Invariant positive forms on generalized (q-)Weyl algebras.

Daniil Klyuev

Plan:

1. Generalized Weyl and q-Weyl algebras. Formulation of a problem.
2. Quantizations and short star-products (Some slides provided by Pavel Etingof from his talk.)
3. Positivity conditions.

Generalized Weyl algebra

Definition

Let P be a polynomial of degree n. A generalized Weyl algebra is an algebra $\mathcal{A}$ with generators u, v, z and relations $[z, u] = -u$, $[z, v] = v$, $vu = P(z - \frac{1}{2})$, $uv = P(z + \frac{1}{2})$.

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2. If $n = 2$, $P(x) = x^2 + C$, this is central reduction of $U(\mathfrak{sl}_2)$ with $e = v$, $f = -u$, $h = 2z$. The Casimir element is $ef + fe + \frac{h^2}{2} = \frac{h^2}{2} + h + 2fe$.

Generalized q-Weyl algebra

Definition

Let P be a Laurent polynomial. A generalized q-Weyl algebra is an algebra $\mathcal{A}$ with generators u, v, Z and relations $ZuZ^{-1} = q^2 u$, $ZvZ^{-1} = q^{-2} v$, $uv = P(q^{-1}Z)$, $vu = P(qZ)$.

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2. If $P(x) = \frac{x+x^{-1}}{(q-q^{-1})^2}$ this is central reduction of $U_q(\mathfrak{sl}_2)$: $u = E$, $v = F$, $Z = K$, the Casimir element is $\Lambda = (q-q^{-1})^2FE + qK + q^{-1}K^{-1}$.

Positive traces

Let $\mathcal{A}$ be a $\mathbb{C}$-algebra with an antilinear involution ρ.

Definition

A linear map $T\colon \mathcal{A} \to \mathbb{C}$ is a trace if $T(ab) = T(ba)$. The trace is positive if $T(a\rho(a)) > 0$ for nonzero $a \in A$.

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- For central reduction of $U(\mathfrak{g})$ this corresponds to spherical unitary representations of G: $\mathcal{A}$ is a complexification of the corresponding $(\mathfrak{g}, K)$-module.

Involution ρ for generalized Weyl algebras

- Recall that $\mathcal{A}$ is generated by u, v, z with $[z, u] = -u$, $[z, v] = v$, $vu = P(z - \frac{1}{2})$, $uv = P(z + \frac{1}{2})$. We define

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- ρ is well-defined when $P(x) = \overline{P}(-x)$, hence the roots are symmetric wrt $i\mathbb{R}$.
- The problem: classify traces T on $\mathcal{A}$ such that $T(a\rho(a)) > 0$ for all nonzero $a \in \mathcal{A}$.

Short star-products

Definition

A ($\mathbb{Z}/2$-equivariant) star-product on A is an associative multiplication $* : A \otimes A \to A$ such that for $a \in A_n$ and $b \in A_m$,

$$a * b = \sum_{k=0}^{\lfloor \frac{n+m}{2} \rfloor} C_k(a, b),$$

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Definition (Beem, Peelaers, Rastelli, 2016)

A star-product $*$ on A is short if for any $m, n \in \mathbb{Z}_{\geq 0}$ and any $a \in A_n$, $b \in A_m$ one has

$$C_k(a, b) = 0, k > \min(n, m).$$

In other words, $a * b$ has no terms in A_d for $d < |n - m|$.

Filtered deformations

- Let A be a commutative graded algebra. Fix a filtered associative algebra $\mathcal{A} = \bigcup_{d\geq 0} F_d\mathcal{A}$ such that its associated graded algebra is identified with the graded algebra A; namely,

$$\mathrm{gr}\mathcal{A} = \bigoplus_{d\geq 0} F_d\mathcal{A}/F_{d-1}\mathcal{A}$$

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- Such $\mathcal{A}$ is called filtered deformation (quantization) of A.
- For example, when $\mathcal{A}$ is a generalized Weyl algebra, $\mathrm{gr}\,\mathcal{A}$ is isomorphic to $A = \mathbb{C}[u, v, z]/(uv - z^n)$, Kleinian singularity of type A. Each deformation of A is isomorphic to a GWA with parameter P of degree n.

Bijection between star-products and quantization maps

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Lemma

This defines a pair of mutually inverse bijections between star-products on A and $(\mathbb{Z}/2$-equivariant) quantizations of A equipped with a quantization map.

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- It follows from the results in Etingof-Stryker that there is a bijection between positive short star-products and pairs $(\mathcal{A}, T)$, where $\mathcal{A}$ is a filtered deformation of A and T is a positive trace.
- This is a physical motivation for considering positive traces. Beem, Peelaers and Rastelli tried to compute positive short star-products on Kleinian singularities of type A.

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Proposition

Let k be the number of good roots. Then the dimension of the cone of positive traces modulo scaling is $k - 4$ if n is divisible by 4 and $k - 2$ otherwise.

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- For example, for $n=2$ a positive trace is unique and exists when $P(x) = x^2 + C$ satisfies $C > -\frac{1}{4}$. If $C \geq 0$ we get spherical principal series representations, if $-\frac{1}{2} < C < 0$ we get complementary series representations.

- Suppose that $\mathcal{A}$ is generalized q-Weyl algebra. Then $\rho(u) = v$, $\rho(v) = u$, $\rho(Z) = Z^{-1}$ and $P(x)$ is real on unit circle.

Case of q-Weyl algebras

- Suppose that $\mathcal{A}$ is generalized q-Weyl algebra. Then $\rho(u) = v$, $\rho(v) = u$, $\rho(Z) = Z^{-1}$ and $P(x)$ is real on unit circle.
- There is a similar notion of good roots ($q < |\alpha| < q^{-1}$), but the answer is always $k - 1$.

Bimodules

- Let M be an $\mathcal{A}-\overline{\mathcal{A}}$-bimodule. For any P we can define ρ in the same way: $\rho(u)=v$, $\rho(v)=u$, $\rho(z)=-z$.

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- The answer is the same: number of good roots minus a constant from 1 to 4.
- The story when $\mathcal{A}$ is a q-deformation is similar.

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Computation of traces

We will restrict our attention to the case of even n for convenience. In this case $s = \mathrm{id}$. An easy computation shows that the space of traces on $\mathcal{A}$ is isomorphic to V^*, where

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Suppose that $\phi \in V^*$. We have $\mathcal{A} = \oplus_{m\in\mathbb{Z}}\mathcal{A}_m$ — an eigenspace of $\mathrm{ad}\, z$ decomposition. The corresponding trace T acts as follows: if $a \in \mathcal{A}_m$, $m \neq 0$, then $T(a) = 0$. If $a \in \mathcal{A}_0 = \mathbb{C}[h]$ then $a = R(h)$ and $T(a) = \phi(R)$.

Analytic formula.

Suppose that all roots of $P(x)$ belong to the open strip $|\operatorname{Re} x| < 1$. Then there is the following formula:

$$T(R) = \int_{i\mathbb{R}} R(x)w(x)dx,$$

where $w(x) = \frac{Q(e^{i\pi x})}{\mathcal{P}(e^{i\pi x})}$, $\mathcal{P}(x) = \prod_{P(\alpha)=0} (x + e^{i\pi\alpha})$.

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Condition $T(S(x)\overline{S}(-x)) > 0$ translates to $w(x) \geq 0$ on $i\mathbb{R}$, similarly for the second condition we get $\pm w(x+1)P(x) \leq 0$ on $i\mathbb{R}$.

Analytic formula, ctd.

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It can be proved that roots x with $|\operatorname{Re} x| > 1$ do not change the cone of positive forms, and each pair of roots $x \pm 1$ with $\operatorname{Re} x = 0$ multiplies the cone by $\mathbb{R}_{\geq 0}$: there exists trace $R \mapsto R(x)$.