Hikita-Nakajima conjecture for ADHM spaces

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Symplectic resolutions of singularities

Setting: symplectic resolution is

$$\pi\colon X \to Y,$$

X - smooth symplectic, Y - (normal) Poisson affine,

π - resolution of singularities, compatible with Poisson structures.

Resolution π is **conical** if

$\mathbb{C}^\times \curvearrowright X, Y,\ \pi$ is $\mathbb{C}^\times$-equivariant,

$\mathbb{C}^\times$ contracts Y to **one** point p and scales ω_X with positive weight.

Example 1: Springer resolution for $\mathfrak{sl}_2$

$$\pi \colon X = T^*\mathbb{P}^1 \to \mathcal{N} = Y,$$

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid \det(A) = 0 \right\} = \{(a, b, c) \mid a^2 + bc = 0\},$$

$$\mathbb{C}^\times \curvearrowright \mathcal{N}, \ t \cdot (a, b, c) = (ta, tb, tc).$$

Example 2: Symmetric powers of Kleinian singularities

Kleinian singularity of type A_{r-1}:

$$\Gamma = \mathbb{Z}/r\mathbb{Z} \curvearrowright \mathbb{A}^2,\ [1]\cdot(x,y) = (e^{\frac{2\pi i}{r}}x, e^{-\frac{2\pi i}{r}}y),$$

$$Y = \mathbb{A}^2/\Gamma = \operatorname{Spec}\mathbb{C}[x,y]^\Gamma = \{(a,b,c) \mid a^r = bc\},$$

$$\pi\colon X = \widetilde{\mathbb{A}^2/\Gamma} \to \mathbb{A}^2/\Gamma = Y.$$

Generalization:

$$\pi\colon X = \mathrm{Hilb}_n(\widetilde{\mathbb{A}^2/\Gamma}) \to S^n(\mathbb{A}^2/\Gamma) = Y,$$

where for a variety Z:

$$S^n(Z) := Z^n/S_n,\ \mathrm{Hilb}_n(Z) = \{\mathcal{I} \subset \mathcal{O}_Z \mid \mathrm{length}(\mathcal{O}_Z/\mathcal{I}) = n\}.$$

Example 3: *ADHM* space $\mathfrak{M}(n, r)$, quiver varieties

Pick $n, r \in \mathbb{Z}_{\geqslant 1}$, $V = \mathbb{C}^n$, $W = \mathbb{C}^r$

$$M = M(n, r) = \mathrm{End}(V)^{\oplus 2} \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W),$$

$$(X, Y, \gamma, \delta) \in M, \ \ \mathrm{GL}(V) \curvearrowright M,$$

then

$$\mathfrak{M}(n, r) := M /\!\!/\!\!/^{\det} \mathrm{GL}(V), \ \mathfrak{M}_0(n, r) := M /\!\!/\!\!/ \mathrm{GL}(V).$$

We have

$$\pi \colon X = \mathfrak{M}(n, r) \to \mathfrak{M}_0(n, r) = Y.$$

Definition generalizes to arbitrary quiver Q, we get Nakajima quiver varieties

$$\mathfrak{M}(Q) \xrightarrow{\pi} \mathfrak{M}_0(Q).$$

Symplectic duality: preliminaries

$$\pi\colon X \to Y,$$

$$S_Y \subset \mathrm{Aut}_{\mathbb{C}^\times,\{\,,\,\}}(Y) \text{ a maximal torus, } \mathfrak{s}_Y := \mathrm{Lie}\, S_Y,$$

$$\mathfrak{t}_Y := H^2(X,\mathbb{C}).$$

By the results of Namikawa there exists the universal (symplectic) deformation

$$\begin{array}{ccc} X^{\mathrm{univ}} & \xrightarrow{\pi^{\mathrm{univ}}} & Y^{\mathrm{univ}} \\ & \searrow \quad \swarrow & \\ & \mathfrak{t}_Y & \end{array}$$

Symplectic duality

$$X \to Y \quad \text{symplectically dual to} \quad X^! \to Y^!,$$

then

$$\mathfrak{s}_Y \simeq \mathfrak{t}_{Y^!},$$
$$\mathfrak{t}_Y \simeq \mathfrak{s}_{Y^!},$$
$$X^{S_Y} \simeq (X^!)^{S_{Y^!}} \dots$$

Examples:

$$\mathfrak{M}_0(n,1) = S^n(\mathbb{A}^2) \text{ is self dual,}$$

$$\mathfrak{M}_0(n,r) \text{ is dual to } S^n(\mathbb{A}^2/(\mathbb{Z}/r\mathbb{Z})),$$

$\mathfrak{M}_0(Q)$ is (expected to be) dual to the BFN Coulomb branch $\mathcal{M}(Q)$.

Schematic fixed points

$$Y = \operatorname{Spec} B \text{ affine variety,}$$

$$\mathbb{C}^\times \curvearrowright Y \Leftrightarrow B = \bigoplus_{i\in\mathbb{Z}} B_i \text{ s.t. } B_i \cdot B_j \subset B_{i+j},$$

$$\mathbb{C}[Y^{\mathbb{C}^\times}] := B/(b_i \in B_i \mid i \neq 0).$$

Example: $\mathbb{C}^\times \curvearrowright \mathcal{N}$ via the conjugation by $\operatorname{diag}(t, t^{-1})$,

$$\mathbb{C}[\mathcal{N}^{\mathbb{C}^\times}] = \mathbb{C}[a,b,c]/(a^2+bc, b, c) = \mathbb{C}[a]/(a^2).$$

Hikita-Nakajima conjecture

Suppose $X \to Y$, $X^! \to Y^!$ are sympl. dual. Pick a generic $\nu \colon \mathbb{C}^\times \to S_{Y^!}$.

Conjecture [Hikita, Nakajima]

There is an isomorphism of (graded) $\mathbb{C}[\mathfrak{s}]$-algebras

$$H^*_{S_Y}(X) \simeq \mathbb{C}[(Y^{!,\mathrm{univ}})^{\nu(\mathbb{C}^\times)}].$$

Theorem [K-Shlykov]

Hikita-Nakajima conjecture holds for ADHM spaces i.e. for

$$X = \mathfrak{M}(n, r),\ Y^! = S^n(\mathbb{A}^2/(\mathbb{Z}/r\mathbb{Z})).$$

Quiver variety side: $H^*_S(\mathfrak{M}(Q))$

$\mathfrak{M} = \mathfrak{M}(Q)$ a quiver variety and $\mathcal{M} = \mathcal{M}(Q)$ is the Coulomb branch. What is known about $H^*_S(\mathfrak{M})$ (Nakajima, McGerty and Nevins):

$$H^*_S(\mathfrak{M}) \text{ is a free module over } H^*_S(\mathrm{pt}).$$

$$\text{Algebra } H^*_S(\mathfrak{M}) \text{ is generated by } c_i(\mathcal{V}_j),\ j \in Q_0.$$

Assumption: $\mathfrak{M}^S$ is finite (for example, if Q is a cyclic quiver, in particular, if $\mathfrak{M} = \mathfrak{M}(n, r)$).
Using the localization theorem, we obtain the embedding

$$\iota^* \colon H^*_S(\mathfrak{M}) \hookrightarrow H^*_S(\mathfrak{M}^S) = \mathbb{C}[\mathfrak{s}]^{|\mathfrak{M}^S|},$$

$$c_i(\mathcal{V}_j) \overset{\iota^*}{\mapsto} (c_i(\mathcal{V}_j|_p))_{p \in \mathfrak{M}^S}.$$

Coulomb side: $\mathbb{C}[(\mathcal{M}^{\text{univ}})^{\mathbb{C}^\times}]$

Goal:
(a) construct an embedding $\mathbb{C}[(\mathcal{M}^{\text{univ}})^{\nu(\mathbb{C}^\times)}] \hookrightarrow \mathbb{C}[\mathfrak{s}]^{|\mathfrak{M}^S|}$.
(b) find generators of $\mathbb{C}[(\mathcal{M}^{\text{univ}})^{\nu(\mathbb{C}^\times)}]$ which images coincide with images of $c_i(\mathcal{V}_j)$.
$(a),(b) \Rightarrow$ Hikita-Nakajima conjecture.
Partial solution of $(a),(b)$.
Consider a symplectic resolution $\pi\colon \widetilde{\mathcal{M}}^{\text{univ}} \to \mathcal{M}^{\text{univ}}$.
Then for a morphism in (a) take

$$\pi^*\colon \mathbb{C}[(\mathcal{M}^{\text{univ}})^{\nu(\mathbb{C}^\times)}] \to \mathbb{C}[(\widetilde{\mathcal{M}}^{\text{univ}})^{\nu(\mathbb{C}^\times)}] = \mathbb{C}[\mathfrak{s}]^{|\mathfrak{M}^S|}.$$

Recall

$$\mathbb{C}[\mathcal{M}^{\text{univ}}] = H_*^{G_V\times S}(\mathcal{R}) = H_*^S(\mathcal{R}/G_V),\ G_V = \prod_{j\in Q_0} \mathrm{GL}(V_j),\ \mathcal{E}_j = \mathcal{R}\times_{G_V} V_j.$$

Proposition [BFN], [FT], [W], [KWWY2]

Functions $c_i(\mathcal{E}_j)$ generate the algebra $\mathbb{C}[(\mathcal{M}^{\text{univ}})^{\nu(\mathbb{C}^\times)}]$.

Example: the case $\mathfrak{M}_0(n,1) = S^n(\mathbb{A}^2) = \mathcal{M}$

Universal deformations of $\mathfrak{M}(n,1)$, $\mathfrak{M}_0(n,1)$:

$$\text{replace } [X,Y] + \gamma\delta = 0 \text{ by } [X,Y] + \gamma\delta = \kappa\,\mathrm{Id}_V,\ \ \kappa \in \mathbb{C}.$$

$$\mathfrak{M}(n,1)^{\mathbb{C}^\times} = \mathrm{Hilb}_n(\mathbb{A}^2)^{\mathbb{C}^\times} = \{\lambda \text{ - partitions of } n\}.$$

Function $c_i(\mathcal{E})$ is

$$\mathfrak{M}(n,1)^{\mathrm{univ}} \ni (X,Y,\gamma,\delta) \mapsto e_i(\alpha_1,\ldots,\alpha_n),$$

$\alpha_1,\ldots,\alpha_n$ are eigenvalues of YX.
Value of $c_i(\mathcal{E})$ at $\lambda(\kappa) \in (\mathfrak{M}(n,1)^{\mathrm{univ}})^{\mathbb{C}^\times}$ is

$$\kappa^i e_i(c_1,\ldots,c_n),\ \ c_k \text{ are contents of } \lambda.$$

To finish the proof of HN, it remains to note that contents c_k are exactly weights of $\mathcal{V}|_\lambda$ ($\mathcal{V}$ - tautological bundle on $\mathrm{Hilb}_n(\mathbb{A}^2)$), so (a), (b) above follow.

Conjectures and concluding remarks

Conjecture 1

Values of $c_i(\mathcal{E}_j)$ at $(\widetilde{\mathcal{M}}^{\mathrm{univ}})^{\nu(\mathbb{C}^\times)}$ are equal to S-characters of $\mathcal{V}_j$ at $\mathfrak{M}^S$.

Conjecture 2

The algebra $\mathbb{C}[(\widetilde{\mathcal{M}}^{\mathrm{univ}})^{\nu(\mathbb{C}^\times)}]$ is flat over $\mathbb{C}[\mathfrak{s}]$.

Proposition

Hikita-Nakajima conjecture follows from Conjectures $1, 2$. Moreover, it is enough to check the Conjecture 1 for generic points of $\mathfrak{s}$.

Remark

A statement closely related to Conjecture 1 in generic point was formulated and proved in [KWWY]. The main tool for the argument are so-called KLRW-algebras (they describe the structure of Coulomb branches by the results of Ben Webster).

References

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