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1. Symplectic duality with examples

There is a certain interesting class of symplectic varieties called *conical symplectic resolutions of singularities*. Informally speaking symplectic resolution of singularities consists of a pair $\pi\colon X \to Y$, where Y is an affine (singular) Poisson variety and X is a symplectic variety with π being a resolution of singularities. A symplectic resolution is *conical* if group $\mathbb{C}^\times$ acts on both X, Y such that π is equivariant, $\mathbb{C}^\times$ contracts Y to the point and scales symplectic form ω_X with some positive weight.

Let us some give examples of conical symplectic resolutions

$$\pi\colon X \to Y.$$

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The simplest example of the symplectic resolution is

$$\pi\colon X = T^*\mathbb{P}^1 \to \mathcal{N} = Y,$$

where $\mathcal{N} \subset \mathfrak{sl}_2$ consists of nilpotent matrices $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ i.e. of matrices with zero determinant so

$$\mathcal{N} = \{(a,b,c) \,|\, bc + a^2 = 0\}.$$

The Poisson strucutre on $\mathcal{N}$ is induced from the Kirillov-Kostant-Souriau Poisson bracket on $\mathfrak{sl}_2 \simeq \mathfrak{sl}_2^*$, the conical $\mathbb{C}^\times$ action on $\mathcal{N}$ is just the scaling action.

We have

$$X = \left\{ \left(\ell \subset \mathbb{C}^2,\ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathcal{N} \right) \,|\, \ell \subset \ker A \right\} = T^*\mathbb{P}^1$$

the map $\pi\colon T^*\mathbb{P}^1 \to \mathcal{N}$ simply forgets the line ℓ.

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Another example (generalizing the above one) is $T^*\mathbb{P}^{n-1}$ that is similar to the example above and resolves the space $\mathbb{O}_n \subset \mathfrak{sl}_n$ consisting of matrices $A \in \mathfrak{sl}_n$ such that $\operatorname{rk} A \leqslant 1$. One can show that

$$T^*\mathbb{P}^{n-1} = \left\{ \left(\ell \subset \mathbb{C}^n,\ A \in \mathfrak{sl}_n \right), \operatorname{Im} A \subset \ell \subset \ker A \right\}.$$

The map

$$\pi\colon T^*\mathbb{P}^{n-1} \to \mathbb{O}_n$$

simply forgets ℓ. For $n = 2$ we get the same example as above.

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Other examples are resolutions of type A singularities (or more generally ADE singularities). Explicitly consider the two-dimensional affine space $\mathbb{A}^2$ and consider the action of the group $\Gamma = \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{A}^2$, where the generator $[1] \in \mathbb{Z}/n\mathbb{Z}$ acts as follows:

$$[1] \cdot (x, y) = (e^{-\frac{2\pi i}{n}} x, e^{\frac{2\pi i}{n}} y).$$

We can consider the quotient $\mathbb{A}^2/\Gamma$ functions on which are

$$\mathbb{C}[\mathbb{A}^2]^\Gamma = \mathbb{C}[xy, x^n, y^n] = \mathbb{C}[a,b,c]/(a^n - bc)$$

so

$$\mathbb{A}^2/\Gamma = \{(a,b,c) \,|\, a^n = bc\}.$$

For $n = 2$ we get our main example of the nilotent cone $\mathcal{N}$ in $\mathfrak{sl}_2$. The variety $\mathbb{A}^2/\Gamma$ is Poisson, bracket is induced from the standard bracket on $\mathbb{A}^2$ ($\{x, y\} = 1$).

It turns out that in general this singularity $a^n = bc$ can be resolved via $n - 1$ blow ups so we obtain the desired symplectic resolution

$$\pi\colon \widetilde{\mathbb{A}^2/\Gamma} \to \mathbb{A}^2/\Gamma.$$

The conical $\mathbb{C}^\times$ action comes from the scaling action on $\mathbb{A}^2$.

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Another example is $\mathrm{Hilb}_n(\mathbb{A}^2)$ (Hilbert scheme of n points on $\mathbb{A}^2$) and more generally the *ADHM* space $\mathfrak{M}(n,r)$ or even more generally quiver varieties $\mathfrak{M}(Q)$. We start from the definition of the *ADHM* space $\mathfrak{M}(n,r)$.

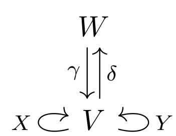
Pick positive integers $n, r \in \mathbb{Z}_{\geqslant 1}$. Let V be a vector space of dimension n and W be a vector space of dimension r:

$$V = \mathbb{C}^n,\; W = \mathbb{C}^r.$$

Consider the space

$$\mathbf{M} = \mathbf{M}(n,r) := \mathrm{Hom}(V,V)^{\oplus 2} \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W) = T^*(\mathrm{Hom}(V,V) \oplus \mathrm{Hom}(W,V)).$$

Elements of $\mathbf{M}$ will be denoted by (X, Y, γ, δ) and called quadruples. They can be considered as representations of the following quiver



Group $\mathrm{GL}(V)$ acts naturally on $\mathbf{M}$ preserving symplectic structure. There is a moment map

$$\mu\colon \mathbf{M} \to \mathfrak{gl}(V),\; (X, Y, \gamma, \delta) \mapsto [X, Y] + \gamma\delta.$$

Definition 1.1. A quadruple $(X, Y, \gamma, \delta) \in \mathbf{M}(n,r)$ is called stable if for every X, Y-invariant subspace $S \subset V$ such that S contains $\operatorname{im}\gamma$ we have $S = V$. We denote by $\mathfrak{M}(n,r)^{\mathrm{st}} \subset \mathfrak{M}(n,r)$ the (open) subset of stable quadruples.

Definition 1.2. The Nakajima quiver varieties $\mathfrak{M}(n,r)$, $\mathfrak{M}_0(n,r)$ that we call *ADHM* spaces or Gieseker varieties are defined as the following quotients

$$\mathfrak{M}(n,r) := \mu^{-1}(0)^{\mathrm{st}} / \mathrm{GL}(V),\; \mathfrak{M}_0(n,r) := \mu^{-1}(0) /\!\!/ \mathrm{GL}(V).$$

Remark 1.3. *This definition can be generalized to define quiver variety $\mathfrak{M}(Q)$ corresponding to arbitrary quiver $Q = (I_0, I_1)$ together with the choice of dimension and framing vectors $(v_i)_{i\in I_0}$, $(w_i)_{i\in I_0}$. In the definition above one should just replace V by $\bigoplus_{i\in I_0} V_i$, $W = \bigoplus_{i\in I_0} W_i$ and $\mathrm{GL}(V)$ by $G_V = \prod_{i\in I_0} \mathrm{GL}(V_i)$. Let me not go deep into this since we will be mostly interested in $\mathfrak{M}(n,r)$ or even $\mathfrak{M}(n,1)$.*

The natural morphism

$$\pi\colon X = \mathfrak{M}(n,r) \to \mathfrak{M}_0(n,r) = Y$$

is our symplectic resolution of singularities. Let us mention that $\mathfrak{M}(n,r)$ is also known as the Uhlenbeck compactification of the moduli space of $SU(r)$ instantons on $\mathbb{C}^2$. Our goal for now is to discuss geometry of $\mathfrak{M}(n,r), \mathfrak{M}_0(n,r)$.

Let $\mathfrak{z}(V) \subset \mathfrak{gl}(V)$ be the center of $\mathfrak{gl}(V)$. Explicitly

$$\mathfrak{z}(V) = \{\mathrm{diag}(t,t,\ldots,t)\, t \in \mathbb{C}\}.$$

Varieties $\mathfrak{M}(n,r), \mathfrak{M}_0(n,r)$ admit certain natural deformations over the space $\mathfrak{z}(V) = \mathbb{C}$.

Definition 1.4. The universal quiver varieties $\mathfrak{M}^{\mathrm{univ}}(n,r)$, $\mathfrak{M}_0^{\mathrm{univ}}(n,r)$ are defined as follows:

$$\mathfrak{M}^{\mathrm{univ}}(n,r) := \mu^{-1}(\mathfrak{z}(V))^{\mathrm{st}} / \mathrm{GL}(V),\ \mathfrak{M}_0^{\mathrm{univ}}(n,r) := \mu^{-1}(\mathfrak{z}(V)) /\!\!/ \mathrm{GL}(V).$$

Let us now give an alternative description of the varieties $\mathfrak{M}(n,1), \mathfrak{M}_0(n,1)$ (describe them more explicitly).

Let

$$S^n(\mathbb{A}^2) := (\mathbb{A}^2)^n/S_n$$

be the variety parametrizing n unordered points $\{p_1,\ldots,p_n\}$ on $\mathbb{A}^2$ (with multiplicities), $p_i \in \mathbb{A}^2$. This is a singular (Poisson) variety.

The variety $S^n(\mathbb{A}^2)$ can be resolved by the so called Hilbert scheme of n-points on $\mathbb{A}^2$.

Definition 1.5. The variety $\mathrm{Hilb}_n(\mathbb{A}^2)$ is the variety whose $\mathbb{C}$-points are ideals $J \subset \mathbb{C}[x,y]$ such that $\dim \mathbb{C}[x,y]/J = n$.

The following proposition is well-known.

Proposition 1.6. *There exist isomorphisms*

$$\mathfrak{M}(n,1) \simeq \mathrm{Hilb}_n \mathbb{A}^2,\ \mathfrak{M}_0(n,1) \simeq S^n(\mathbb{A}^2). \tag{1.1}$$

Recall that the isomorphism $\mathfrak{M}(n,1) \xrightarrow{\sim} \mathrm{Hilb}_n \mathbb{A}^2$ above can be constructed as follows: starting from the ideal $J \in \mathrm{Hilb}_n(\mathbb{A}^2)$ we can construct the quadruple (X,Y,γ,δ) as follows: $V = \mathbb{C}[x,y]/J$, X is the multiplication by x, Y is the multiplication by y, γ corresponds to the embedding $\mathbb{C} \subset V$ that sends 1 to $1 \in \mathbb{C}[x,y]/J = V$ and $\delta = 0$.

The natural question is how to describe the whole deformation $\mathfrak{M}_0^{\mathrm{univ}}(n,1)$ of $\mathfrak{M}_0(n,1) = S^n(\mathbb{A}^2)$ in terms similar to $S^n(\mathbb{A}^2)$ (without using quiver description). Recall first that

$$\mathbb{C}[S^n(\mathbb{A}^2)] = \mathbb{C}[\mathbb{A}^{2n}]^{S_n} = \mathbb{C}[x_1,y_1,\ldots,x_n,y_n]^{S_n} = \mathbf{e}\Big(\mathbb{C}S_n \ltimes \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]\Big)\mathbf{e},$$

where $\mathbf{e} := \frac{1}{n!}\sum_{g \in S_n} g \in \mathbb{C}S_n$ is the symmetrizing idempotent. It is also easy to see that the algebra above can be identified with the center of $\mathbb{C}S_n \ltimes \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. The algebra $\mathbb{C}S_n \ltimes \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ has a one-parametric deformation called rational Cherednik algebra H_n that can be defined as follows.

Definition 1.7. Algebra H_n is a quotient of the semidirect product

$$\Big(\mathbb{C}S_n \ltimes \mathbb{C}\langle x_1,\dots,x_n,y_1,\dots,y_n\rangle\Big) \otimes \mathbb{C}[\kappa]$$

subject to the relations

$$[x_i,x_j]=[y_i,y_j]=0,$$
$$[x_i,y_i]=\kappa\sum_{j\neq i}(ij),$$
$$[x_i,y_j]=-\kappa(ij),\ i\neq j$$

where $(ij)\in S_n$ is the transposition $i\longleftrightarrow j$.

Let $Z_n \subset H_n$ be the center of H_n. This proposition follows from the results of Etingof-Ginzburg.

Proposition 1.8. *We have an isomorphism of algebras*

$$\mathbb{C}[\mathfrak{M}_0^{\mathrm{univ}}(n,1)] \simeq Z_n.$$

So for $r=1$ we have some (rather explicit) description of the varieties $\mathfrak{M}(n,1)$, $\mathfrak{M}_0(n,1)$ and their deformations.

It would be natural if we have some generalization of $Z_n \subset H_n$ to the case of arbitrary r. Indeed we have: we just need to replace S_n with the semidirect product $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ and then we obtain the algebra $H_{n,r}$ with center $Z_{n,r}$. This center will be the deformation of the algebra of functions of the Poisson variety

$$(\mathbb{A}^{2n})/(S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n) = (\mathbb{A}^2)^n/(S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n),$$

here S_n acts via permutations and each $\mathbb{Z}/r\mathbb{Z}$ acts on its copy of $\mathbb{A}^2$ via $[1]\cdot(x,y) = (e^{\frac{2\pi i}{r}}x, e^{-\frac{2\pi i}{r}}y)$. So this is a "biproduct" of the examples $(\mathbb{A}^2)^n/S_n$ and $\mathbb{A}^2/(\mathbb{Z}/r\mathbb{Z})$.

Remark 1.9. *It is NOT true that $(\mathbb{A}^2)^n/(S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n)$ is isomorphic to $\mathfrak{M}_0(n,r)$ (this is only true for $r=1$). Consider the simplest case $n=1$. It is easy to see then that we have isomorphisms*

$$\mathfrak{M}(1,r) \simeq T^*\mathbb{P}^{r-1}\times\mathbb{C}^2,\ \mathfrak{M}_0(1,r)\simeq \mathbb{O}_r\times\mathbb{C}^2.$$

So $\mathfrak{M}_0(1,r)$ is $\mathbb{O}_r\times\mathbb{C}^2$ but not $\mathbb{A}^2/(\mathbb{Z}/r\mathbb{Z})$. Note that these varieties have different dimensions for $r>1$!

The (Poisson) varieties

$$\mathfrak{M}_0(n,r),\ (\mathbb{A}^2)^n/(S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n)$$

are symplectically dual and the identification $\mathfrak{M}_0(n,r) \simeq (\mathbb{A}^2)^n/S_n$ above corresponds to the fact that $S^n(\mathbb{A}^2)$ is selfdual. Let us briefly discuss the concept of symplectic duality.

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Assume that we have some general conical symplectic resolution $X\to Y$. One can attach to it the following spaces. First of all let $\mathrm{Aut}_{\mathbb{C}^\times}(Y)$ be the group of Poisson authomorphisms of Y that commute with $\mathbb{C}^\times$. Let $S_Y \subset \operatorname{Aut}_{\mathbb{C}^\times}(Y)$ be a maximal torus. Set

$$\mathfrak{s}_Y := \operatorname{Lie} S_Y,\ \mathfrak{t}_Y := H^2(X, \mathbb{C}).$$

It turns out that the space $\mathfrak{t}_Y$ is the base of a certain canonical symplectic/Poisson deformation of $X \to Y$ i.e. there are families $\pi^{\mathrm{univ}}\colon X^{\mathrm{univ}} \to Y^{\mathrm{univ}}$ over $\mathfrak{t}_Y$ such that the fiber over $0 \in \mathfrak{t}_Y$ is our $\pi\colon X \to Y$ (we have already seen such families for quiver varieties $\mathfrak{M}(n,r)$).

Symplectic duality predicts that (every) symlectic resolution $X \to Y$ should have *dual* one $X^! \to Y^!$ and that these two resolutions are closely related. The basic prediction of symplectic duality is:

$$\mathfrak{t}_Y \simeq \mathfrak{s}_{Y^!},\ \mathfrak{s}_Y \simeq \mathfrak{t}_{Y^!}.$$

The main example of symplectically dual varieties for us are:

$$Y = \mathfrak{M}_0(n,r),\ Y^! = (\mathbb{A}^2)^n/(S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n).$$

Recall that the universal deformation of $(\mathbb{A}^2)^n/(S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n)$ is the spectrum of the center $Z_{n,r}$ of the rational Cherednik algebra $H_{n,r}$ corresponding to the group $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$.

Remark 1.10. *From the general prospective of symplectic reflection algebras the number of parameters of the deformation $H_{n,r}$ is equal to the number of conjugacy classes of reflections in $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ so is equal to r (we have conjugacy class of any transposition in S_n and also $r-1$ conjugacy classes of $[1],[2],\ldots,[r-1] \in \mathbb{Z}/r\mathbb{Z}$). So*

$$\operatorname{Spec} Z_{n,r} \text{ is } r \text{ parametric deformation of } (\mathbb{A}^{2n})/(S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n)$$

i.e.

$$\mathfrak{t}_{Y^!} = \mathbb{C}^r.$$

The torus $S_{Y^!}$ acting on $Y^!$ is $\mathbb{C}^\times$ acting on $\mathbb{A}^2$ via $t \cdot (x,y) = (tx, t^{-1}y)$. So

$$\mathfrak{s}_{Y^!} = \mathbb{C}.$$

Variety $Y = \mathfrak{M}_0(n,r)$ has 1-parametric deformation $\mathfrak{z}(V)$ so

$$\mathfrak{t}_Y = \mathbb{C}$$

and $\mathfrak{M}_0(n,r)$ equipped with the action of r-dimensional torus S which can be described as follows: group $\mathrm{PGL}(W)$ acts naturally on $\mathfrak{M}_0(n,r)$ so a maximal torus $A \subset \mathrm{PGL}(W)$ acts. We also have the action of $\mathbb{C}^\times$ given by $t \cdot (X,Y,\gamma,\delta) = (tX, t^{-1}Y, \gamma, \delta)$. Alltogether we get $S = A \times \mathbb{C}^\times \simeq (\mathbb{C}^\times)^r$. So

$$\mathfrak{s}_Y = \mathbb{C}^r.$$

So we see that for our example we indeed have

$$\mathfrak{t}_Y = \mathbb{C} = \mathfrak{s}_{Y^!},\ \mathfrak{s}_Y = \mathbb{C}^r = \mathfrak{t}_{Y^!}.$$

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A much more unexpected (from the first glance) prediction of symplectic duality is the so-called (equivariant) Hikita-Nakajima conjecture that will be the main topic of our talk.

Recalling what we have already discussed:

$$T^*(\mathbb{P}^{r-1}) \to \mathbb{O}_r \text{ is symplectically dual to } \widetilde{\mathbb{A}^2/\Gamma} \to \mathbb{A}^2/\Gamma,$$

$\Gamma = \mathbb{Z}/r\mathbb{Z}$.

$$\mathrm{Hilb}_n(\mathbb{A}^2) \to S^n(\mathbb{A}^2) \text{ is symplectically dual to } \mathrm{Hilb}_n(\mathbb{A}^2) \to S^n(\mathbb{A}^2) = (\mathbb{A}^{2n})/S_n$$

and more generally

$$\mathfrak{M}(n,r) \to \mathfrak{M}_0(n,r) \text{ is symplectically dual to } (\mathbb{A}^2)^n/(S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n),$$

Remark 1.11. *Recall that the universal deformation of* $(\mathbb{A}^{2n})/(S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n)$ *is* $\mathrm{Spec}\, Z_{n,r}$*, where* $Z_{n,r} \subset H_{n,r}$ *is the center of the rational Cherednik algebra corresonding to* $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$.

$$\mathfrak{M}(Q) \text{ is symplectically dual to the Coulomb branch corresponding to } Q.$$

2. Equivariant Hikita-Nakajima conjecture (general version)

Let $X \to Y$, $X^! \to Y^!$ be a pair of symplectically dual varieties. Equivariant Hikita-Nakajima conjecture is a conjecture that relates the topology of X with the algebraic structure of $Y^!$.

We start from some general notion. Assume that Y is an affine variety equipped with an action of the group $\mathbb{C}^\times$. The action of $\mathbb{C}^\times$ on Y is the same as the action of $\mathbb{C}^\times$ on $B := \mathbb{C}[X]$ by algebra authomorphisms so is the same as the grading

$$B = \bigoplus_{k\in\mathbb{Z}} B_k$$

such that $B_k \cdot B_l \subset B_{k+l}$.

We define the *schematic fixed points* $Y^{\mathbb{C}^\times}$ as follows

$$Y^{\mathbb{C}^\times} = \mathrm{Spec}\left(B/(f_i,\, f_i \in B_i,\, i \neq 0)\right).$$

Remark 2.1. *Consider the simplest example: assume that* $Y = \mathbb{A}^1$ *and* $\mathbb{C}^\times$ *acts on* X *via multiplication* $t \cdot y = ty$. *Then* $B = \mathbb{C}[\mathbb{A}^1] = \mathbb{C}[y]$ *and the grading is given by* $\deg y^k = k$. *We see that passing to* $\mathbb{C}^\times$*-fixed points we just mod out by all positive powers of* y *so we get*

$$(\mathbb{A}^1)^{\mathbb{C}^\times} = \mathrm{Spec}(\mathbb{C}) = \{0\} \subset \mathbb{A}^1$$

so this is just one point as expected.

Consider now the following example (our main example):

$$Y := \mathcal{N} = \{(a,b,c) \,|\, a^2 = bc\}$$

so

$$\mathbb{C}[\mathcal{N}] = \mathbb{C}[a,b,c]/(a^2 - bc).$$

We have an action of $\mathbb{C}^\times$ on $\mathcal{N}$ given by

$$t \cdot (a, b, c) = (t^{-1}a, tb, c).$$

Then as a set $\mathcal{N}^{\mathbb{C}^\times}$ consists only of one point $(0,0,0)$. Indeed since point is fixed we must have $a = b = 0$ but then $0 = bc = a^2$ so $a = 0$ and we get the unique fixed point $(0,0,0)$. But as an algebra we get something two-dimensional:

$$\mathbb{C}[\mathcal{N}^{\mathbb{C}^\times}] = \mathbb{C}[\mathcal{N}]/(b,c) = \mathbb{C}[a,b,c]/(b,c,bc-a^2) = \mathbb{C}[a]/(a^2).$$

Note that something interesting happens at the point $(0,0,0) \in \mathcal{N}$ that is distinguished since this is a *singular* point of $\mathcal{N}$.

Proposition 2.2. *If X is smooth then $X^{\mathbb{C}^\times}$ is smooth. In particular, if X is smooth and the set of $\mathbb{C}^\times$-fixed points of X is finite then the schematic fixed points $X^{\mathbb{C}^\times}$ is just the spectrum of $\bigoplus_{p \in X^{\mathbb{C}^\times}} \mathbb{C}$ i.e. just the set of $\mathbb{C}^\times$-fixed points of X without any interesting scheme structure.*

Assume now that we have a pair $X \to Y$, $X^! \to Y^!$ of dual symplectic resolutions. Let $\nu \colon \mathbb{C}^\times \to S_{Y^!}$ be a generic cocharacter of the torus $S_{Y^!}$. This conjecture belongs to Hikita and Nakajima, we will call it *Hikita-Nakajima conjecture*.

Conjecture 2.3. *There is an isomorphism of algebras over $\mathbb{C}[\mathfrak{s}_Y]$:*

$$H^*_{S_Y}(X, \mathbb{C}) \simeq \mathbb{C}[(Y^{!,\mathrm{univ}})^{\nu(\mathbb{C}^\times)}].$$

In nonequivariant form it says that we should have an isomorphism of algebras

$$H^*(X, \mathbb{C}) \simeq \mathbb{C}[(Y^!)^{\nu(\mathbb{C}^\times)}].$$

In the example $T^*\mathbb{P}^1 \to \mathcal{N}$ that we like we have already computed $\mathbb{C}[\mathcal{N}^{\mathbb{C}^\times}]$ and got $\mathbb{C}[a]/a^2$. Note now that

$$H^*(T^*\mathbb{P}^1, \mathbb{C}) = H^*(\mathbb{P}^1, \mathbb{C}) \simeq H^*(S^2, \mathbb{C}) \simeq \mathbb{C}[a]/a^2$$

so the Hikita conjecture is clear in this case.

Let me now formulate the main theorem of this talk. Recall that we have the symplectically dual pair

$$X = \mathfrak{M}(n,r),\ Y^! = (\mathbb{A}^2)^n/(S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n)$$

and $Y^{!,\mathrm{univ}} = \operatorname{Spec} Z_{n,r}$ is the spectrum of the center of $H_{n,r}$ (rational Cherednik algbera corresponding to $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$).

Theorem 2.4. *[K-Shlykov] Hikita-Nakajima conjecture holds for $X = \mathfrak{M}(n,r)$, the ADHM space. More detailed we have an isomorphism of algebras*

$$H^*_S(\mathfrak{M}(n,r), \mathbb{C}) \simeq \mathbb{C}[(\operatorname{Spec} Z_{n,r})^{\mathbb{C}^\times}].$$

Remark 2.5. *For $r = 1$ (i.e. when $\mathfrak{M}(n,r) = \mathrm{Hilb}_n(\mathbb{A}^2)$) the nonequivariant version of this theorem was proved by Hikita. For $n = 1$ the nonequivariant version was again proved by Hikita, the equivariant version (for $n = 1$) follows from the results of Kamnitzer, Tingley, Webster, Weekes, and Yacobi.*

3. Equivariant Hikita-Nakajima conjecture for $\mathrm{Hilb}_n(\mathbb{A}^2)$

Consider the case $r = 1$ so $\mathfrak{M}(n,1) = \mathrm{Hilb}_n(\mathbb{A}^2)$. We want to identify the algebras

$$H^*_{\mathbb{C}^\times}(\mathrm{Hilb}_n(\mathbb{A}^2)),\ \mathbb{C}[\mathfrak{M}_0^{\mathrm{univ}}(n,1)^{\mathbb{C}^\times}].$$

Let us, first of all, study the algebra of equivariant cohomology. Here (since quiver varieties are very well studied) it is not so important that we are dealing with the Hilbert scheme so let us consider any quiver variety $\mathfrak{M}(Q)$ corresponding to some dimension and framing vectors $(v_i)_{i\in I_0}$, $(w_i)_{i\in I_0}$. Recall that to every vertex $i \in I_0$ of Q we can associate the tautological vector bundle on $\mathfrak{M}(Q)$ to be denoted by $\mathcal{V}_i$, the rank of this vector bundle is equal to v_i. For example for $\mathfrak{M}(n,r)$ we have only one tautological bundle $\mathcal{V}$ and its rank is equal to n, this tautological bundle just associates to (X,Y,γ,δ) the vector space V that we use in the definition of $\mathfrak{M}(n,r)$. This is the reason why it is called tautological. Recall also that S is the torus acting symplectically on $\mathfrak{M}(Q)$.

The following facts are known about $H^*_S(\mathfrak{M})$ and follow from the results of Nakajima, MacGerty and Nevins:

(1) $H^*_S(\mathfrak{M}(Q))$ is a free module over $H^*_S(\mathrm{pt})$

(2) As a module over $H^*_S(\mathrm{pt})$ algebra $H^*_S(\mathfrak{M}(Q))$ is generated by $c_i(\mathcal{V}_j)$, $i = 1,\ldots,v_j$.

Consider now the set of S-fixed points of $\mathfrak{M}(Q)$ and denote by ι the embedding

$$\iota\colon \mathfrak{M}(Q)^S \subset \mathfrak{M}(Q).$$

The localization theorem tells us that the pull back homomorphism $H^*_S(\mathfrak{M}(Q)) \to H^*_S(\mathfrak{M}(Q)^S)$ becomes isomorphism after tensoring by the field $\mathbb{C}(\mathfrak{s})$ of the rational functions on $\mathfrak{s} = \mathrm{Lie}\, S$. Since $H^*_S(\mathfrak{M}(Q))$ is a free $\mathbb{C}[\mathfrak{s}]$-module (fact (1)) we conclude that ι^* induces the embedding:

$$\iota^*\colon H^*_S(\mathfrak{M}(Q)) \subset H^*_S(\mathfrak{M}(Q)^S) = \mathbb{C}[\mathfrak{s}]^{|\mathfrak{M}(Q)^S|},\ c_i(\mathcal{V}_j) \mapsto (c_i(\mathcal{V}_j|_p))_{p\in\mathfrak{M}(Q)^S},$$

where $c_i(\mathcal{V}|_p)$ is simply $e_i(\alpha_1,\ldots,\alpha_{v_j})$, here α_k are weights of $\mathfrak{s}$ acting on the fiber $\mathcal{V}_j|_p$ (of $\mathcal{V}_j$ at p) and e_i is the elementary symmetric polynomial.

In the case of $\mathfrak{M}(Q) = \mathrm{Hilb}_n(\mathbb{A}^2)$ the fiber of the tautological bundle $\mathcal{V}$ at the point (ideal) $J \in \mathrm{Hilb}_n(\mathbb{A}^2)$ is the vector space $\mathbb{C}[x,y]/J$. Recall also that $(\mathrm{Hilb}_n(\mathbb{A}^2))^{\mathbb{C}^\times}$ is parametrized by Young diagrams $\mathcal{P}(n)$ with n boxes (partitions of n) and this parametrization is very explicit: it associates to $\mathbb{Y} \in \mathcal{P}(n)$ certain explicit monomial ideal $J_{\mathbb{Y}} \in \mathbb{C}[x,y]$. Using this explicit description of $\mathrm{Hilb}_n(\mathbb{A}^2)^{\mathbb{C}^\times}$ it is easy to see that

$$\iota^*\colon H^*_{\mathbb{C}^\times}(\mathrm{Hilb}_n(\mathbb{A}^2),\mathbb{C}) \subset H^*_{\mathbb{C}^\times}(\mathrm{Hilb}_n(\mathbb{A}^2)^{\mathbb{C}^\times},\mathbb{C}) = \mathbb{C}[\kappa]^{|\mathcal{P}(n)|},$$

is given by

$$c_i(\mathcal{V}) \mapsto (e_i(\kappa\,\mathrm{ct}_1(\mathbb{Y}),\ldots,\kappa\,\mathrm{ct}_n(\mathbb{Y})))_{\mathbb{Y}\in\mathcal{P}(n)},$$

where $\mathrm{ct}_1(\mathbb{Y}),\ldots,\mathrm{ct}_n(\mathbb{Y})$ is the multiset of contents of boxes of $\mathbb{Y}$.

So we see that the algebra $H^*_{\mathbb{C}^\times}(\mathrm{Hilb}_n(\mathbb{A}^2))$ that we are interested in is embedded inside very simple algebra $\mathbb{C}[\kappa]^{|\mathcal{P}(n)|}$ and the embedding is very explicit on generators.

So in order to identify it with $\mathbb{C}[\mathfrak{M}_0^{\mathrm{univ}}(n,1)^{\mathbb{C}^\times}]$ it is enough to:

(a) Construct some embedding $\mathbb{C}[\mathfrak{M}_0^{\mathrm{univ}}(n,1)^{\mathbb{C}^\times}] \subset \mathbb{C}[\kappa]^{|\mathcal{P}(n)|}$

(*b*) Find generators of $\mathbb{C}[\mathfrak{M}_0^{\mathrm{univ}}(n,1)^{\mathbb{C}^\times}]$ such that their images under the embedding above coincide with the images of $c_i(\mathcal{V})$.

Let us describe briefly how to deal with (*a*), (*b*).

We have the resolution morphism

$$\mathfrak{M}^{\mathrm{univ}}(n,1) \to \mathfrak{M}_0^{\mathrm{univ}}(n,1).$$

This morphism is $\mathbb{C}^\times$-invariant so induces morphism at the level of fixed points:

$$(\mathfrak{M}^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times} \to (\mathfrak{M}_0^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times}$$

and so the homomorphism of algebras

$$\mathbb{C}[(\mathfrak{M}_0^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times}] \to \mathbb{C}[(\mathfrak{M}^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times}].$$

Note now that on the LHS we have exactly the algebra that we are interested in and on the RHS we have schematic fixed points of *smooth* variety! Torus $\mathbb{C}^\times$ acts fiberwisely on our deformation $\mathfrak{M}^{\mathrm{univ}}(n,1) \to \mathbb{A}^1$ and fixed points of every fiber can be easily described and are parametrized by $\mathcal{P}(n)$ (as they do for the zero fiber that is $\mathrm{Hilb}_n(\mathbb{A}^2)$).

One can show that the fiber of $\mathfrak{M}^{\mathrm{univ}}(n,1) \to \mathbb{A}^1$ over nonzero point can be identified with so-called Calogero-Moser variety $\mathcal{C}(n)$ that parametrizes pairs of matrices $(X,Y) \in \mathrm{End}(\mathbb{C}^n)$ such that $[X,Y] - \mathrm{id}$ has rank 1:

$$\mathcal{C}(n) := \{(X,Y) \in \mathrm{End}(\mathbb{C}^n)^{\oplus 2} \mid \mathrm{rk}\Big([X,Y] - \mathrm{Id}\Big) = 1\}/\,\mathrm{GL}_n\,.$$

This is a smooth variety and its $\mathbb{C}^\times$-fixed points can be explicitly described (this is the result of Wilson).

We conclude that $\mathbb{C}[(\mathfrak{M}^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times}] = \mathbb{C}[\kappa]^{|\mathcal{P}(n)|}$ and the desired embedding $\mathbb{C}[(\mathfrak{M}_0^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times}] \subset \mathbb{C}[\kappa]^{|\mathcal{P}(n)|}$ is just the pull back of function on $(\mathfrak{M}_0^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times}$ to $(\mathfrak{M}^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times}$. So we have figured out with (*a*).

To deal with (*b*) we just need to find some generators of $\mathbb{C}[(\mathfrak{M}_0^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times}]$ and to compute their values on fixed points. Using results of Lusztig on the generators of $\mathbb{C}[(\mathfrak{M}_0(n,1)]$ one can show that the following functions are the desired generators:

Lemma 3.1. *The algebra* $\mathbb{C}[(\mathfrak{M}_0^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times}]$ *is generated by functions*

$$(X,Y,\gamma,\delta) \mapsto e_i(\alpha_1,\ldots,\alpha_n),$$

here $\alpha_1,\ldots,\alpha_n$ *is the multiset of eigenvalues of the operator* YX *(in other words, our functions above are* $\pm$ *coefficients of the characteristic polynomial of* YX*).*

In order to prove (*b*) and so finish the proof of Hikita-Nakajima conjecture for $\mathfrak{M}(n,1)$ it remains to compute the values of the functions above on the fixed points $(\mathfrak{M}^{\mathrm{univ}}(n,1))^{\mathbb{C}^\times}$. This computation reduces to the calculation of eigenvalues of the operator YX for

$$(X,Y) \in \mathcal{C}(n)^{\mathbb{C}^\times}.$$

Recall that the set $\mathcal{C}(n)^{\mathbb{C}^\times}$ is parametrized by $\mathcal{P}(n)$. It is easy to see that the eigenvalues of YX corresponding to $\mathbb{Y} \in \mathcal{P}(n)$ are precisely the contents of the Young diagram $\mathbb{Y}$ (this was implicitly observed already in Wilson's paper [W]). This finishes the proof.

Consider the example. Assume that $n = 3$. Let us describe the $\mathbb{C}^\times$ fixed points of $\mathcal{C}(3)$. Recall that

$$\mathcal{P}(3) = \left\{ \yng(1,1,1),\ \yng(2,1),\ \yng(3) \right\}.$$

The corresponding $\mathbb{C}^\times$-fixed points of $\mathcal{C}(3)$ can be described as follows:

$$Y_{\yng(1,1,1)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},\ X_{\yng(1,1,1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow YX = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$Y_{\yng(2,1)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},\ X_{\yng(2,1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow YX = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$Y_{\yng(3)} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},\ X_{\yng(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow YX = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We see that the diagonal terms of XY are precisely the contents of the corresponding Young diagrams!

4. Equivariant Hikita-Nakajima conjecture for *ADHM* spaces

Let us now discuss the proof of Hikita-Nakajima conjecture for $\mathfrak{M}(n,r)$ (now we do not assume that $r = 1$). So we want to prove that

$$H^*_S(\mathfrak{M}(n,r),\mathbb{C}) \simeq \mathbb{C}[(\operatorname{Spec} Z_{n,r})^{\mathbb{C}^\times}],$$

where $Z_{n,r} \subset H_{n,r}$ is the center.

The set of S fixed points of $\mathfrak{M}(n,r)$ are now parametrized by the r-multipartitions of n (to be denoted $\mathcal{P}(r,n)$) and as in the Hilbert scheme case we have the embedding

$$H^*_S(\mathfrak{M}(n,r),\mathbb{C}) \subset H^*_S(\mathfrak{M}(n,r)^S,\mathbb{C}) = \mathbb{C}[\mathfrak{s}]^{|\mathcal{P}(r,n)|}$$

which sends generators $c_i(\mathcal{V})$ to certain very explicit elements of $\mathbb{C}[\mathfrak{s}]^{|\mathcal{P}(r,n)|}$ ("shifted multicontents" of $\mathcal{P}(r,n)$). So as in the Hilbert case it remains to construct the corresponding embedding

$$\mathbb{C}[(\operatorname{Spec} Z_{n,r})^{\mathbb{C}^\times}] \subset \mathbb{C}[\mathfrak{s}]^{|\mathcal{P}(r,n)|}$$

and find generators corresponding to $c_i(\mathcal{V})$. Again the same approach as for the Hilbert scheme works to construct the embedding (consider the resolution of Spec $Z_{n,r}$ and pull back to its fixed points). To make it more explicit let us reinterpret this approach in terms of representation theory of $H_{n,r}$. So we want (using representation theory) construct the embedding above, find generators and compute their images (and this will finish the proof of Hikita-Nakajima conjecture for $\mathfrak{M}(n,r)$).

The Rational Cherednik algebra $H_{n,r}$ (corresponding to the group $\Gamma = S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$) is generated by

$$x_1, \dots, x_n, y_1, \dots, y_n, S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$$

subject to certain relations. The $\mathbb{C}^\times$-action with respect to which we are taking (schematic) fixed points corresponds to the following grading on $H_{n,r}$:

$$\deg x_i = 1, \, \deg y_i = -1, \, \deg \Gamma = 0.$$

Recall that the algebra of schematic points that we are studying is defined as

$$Q_{n,r} := Z_{n,r}/(f_i \deg f_i = i \neq 0).$$

The set of irreducible representations of Γ are in bijection with $\mathcal{P}(n,r)$, let $S(\boldsymbol{\lambda})$ be the representation corresponding to $\boldsymbol{\lambda}$. Consider the induced module

$$\Delta(\boldsymbol{\lambda}) := \operatorname{Ind}^{H_{n,r}}_{(\mathbb{C}[\mathfrak{s}]\otimes\mathbb{C}\Gamma)\ltimes\mathbb{C}[x_1,\dots,x_n]}(S(\boldsymbol{\lambda}) \otimes \mathbb{C}[\mathfrak{s}])$$

that can be considered as a (universal) Verma module for $H_{n,r}$ (with highest weight $S(\boldsymbol{\lambda})$) and let $L(\boldsymbol{\lambda})$ be the irreducible quotient of $\Delta(\boldsymbol{\lambda})$.

The center $Z_{n,r}$ acts on $L(\boldsymbol{\lambda})$ via some scalar, this scalar is determined by the action on the highest component and so factors through $Q_{n,r}$. Summing through all representations we get the desired embedding.

$$Q_{n,r} \subset \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}(r,n)} \operatorname{End}_{H_{n,r}}(L(\boldsymbol{\lambda})) = \mathbb{C}[\mathfrak{s}]^{|\mathcal{P}(r,n)|}.$$

Generators of $Q_{n,r}$ come from so called Dunkl-Opdam elements in $H_{n,r}$. They are defined as follows:

$$u_i = \frac{1}{r} y_i x_i + JM_i + ?,$$

where $JM_i \in \mathbb{C}\Gamma$ are the Jucys-Murphy elements of Γ (that act on irreducible representations of Γ via the multiplication by the content of i'th box) and ? corresponds to certain linear combination of generators of $(\mathbb{Z}/r\mathbb{Z})^n$.

The following lemma finishes the proof of the Hikita-Nakajima conjecture for $\mathfrak{M}(n,r)$.

Lemma 4.1. *The algebra $Q_{n,r}$ is generated by the classes of $e_i(u_1,\dots,u_n)$. Via the embedding above the element $e_i(u_1,\dots,u_n)$ maps to the same element as $c_i(\mathcal{V})$.*

5. Possible generalizations and conjectures

Recall that if Q is a quiver and $\mathfrak{M}(Q)$ is the quiver variety then the symplectically dual variety is the Coulomb branch $\mathcal{M}(Q)$ that is defined as the spectrum of a certain algebra of equivariant homology $H_*^{G_V}(\mathcal{R})$ of some space $\mathcal{R} = \mathcal{R}(Q)$ (space of triples). The universal deformation $\mathcal{M}^{\mathrm{univ}}(Q)$ also can be realized in this terms: we need to consider the algebra $H_*^{G_V\times S}(\mathcal{R})$ i.e.

$$\mathcal{M}^{\mathrm{univ}}(Q) = \operatorname{Spec} H_*^{G_V\times S}(\mathcal{R}).$$

We want to identify the algebras

$$H_S^*(\mathfrak{M}(Q), \mathbb{C}), \; \mathbb{C}[(\operatorname{Spec} H_*^{G_V\times S}(\mathcal{R}))^{\mathbb{C}^\times}].$$

Recall that the LHS has generators $c_i(\mathcal{V}_j)$. Note now that the RHS also have similar generators! Indeed for every vertex $i \in I_0$ one can consider the trivial vector $\mathcal{E}_j := \mathcal{R} \times V_j$ with $G_V \times S$-equivarinat structure being induced by the action natural of $G_V = \prod_l \mathrm{GL}(V_l) \twoheadrightarrow \mathrm{GL}(V_j)$ on V_j. Then $c_i(\mathcal{E}_j)$ are interesting elements of $H_*^{G_V \times S}(\mathcal{R})$.

Remark 5.1. *the fact that $[c_i(\mathcal{E}_j)]$ are indeed generators of schematic fixed points follows from the results of Finkelberg, Tsymbaluk, Weekes, and the detailed proof will appear in the work of Kamnitzer, Webster, Weekes and, Yacobi.*

One can show that in the case of $X = \mathfrak{M}(n,r)$ the Hikita-Nakajima isomorphism identifies $c_i(\mathcal{V}_j)$ with $c_i(\mathcal{E}_j)$.

So we can formulate the conjecture:

Conjecture 5.2. *There exists the isomorphism algebras*

$$H_S^*(\mathfrak{M}(Q), \mathbb{C}) \simeq \mathbb{C}[\mathrm{Spec}(H_*^{G_V \times S}(\mathcal{R}(Q))^{\mathbb{C}^\times}] \tag{5.1}$$

that sends $c_i(\mathcal{V}_j) \in H_S^*(\mathfrak{M}(Q), \mathbb{C})$ to $[c_i(\mathcal{E}_j)] \in \mathbb{C}[\mathrm{Spec}(H_*^{G_V \times S}(\mathcal{R}))^{\mathbb{C}^\times}]$, here $i \in I_0$, $k = 1, \ldots, v_i$.

One can try to use the same approach to the proof of this conjecture as we did. Elements $c_i(\mathcal{E}_j)$ can be considered as functions on $\mathcal{M}^{\mathrm{univ}}(Q)$. The goal is to show that their value on $\mathbb{C}^\times$-fixed points of $\mathcal{M}^{\mathrm{univ}}(Q)$ coincide (in the appropriate sense) with the S-characters of $\mathcal{V}_i$ at the S-fixed points of $\mathfrak{M}(Q)$.

Let us also mention that there are many directions in which one can generalize the results that we discussed in this talk. One direction is to still deal with $X = \mathfrak{M}(n,r)$ but replace $H_S^*(\mathfrak{M}(n,r))$ by $K_S(\mathfrak{M}(n,r))$-theory, another direction is to add additional equivariance with respect to the contracting $\mathbb{C}^\times$-action. Another direction is to replace $\mathfrak{M}(n,r)$ with some other quiver variety (for example consider Q to by the cyclic quiver with m vertices).

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