Shape factors for heat conduction inside and outside two-dimensional bodies

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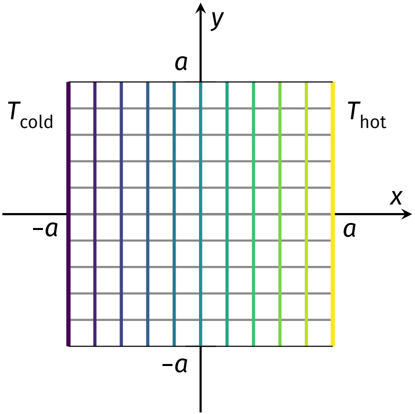
Shape factor inside a square of side length $2a$

One side hot, opposite side cold, other sides adiabatic



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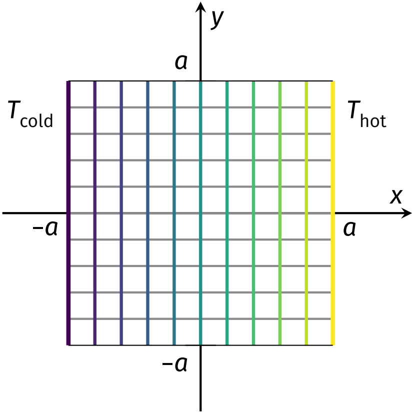
Simple conduction

$Q = k\,2a\,(\Delta T/2a)$

$Q \equiv k\,S\,\Delta T$

so $S = 1$



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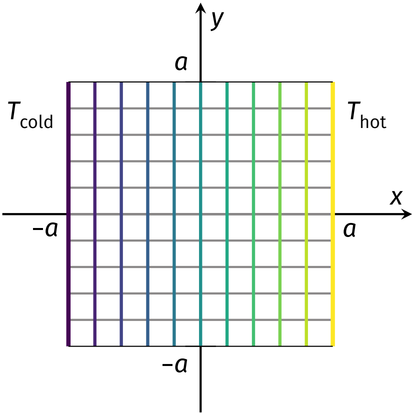
As a flux plot

$n_a = 10$ adiabat increments

$n_i = 10$ isotherm increments

so $S = n_a/n_i = 10/10 = 1$



Flux plot by Jakob and Dow (*Trans. ASME*, 1946)

Checking whether a convection test surface will be isothermal

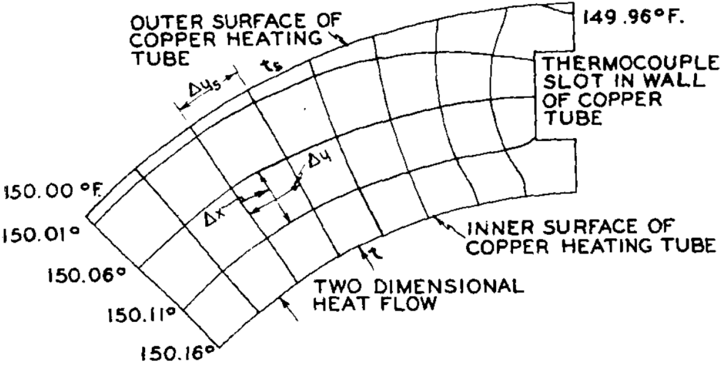
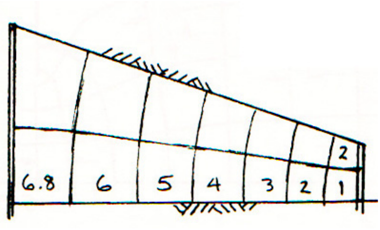


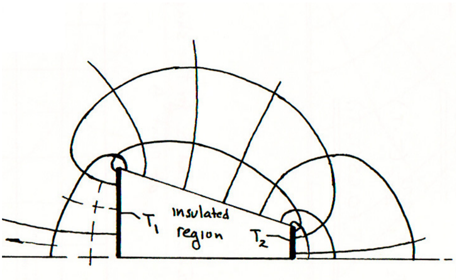
FIG. 8 TEMPERATURE FIELD IN COPPER TUBE WALL

Shape factors for a wedge (AHTT Problem 5.22)

Graphical results are almost equal. **Why?!**



$$S = \frac{2}{6.8} = 0.29$$



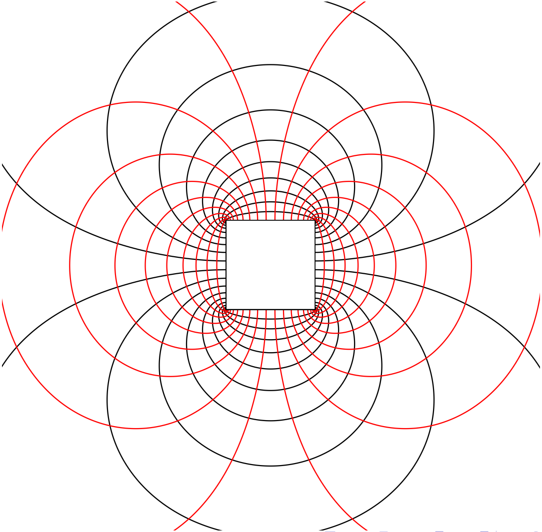
$$S = \frac{2.4}{8} = 0.30$$

Shape factor *outside* a square of side *a*

One side hot, opposite side cold, other sides adiabatic

Not so easy!

FEM solution shown here



Shape factor *outside* a square of side *a*

One side hot, opposite side cold, other sides adiabatic

Not so easy!

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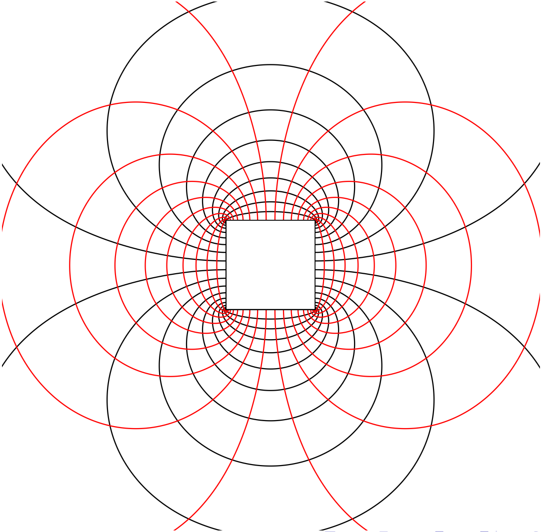
Graphically

n_a = 19 adiab. increments

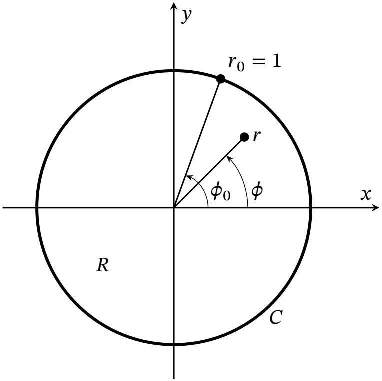
n_i = 19 isoth. increments

so $S = n_a/n_i = 19/19 = 1$



Solve conduction equation inside and outside a disk

Apply Poisson integral formula (from complex variables). Dimensionless temperature: $0 \leqslant \theta \leqslant 1$



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Apply Poisson integral formula (from complex variables). Dimensionless temperature: $0 \leqslant \theta \leqslant 1$

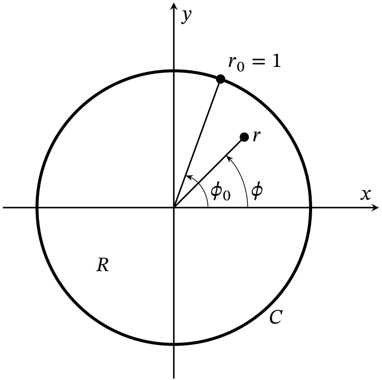
Interior

$$\nabla^2\theta = 0 \qquad \text{for } z \text{ in } R$$
$$\theta(1,\phi) = h(\phi) \quad \text{for } z = (r,\phi) \text{ on } C$$

$$\theta(r,\phi) = \int_{-\pi}^{\pi} P(r,\phi,\phi_0)h(\phi_0)\,d\phi_0$$

Poisson kernel

$$P(r,\phi,\phi_0) = \frac{1}{2\pi}\,\frac{1-r^2}{1+r^2-2r\cos(\phi-\phi_0)}$$



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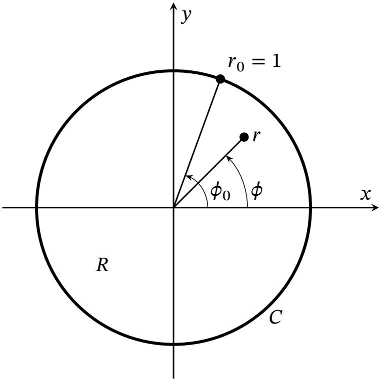
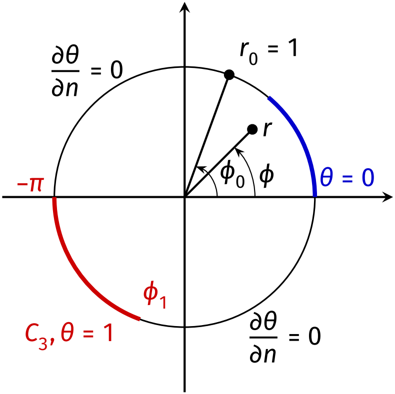
Exterior

$$\theta^e(r,\phi) = -\int_{-\pi}^{\pi} P(r,\phi,\phi_0)h(\phi_0)\,d\phi_0$$



Shape factor for a disk

Integrate heat flux along one isothermal boundary, C_3



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$$\frac{\dot{Q}^i}{k^i(T_3 - T_1)} = S^i = \int_{C_3} \frac{\partial \theta}{\partial n}\, dl = \int_{-\pi}^{\phi_1} \left.\frac{\partial \theta}{\partial r}\right|_{r=1} d\phi$$

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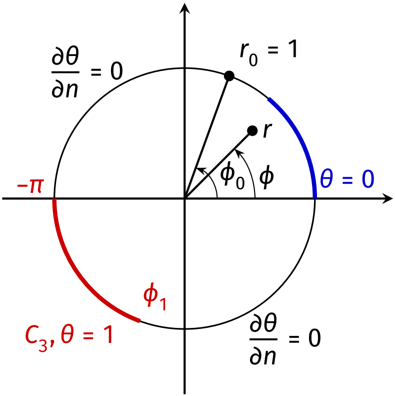

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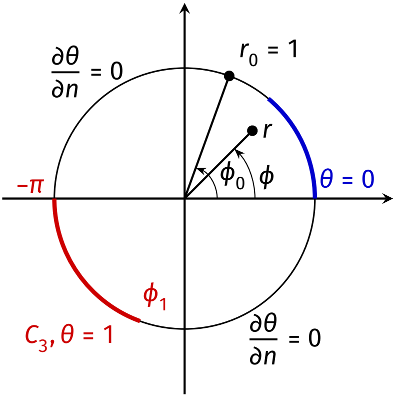

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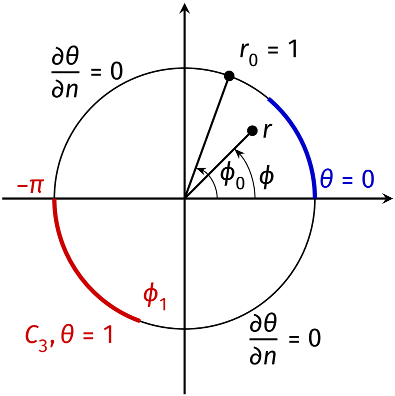

$S^i = S^e$: The interior and exterior shape factors of the disk are always equal!

Boundary condition for $|z| \to \infty$

Note that θ^e has a specific and finite limit:

$$\theta^e_\infty \equiv \lim_{r\to\infty} \theta^e(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi_0)\, d\phi_0$$

Temperature at infinity is average temperature around disk's perimeter.

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Further:

$$\theta^e(z) \sim \theta^e_\infty + a_1/z + a_2/z^2 + \cdots \qquad \text{as } r = |z| \to \infty$$

Heat flux $q \sim 1/r^2$ and heat flow $\dot{Q} \sim 2\pi r/r^2 = 2\pi/r$: **no heat transfer to infinity**.

Conformal maps

Analytic functions that map solutions of Laplace's equation from one shape to another in the complex plane

Riemann mapping theorem: Any simply-connected region R in z-plane can be conformally mapped, one-to-one, onto the unit disk in the w-plane.

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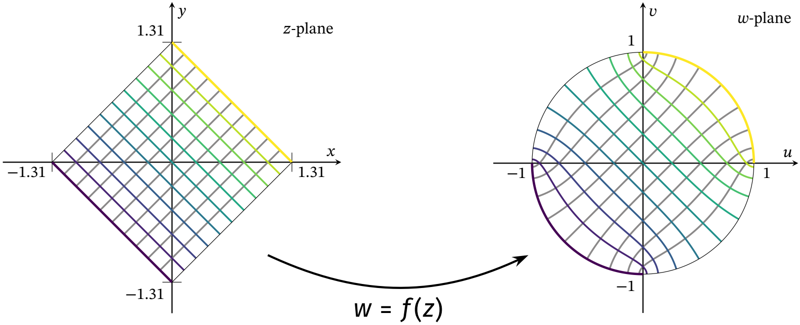
- Isotherms map to isotherms and adiabats map to adiabats
- The boundary of R maps to the disk's circumference.
- The mapped temperature field still satisfies $\nabla^2\theta = 0$

Conformal mapping of square to disk

sn$(u|m)$ and dn$(u|m)$ are complex-valued Jacobi elliptic functions (Schwarz, 1869)

$$w = f(z) = \sqrt{m}\,\frac{\mathrm{sn}\left(z/\sqrt{m} \mid m\right)}{\mathrm{dn}\left(z/\sqrt{m} \mid m\right)}, \quad \text{where } m = 1/2$$



$S = 1$ $\qquad$ $S = 1$

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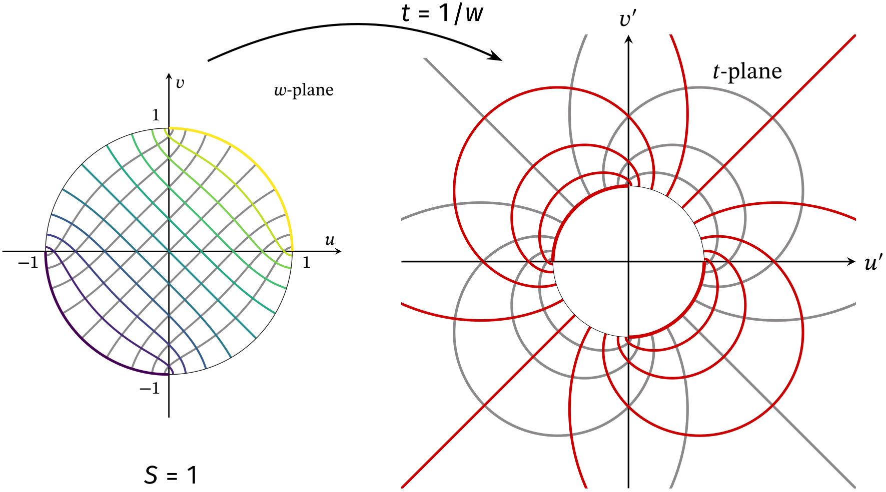
- The mapping is unique to within an arbitrary rotation of the disk.
- The region E exterior to R can also be mapped to the unit disk

Mapping of disk interior to disk exterior

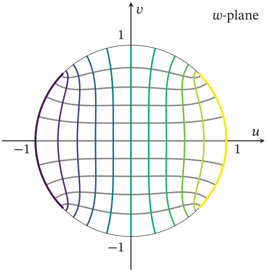
Confirms what we'd already shown with Poisson integral formula



S = 1

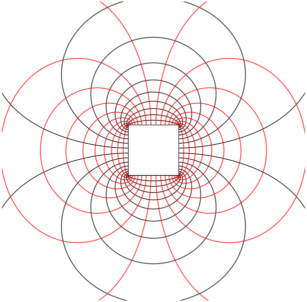
Conformal map of disk interior to exterior of a square

We can map, one-to-one, the square's interior to the disk's interior to the square's exterior



$S = 1$

$z = f(w)$



$S = 1$

$$z = f(w) = \int_{w_0}^{w} \frac{\sqrt{1 - t^4}}{t^2}\, dt$$

Here $|w| \leq 1$, and $|w_0| \to 0$ maps to the point at infinity in extended z-plane.

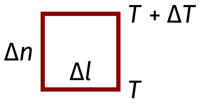
Shape factors are invariant under conformal maps

At each point, the coordinates stretch by a factor *J*

Two isotherms a distance Δn apart are at T and $T + \Delta T$.

The heat flow through a section of length Δl is:

$$\Delta \dot{Q} = k \frac{\Delta T}{\Delta n} \Delta l$$



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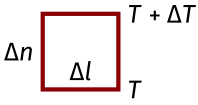
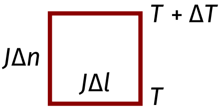


$$\Delta \dot{Q} = k \frac{\Delta T}{\Delta n} \Delta l$$

After mapping, the section has length $J\,\Delta l$. The isotherms have same temperatures but are a distance $J\,\Delta n$ apart:



$$\Delta \dot{Q} = k \frac{\Delta T}{J \Delta n} (J \Delta l) = k \frac{\Delta T}{\Delta n} \Delta l$$

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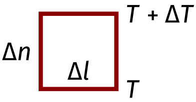

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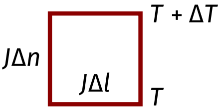

$$\Delta \dot{Q} = k \frac{\Delta T}{J\Delta n}(J\Delta l) = k \frac{\Delta T}{\Delta n} \Delta l$$

$\Delta \dot{Q}$ is the same. Summing over all sections on the unmapped or mapped boundaries (i.e., integrating), gives the same total heat flow, $\dot{Q}$, for each.

Thus, *S* is the same before and after conformal mapping.

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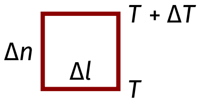

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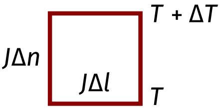

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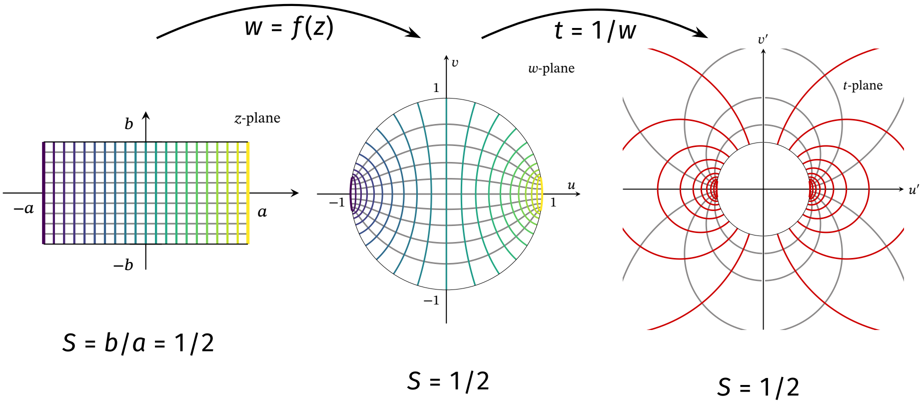
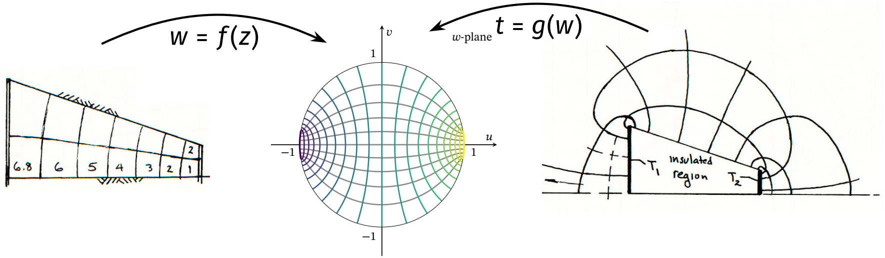
With more math, we can prove: $$\int_{\sigma_z} \vec{n} \cdot \nabla_z T \, dl_z = \int_{\sigma_w} \vec{n} \cdot \nabla_w T \, dl_w$$

Conformal mapping of rectangle to disk ($a : b = 2 : 1$)

$$w = f(z) = \frac{\mathrm{sn}(\lambda z \mid m)\,\mathrm{dn}(\lambda z \mid m)}{\mathrm{cn}(\lambda z \mid m)}$$ for $\lambda = K/2a$ with $K(m)$ the real quarter period



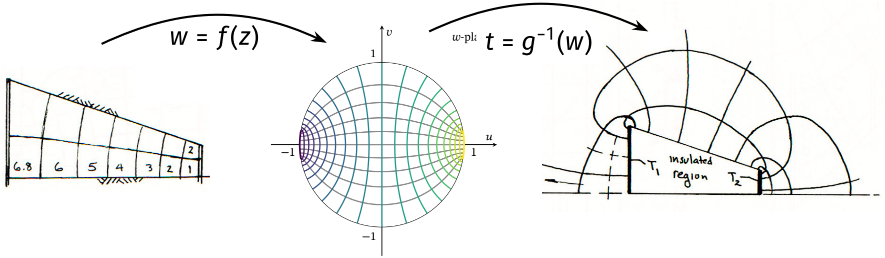
What we have learned from conformal mapping



Riemann: the inside and outside can be mapped to the unit disk.

Disk in center is only schematic, not computed.

What we have learned from conformal mapping

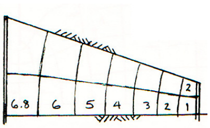


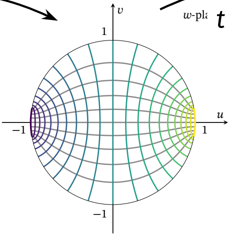
These mappings are 1-1 and so have inverse mappings.

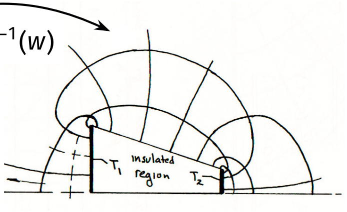
Disk in center is only schematic, not computed.

What we have learned from conformal mapping

$w = f(z)$ $\qquad$ $t = g^{-1}(w)$







$S = 0.3$ $\qquad$ $S = 0.3$ $\qquad$ $S = 0.3$

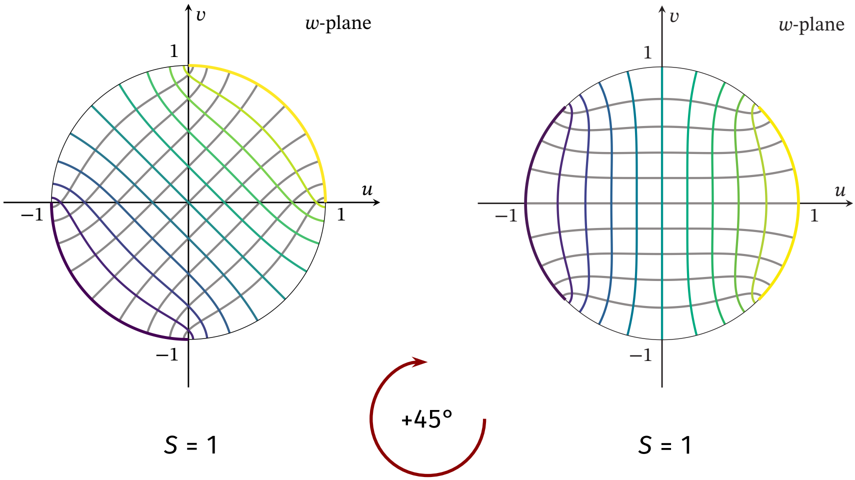
The shape factors are the same before and after mappings.

Therefore, the interior and exterior shape factors are equal.

Disk in center is only schematic, not computed.

Shape factors are invariant under rotation of b.c.s

The disk is conformally mapped to the square by: $z = \int_0^w \frac{ds}{\sqrt{1 - s^4}}$



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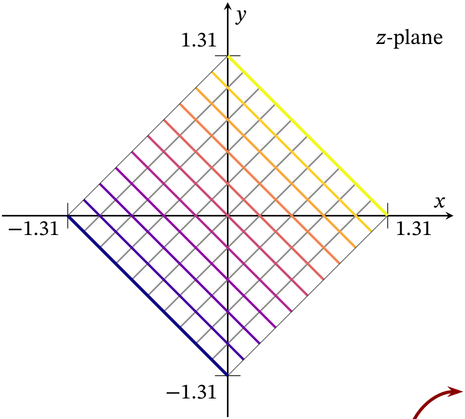
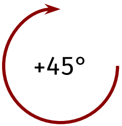
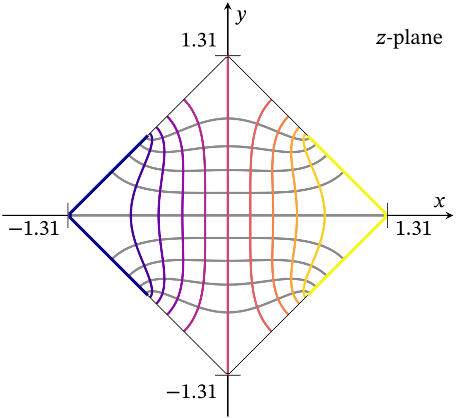
Yin-Yang bodies are equivalent through rotation of a 90° unit disk prior to mapping



$S = 1$





$S = 1$

Shape factors for "Yin-Yang" bodies are $S = 1$

Isothermal and adiabatic edges are interchanged across an axis of symmetry

J.H. Lienhard (IV), 1981, *J. Heat Transfer*, **103**(3):600–1

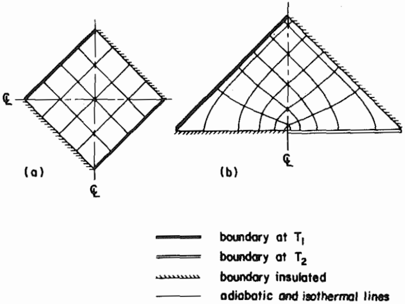


Fig. 1 Two Yin-Yang figures

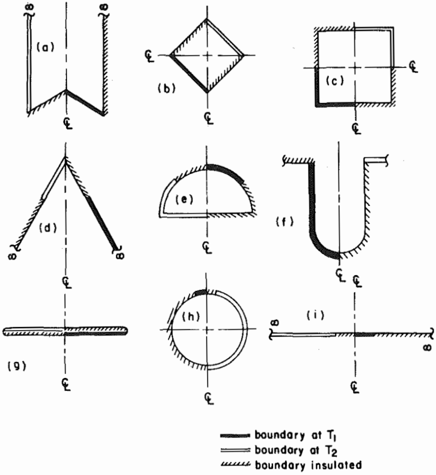
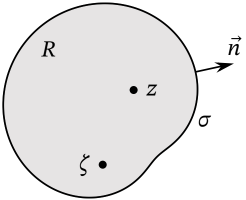
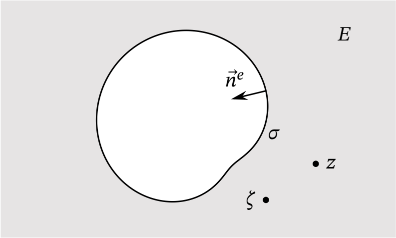


Fig. 2 Nine more Yin-Yang figures

Green's functions an arbitrary 2D exterior region

See paper for details!!





$$\theta(z) = -\int_\sigma \frac{\partial g(z|\zeta)}{\partial n_\zeta}\, h(\zeta)\, dl_\zeta \equiv \int_\sigma I(z|\zeta)\, h(\zeta)\, dl_\zeta$$

The boundary influence function, $I(z|\zeta)$, is the temperature at z produced by a delta-function boundary temperature at ζ (a unit-strength point source).

If $f(z,\zeta)$ takes $z \in R$ to the unit disk and the point $\zeta \in R$ to $w = 0$:

$$g(z|\zeta) = -\frac{1}{2\pi}\log|f(z,\zeta)|$$

Summary

Interior shape factors in two-dimensions are the same as exterior shape factors

1. *Shape factors for conduction inside an object are equal to those for conduction through the material outside the object,*
   a. if the only heat sources and sinks are the isothermal boundary sections
   b. no net heat transfer to the exterior region far away
   c. interior and exterior conductivities must be uniform, but not equal

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    a. For the unit disk using the Poisson integral formula
    b. For arbitrary shapes by conformally mapping to and from the unit disk
    c. And for arbitrary shapes using Green's functions (see paper for details)

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3. A simple geometrical proof shows that shape factors are invariant under conformal mapping (a mathy proof is in the paper)
4. The "Yin-Yang" shape factors with $S = 1$, described in 1981, have been explained as rotations of the unit disk prior to mapping.

Thank you!

To read more, see this paper:

J. H. Lienhard V, “Exterior shape factors from interior shape factors,” *J. Heat Transfer*, **141**(6):061301, June 2019.

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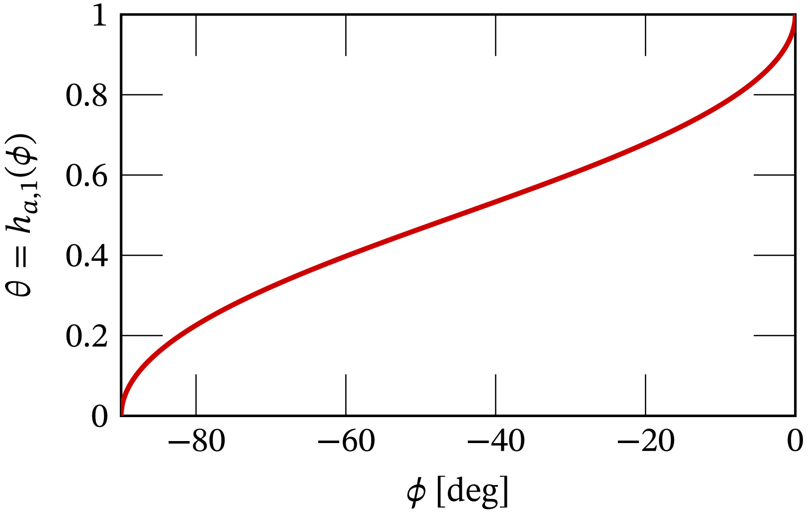
Supplementary slides

Temperature distribution on 90° adiabatic edge of disk



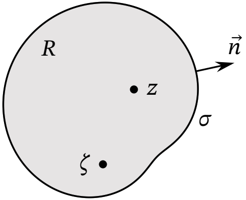
Green's functions for an arbitrary 2D region, R

The Green's function $g(z|\zeta)$ is the solution of:

$$-\nabla^2 g = \delta(\zeta - z)$$

z and ζ in R

$g = 0$ for ζ on σ



Green's second identity

$$\int_R \left[g\nabla^2\theta - \theta\nabla^2 g \right] dR_\zeta = \int_\sigma \left[g\frac{\partial\theta}{\partial n_\zeta} - \theta\frac{\partial g}{\partial n_\zeta} \right] dl_\zeta$$

where subscript ζ means differentiation/integration w.r.t. ζ.

Green's functions for an arbitrary 2D region, R

The Green's function $g(z|\zeta)$ is the solution of:

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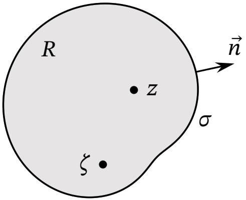
$$z \text{ and } \zeta \text{ in } R$$
$$g = 0 \text{ for } \zeta \text{ on } \sigma$$



Green's second identity

$$\int_R \left[\cancelto{0}{g\nabla^2\theta} - \theta\nabla^2 g\right] dR_\zeta = \int_\sigma \left[\cancelto{0}{g\frac{\partial\theta}{\partial n_\zeta}} - \theta\frac{\partial g}{\partial n_\zeta}\right] dl_\zeta$$

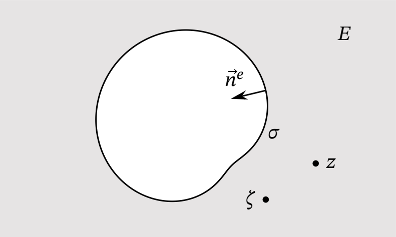
where subscript ζ means differentiation/integration w.r.t. ζ. Then:

$$\theta(z) = -\int_\sigma \frac{\partial g(z|\zeta)}{\partial n_\zeta} h(\zeta)\, dl_\zeta \equiv \int_\sigma I(z|\zeta)\, h(\zeta)\, dl_\zeta$$

The boundary influence function, $I(z|\zeta)$, is the temperature at z produced by a delta-function boundary temperature at ζ (a unit-strength point source).

Green's functions an arbitrary 2D exterior region, E

For the exterior region E, the outward normal direction, $\vec{n}^e$, is opposite $\vec{n}$. The rest is the same. (We also require $g^e(z,\zeta)$ to give bounded solution for θ^e as $|z| \to \infty$.)



$$\theta^e(z) = -\int_\sigma \frac{\partial g^e(z|\zeta)}{\partial n^e_\zeta}\, h(\zeta)\, dl_\zeta = \int_\sigma \frac{\partial g^e(z|\zeta)}{\partial n_\zeta}\, h(\zeta)\, dl_\zeta$$

Since the only temperature sources are on the boundary σ, the exterior solution is also given by the boundary influence function, respecting the change in normal direction:

$$\theta^e(z) = -\int_\sigma I(z|\zeta)\, h(\zeta)\, dl_\zeta$$

Interior and exterior shape factors of R are equal

Let the boundary σ be a chain four curves: σ_1 isothermal at $\theta = 0$; σ_2 and σ_4 adiabatic; and $\boldsymbol{\sigma_3}$ **isothermal at** $\boldsymbol{\theta} = \mathbf{1}$.

The shape factor for the interior is:

$$S^i = \int_{\sigma_3} \frac{\partial \theta}{\partial n_z}\, dl_z = -\int_{\sigma_3} \frac{\partial}{\partial n_z} \int_{\sigma} \frac{\partial g(z|\zeta)}{\partial n_\zeta}\, h(\zeta)\, dl_\zeta dl_z = +\int_{\sigma_3} \frac{\partial}{\partial n_z} \int_{\sigma} I(z|\zeta)\, h(\zeta)\, dl_\zeta dl_z$$

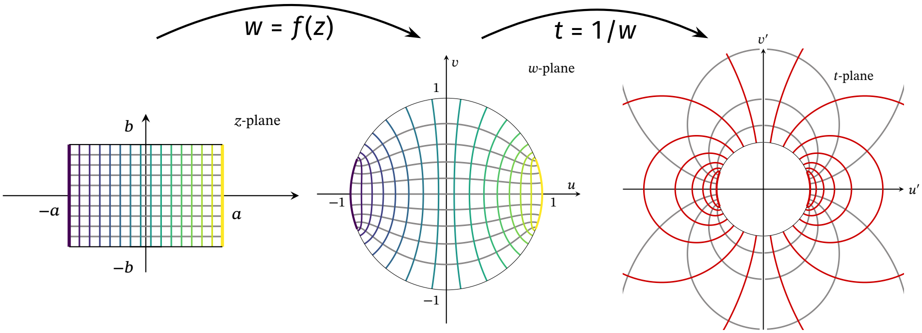
For the exterior, the normal direction is reversed

$$S^e = \int_{\sigma_3} \frac{\partial \theta^e}{\partial n_z^e}\, dl_z = -\int_{\sigma_3} \frac{\partial}{\partial n_z^e} \int_{\sigma} I(z|\zeta)\, h(\zeta)\, dl_\zeta dl_z = +\int_{\sigma_3} \frac{\partial}{\partial n_z} \int_{\sigma} I(z|\zeta)\, h(\zeta)\, dl_\zeta dl_z$$

Comparing eqn. (5) to eqn. (5), we see again that $\boldsymbol{S^e} = \boldsymbol{S^i}$.

Conformal mapping of rectangle to disk ($a : b = 3 : 2$)

$$w = f(z) = \frac{\mathrm{sn}(\lambda z \mid m)\, \mathrm{dn}(\lambda z \mid m)}{\mathrm{cn}(\lambda z \mid m)}$$ for $\lambda = K/2a$ with $K(m)$ the real quarter period



$S = b/a = 2/3$

$S = 2/3$

$S = 2/3$

Line integral of a normal derivative

The integral defining the shape factor is unchanged by a conformal map

Consider an integral in the mapped w-plane

$$\int_\sigma \frac{\partial T}{\partial n}\, dl = \int_\sigma \vec{n} \cdot \nabla T\, dl = \int_\sigma \nabla^\perp T \cdot d\vec{w}$$

in which the skew gradient is

$$\nabla^\perp T \equiv \begin{pmatrix} \partial T/\partial v \\ -\partial T/\partial u \end{pmatrix}$$

The transformation of $d\vec{w}$ to the z-plane is

$$d\vec{w} = \begin{pmatrix} du \\ dv \end{pmatrix} = \underbrace{\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}}_{=J_1} \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad (1)$$

With the Cauchy-Riemann conditions, $|J_1| = (\partial u/\partial x)^2 + (\partial u/\partial y)^2$

$$J_1 = |J_1| \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

where $\alpha^2 + \beta^2 = 1$. $|J_1|$ is:

$$\left|\frac{\partial u}{\partial z}\right|^2 = \frac{\partial u}{\partial z}\overline{\frac{\partial u}{\partial z}} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Similarly,

$$\begin{aligned} \nabla^\perp T &= \begin{pmatrix} \partial T/\partial v \\ -\partial T/\partial u \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} \partial y/\partial v & -\partial x/\partial v \\ -\partial y/\partial u & \partial x/\partial u \end{pmatrix}}_{=J_2} \begin{pmatrix} \partial T/\partial y \\ -\partial T/\partial x \end{pmatrix} \qquad (2) \\ &= |J_2| \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \partial T/\partial y \\ -\partial T/\partial x \end{pmatrix} \end{aligned}$$

Line integral of $\vec{n} \cdot \nabla T$ is unchanged by conformal mapping

$|J_2| = (\partial x/\partial u)^2 + (\partial y/\partial u)^2$ and

$$\left|\frac{\partial z}{\partial u}\right|^2 = \frac{\partial z}{\partial u}\frac{\partial \bar{z}}{\partial u} = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2$$

Thus, $|J_1|\,|J_2| = |\partial u/\partial z|^2\,|\partial z/\partial u|^2 = 1$.
For vectors $\vec{a}$ and $\vec{b}$ and matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\begin{aligned}(\mathbf{A}\vec{a}) \cdot \left(\mathbf{B}\vec{b}\right) &= (\mathbf{A}\vec{a})^T \left(\mathbf{B}\vec{b}\right) \\ &= \vec{a}^T\mathbf{A}^T \left(\mathbf{B}\vec{b}\right) = \vec{a}^T \left(\mathbf{A}^T\mathbf{B}\right)\vec{b}\end{aligned}$$

Then, using eqns. (1) and (2),

$$\nabla^{\perp} T \cdot d\vec{w} = \begin{pmatrix} \partial T/\partial y \\ -\partial T/\partial x \end{pmatrix}^T J_2^T\, J_1 \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Multiplication of the Jacobian matrices produces a considerable simplification:

$$\begin{aligned} J_2^T\, J_1 &= |J_2|\,|J_1| \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \\ &= |J_1|\,|J_2| \begin{pmatrix} \alpha^2+\beta^2 & 0 \\ 0 & \alpha^2+\beta^2 \end{pmatrix} \\ &= |J_1|\,|J_2| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Putting these pieces together, denoting the w and z planes by subscripts, we find that:

$$\boxed{\int_{\sigma_w} \vec{n} \cdot \nabla_w T\, dl_w = \int_{\sigma_w} \nabla_w^{\perp} T \cdot d\vec{w} = \int_{\sigma_z} \nabla_z^{\perp} T \cdot d\vec{z} = \int_{\sigma_z} \vec{n} \cdot \nabla_z T\, dl_z}$$

Riemann mapping theorem in full

Riemann mapping theorem: for a plane simply-connected region R with boundary σ containing an interior point ζ, there exists a function $w = f(z, \zeta)$, analytic on R, that conformally maps R one-to-one onto the unit disk in the w-plane, taking σ to the disk's circumference and ζ to $w = 0$. When $\zeta = 0$, we will simply write $w = f(z)$. The mapping is unique to within an arbitrary rotation of the disk.