Quantum Physics II (8.05) Fall 2002
SAMPLE MIDTERM

Quantum Physics II (8.05) Fall 2000
MIDTERM
Thurs. Nov. 2, 7:30pm
You have 90 minutes.

Answer all problems in the blue books provided. Write YOUR NAME and YOUR SECTION on EACH blue book you use.

There are five problems, totalling 90 points. Do all five. The problems are worth 10, 10, 20, 20 and 30 points. Budget your time accordingly.

None of the problems requires extensive algebra. If you find yourself lost in a calculational thicket, stop and think.

No books, notes or calculators allowed.

Some potentially useful information

- Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

- Position and momentum representations

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar)$$

$$\psi(x) = \langle x|\psi\rangle \ ; \quad \phi(p) = \langle p|\psi\rangle \ ; \quad \langle x|\hat{p}|\psi\rangle = -i\hbar\frac{d}{dx}\psi(x)$$

- Properties of Dirac delta function

$$\delta(\lambda x) = \frac{1}{\lambda}\delta(x) \ ; \qquad \int_{-\infty}^{\infty}\exp(ikx)dx = 2\pi\delta(k)$$

- Ehrenfest Equation. (Equation of motion for Heisenberg picture operator $\hat{A}_H$, assuming that the corresponding operator in the Schrödinger picture is time independent.)

$$i\hbar\frac{d}{dt}\hat{A}_H = [\hat{A}_H, \hat{H}]$$

- Operators for Spin-1/2 particle

$$\hat{S}_i = \frac{\hbar}{2}\sigma_i$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ; \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ ; \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Pauli matrix identities

$$\begin{aligned} \exp(i\theta\sigma_3)\,\sigma_1\,\exp(-i\theta\sigma_3) &= \sigma_1\cos(2\theta) - \sigma_2\sin(2\theta) \\ \exp(i\theta\sigma_3)\,\sigma_2\,\exp(-i\theta\sigma_3) &= \sigma_2\cos(2\theta) + \sigma_1\sin(2\theta) \ . \end{aligned}$$

Similar expressions hold if you cyclically permute the indices 1, 2 and 3.

- Identities valid for any two operators $\hat{A}$ and $\hat{B}$ whose commutator is given by $[\hat{A}, \hat{B}] = C$, *where C is a complex number times the identity operator.*

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + C$$

$$e^{\hat{A}+\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{C/2} = e^{\hat{A}}e^{\hat{B}}e^{-C/2} \ ,$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

where

$$[\hat{x}, \hat{p}] = i\hbar \ .$$

This Hamiltonian can be rewritten as

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$, and the operators $\hat{a}$ and $\hat{a}^\dagger$ are given by

$$\hat{a} = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} + i\hat{p}\right) \ , \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} - i\hat{p}\right) \ ,$$

and satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 \ .$$

The action of $\hat{a}$ and $\hat{a}^\dagger$ on eigenstates of $\hat{N}$ is given by

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ , \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \ .$$

The ground state wave function is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \ .$$

- Gaussian integral

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

1. **Hermitian Operators (10 points)**

   (a) Which of the following operators *must* be hermitian, assuming that the operators $\hat{A}$ and $\hat{B}$ are hermitian?

      i. $\hat{A}^\dagger \hat{B}$

      ii. $\hat{A}^\dagger \hat{A}$

      iii. $[\hat{A}, \hat{B}]$

      iv. $i\left(|\psi\rangle\langle\phi| - |\phi\rangle\langle\psi|\right)$

   (b) Consider a state $|\psi\rangle = \sum_n c_n |\psi_n\rangle$, where the states $|\psi_n\rangle$ are normalized eigenstates of an observable $\hat{A}$, with eigenvalue a_n. When $\hat{A}$ is measured, what is the probability that a_5 is obtained? What is the state of the system immediately after a_5 is obtained?

   (c) We have seen that the description of the most general quantum mechanical two-state system requires four Hermitian operators, represented as matrices by the identity matrix and the three Pauli matrices. How many independent Hermitian operators are required to describe the most general quantum mechanical three-state system? Justify your answer.

2. **Sequential Stern-Gerlach Experiment (10 points)**

   An obstinately classical friend of yours comes to you and says "I don't believe what you are telling me about superposition of states in quantum mechanics. Lets talk about these spin-1/2 particles which you keep telling me about. What *I* think is that every particle has a label painted on it, stating whether it has $S_z = \pm\hbar/2$. Furthermore, every particle has a label painted on it, stating whether it has $S_x = \pm\hbar/2$. So, when *you* say you have a beam of particles, every one of which is in a state $|+; x\rangle$ which you claim is a $50:50$ superposition of states $|+; z\rangle$ and $|-; z\rangle$, I don't believe you. What *I* think is that you have a beam of particles every one of which has the $S_x = +\hbar/2$ label. Furthermore, 50% of the particles in your beam have the $S_z = +\hbar/2$ label and 50% of the particles have the $S_z = -\hbar/2$ label. Voilà."

   Your junior lab partner has prepared a beam of spin-1/2 particles all of which are in fact in the $|+; x\rangle$ state. Describe a sequence of several Stern-Gerlach experiments which you can do beginning with this beam, which will prove that your obstinately classical friend is wrong. You may want to sketch your sequence of experiments, using suitably labelled black boxes to represent each Stern-Gerlach device. Describe the outcome of each of the Stern-Gerlach experiments you propose, and how these outcomes prove your classical friend wrong.

3. **A Simple Harmonic Oscillator State (20 points)**

Consider a particle of mass m in a one-dimensional simple harmonic oscillator potential, with Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

as given in the information sheet.

Suppose that the particle is in the state

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}\hat{a}^\dagger|0>$$

where $|0\rangle$ is the ground state of the two-dimensional harmonic oscillator and the raising operator $\hat{a}^\dagger$ is defined in the standard fashion in terms of $\hat{x}$ and $\hat{p}$, as given in the information sheet.

(a) If you measure the energy in this state, what are the possible outcomes? What is the expectation value of the energy?

(b) Calculate $\langle\hat{x}\rangle$ in this state.

(c) Calculate the wave function which describes this state.

4. **Dynamics in a two-dimensional harmonic oscillator potential (20 points)**

Suppose a particle of mass m is free to move in the (x, y) plane subject to a harmonic potential centered at the origin. The Hamiltonian for this system can be written

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{m\omega^2}{2}\left(\hat{x}^2 + \hat{y}^2\right) \ .$$

Suppose that at $t = 0$ the particle is in the state

$$|\psi(t=0)\rangle = e^{-ib\hat{p}_y/\hbar}\, e^{-iq\hat{x}/\hbar}\, |0\rangle \tag{1}$$

Here, b and q are constants with dimensions of length and momentum respectively.

(a) What are the expectation values of $\hat{x}$, $\hat{y}$, $\hat{p}_x$, and $\hat{p}_y$ at $t = 0$? [Hint: you may find an identity from the information sheet useful.]

(b) In parts (b) and (c), assume that $b \gg \sqrt{\hbar/m\omega}$ and $q \gg \sqrt{m\omega\hbar}$. Explain in a sentence or two what physical situation is described by a harmonic oscillator which is in the state (1) at time $t = 0$.

(c) Describe the time evolution of this state at times $t \geq 0$. [Your description should consist of a sentence or two and perhaps a sketch; no calculations.]

5. **Sequential Stern Gerlach Measurements, with a Twist (30 points)**

Your junior lab partner has once again done a Stern-Gerlach experiment which has allowed her to provide you with a beam of spin-1/2 particles all of which are in the $|+;z\rangle \equiv |+\rangle$ state. Thus, at time $t = 0$

$$|\psi(0)\rangle = |+\rangle \ .$$

Beginning at $t = 0$, you allow these particles to travel through a region in which there is a uniform, static, magnetic field $\vec{B}$. While the particles are in the field, the Hamiltonian describing their time evolution is given by

$$\hat{H} = b_1\hat{\sigma}_1 + b_2\hat{\sigma}_2 + b_3\hat{\sigma}_3 \ , \tag{2}$$

in the $\hat{S}_z$ basis with basis states $|+\rangle$ and $|-\rangle$. The three numbers (b_1, b_2, b_3) are conveniently summarized by the notation: $\vec{b}$. They are related to the magnetic field $\vec{B}$ by $\vec{b} = -(\gamma\hbar/2)\vec{B}$. [You may leave your answers in terms of $\vec{b}$, rather than converting back to $\vec{B}$.]

(a) In parts (a), (b) and (c) of this problem, assume that the magnetic field is in the x-direction. That is, take $b_1 \neq 0$ and $b_2 = b_3 = 0$. Write the Schrödinger equation as a matrix equation in the $\hat{S}_z$ basis. Solve the Schrödinger equation, and find the state $|\psi(t)\rangle$ for all times $t \geq 0$.

(b) You arrange your experiment so that the particles spend a time T in the magnetic field, and then enter a second Stern-Gerlach apparatus, oriented in the z-direction. What are the possible outcomes of this measurement of $\hat{S}_z$? What is the probability of obtaining each possible outcome, in terms of b_1 and T?

(c) Repeat (b), this time with the second Stern-Gerlach apparatus oriented in the y-direction: What are the possible outcomes of this measurement of $\hat{S}_y$? What is the probability of obtaining each possible outcome, in terms of b_1 and T?

(d) For what choice of field direction, if any, (i.e. for what choice of $\vec{b}$, if any) will the outcomes of *any* sequence of Stern-Gerlach experiments performed after the particles pass through the magnetic field described by $\vec{b}$ be exactly as they would have been if $\vec{b}$ were zero? [Speaking loosely, what choice of $\vec{b}$, if any, "does nothing" to $|\psi(0)\rangle$?]

(e) Now, consider a magnetic field pointing in an arbitrary direction. The vector $\vec{b}$ can be parametrized in "polar coordinates":

$$\begin{aligned} b_3 &= b\cos\theta \\ b_2 &= b\sin\theta\sin\phi \\ b_1 &= b\sin\theta\cos\phi \ . \end{aligned}$$

What is the probability that the system is in the state $|+\rangle$ after a time T, in terms of T and the parameters which describe $\vec{b}$?