.

- (a) Consider the states which have the maximum angular momentum for a given principal quantum number n . [These have $\ell = n - 1$ and are just the $\nu = 0$ states.] For these states, find the r at which $|u(r)|^2$ has its maximum value. Express your result in terms of n and the Bohr radius. Compare to Bohr's result.
- (b) For the same states as in part (a), it is in fact the case that $\langle r \rangle = a_0 n(n + 1/2)$ and $\langle r^2 \rangle = a_0^2 n^2(n + 1)(n + 1/2)$. Evaluate Δr and comment on the large n limit. [I really should have asked you to evaluate $\langle r \rangle$ and $\langle r^2 \rangle$ yourself. Thank me.]

6. Qualitative Features of Radial Wavefunctions (12 points)

It is often possible to understand many features of quantum systems without solving differential equations. In this problem, you will map out some features of the bound and scattering states in a Yukawa potential. All parts of this problem can be solved by carefully drawing and labelling graphs or pictures.

Suppose a particle moves in a central potential $V(r) = -V_0 \frac{e^{-\alpha r}}{r}$. V_0 and α are constants. The radial Schrödinger equation for this potential can be written

$$\left\{ -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} - g \frac{e^{-\alpha r}}{r} \right\} u_\ell(r) = k^2 u_\ell(r) \quad (5)$$

where g is a new constant (what is it?) and where $k^2 = 2mE/\hbar^2$. $k^2 > 0$ corresponds to a continuum state, describing scattering off the potential, while $k^2 \equiv -\kappa^2 < 0$ corresponds to a bound state.

- (a) Draw the effective potential $V_{eff}(r) = \frac{\ell(\ell+1)}{r^2} - g \frac{e^{-\alpha r}}{r}$ for various choices of ℓ , g and α .
- (b) Show graphically that the parameters g and α can be chosen so that this potential has
 - No bound states
 - Many bound states

(c) Suppose ℓ and the parameters g and α are chosen such that this potential has two and only two bound states. Draw the radial wave functions $u_\ell(r)$ for the two bound states. Be sure to label

- The functional behavior of $u(r)$ near $r = 0$.
- The functional behavior of $u(r)$ near $r = \infty$.
- The regions of r for which the curvature of u (Curvature = u''/u) is positive and negative

Pay attention in your drawing to the changes in the deBroglie wavelength and to the relative magnitude of u for different values of r .

(d) For the same choice of parameters as in part (c), consider a state with energy just slightly greater than zero. Repeat the steps of part (c) above.