Scattering matrix derivation from GSTC

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1 Matrix form of GSTCs

The Generalized Sheet Transition Conditions (GSTCs) on the electromagnetic fields due to an infinitesimally thin conducting sheet placed in the $z = 0$ plane as given in [1] are equivalent to:

$$\hat{\boldsymbol{z}} \times [[\boldsymbol{H}_{\parallel}]] = \frac{\boldsymbol{\sigma}^e}{2} \{\{\boldsymbol{E}_{\parallel}\}\} \tag{1}$$

$$-\hat{\boldsymbol{z}} \times [[\boldsymbol{E}_{\parallel}]] = \frac{\boldsymbol{\sigma}^m}{2} \{\{\boldsymbol{H}_{\parallel}\}\} \tag{2}$$

where for any vector tangential to the sheet, $\boldsymbol{v}_{\parallel} = (v_x, v_y)^T$, we define

$$[[\boldsymbol{v}_{\parallel}]] = \lim_{z \to 0^+} \boldsymbol{v}_{\parallel}(z) - \lim_{z \to 0^-} \boldsymbol{v}_{\parallel}(z) \tag{3}$$

$$\{\{\boldsymbol{v}_{\parallel}\}\} = \lim_{z \to 0^+} \boldsymbol{v}_{\parallel}(z) + \lim_{z \to 0^-} \boldsymbol{v}_{\parallel}(z). \tag{4}$$

In addition, we define $\boldsymbol{E}_{\parallel}$ as the tangential electric field, $\boldsymbol{H}_{\parallel}$ as the tangential auxiliary field, $\boldsymbol{\sigma}^e$ the electric sheet conductivity, and $\boldsymbol{\sigma}^m$ the magnetic sheet conductivity.

For convenience, we may also label each face of the sheet by a port. Let's also enumerate the ports so that they increment along the +z axis, so let $\boldsymbol{v}_{\parallel}^1 = \lim_{z \to 0^-} \boldsymbol{v}_{\parallel}(z)$ and $\boldsymbol{v}_{\parallel}^2 = \lim_{z \to 0^+} \boldsymbol{v}_{\parallel}(z)$.

Also observe that the curl operator $\hat{\boldsymbol{z}}\times$ can be written as the $\pi/2$ rotation matrix $\boldsymbol{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which has $\boldsymbol{R}^{-1} = -\boldsymbol{R}$. Thus we can also write the GSTCs as

$$\boldsymbol{R}(\boldsymbol{H}_{\parallel}^2 - \boldsymbol{H}_{\parallel}^1) = \frac{\boldsymbol{\sigma}^e}{2} (\boldsymbol{E}_{\parallel}^1 + \boldsymbol{E}_{\parallel}^2) \tag{5}$$

$$\boldsymbol{R}^{-1}(\boldsymbol{E}_{\parallel}^2 - \boldsymbol{E}_{\parallel}^1) = \frac{\boldsymbol{\sigma}^m}{2} (\boldsymbol{H}_{\parallel}^1 + \boldsymbol{H}_{\parallel}^2) \tag{6}$$

2 Planewave decomposition in periodic sheet

Assuming that the sheet is periodic along both orthogonal, tangential axes, Bloch's theorem implies that the solution of the fields along those axes can be decomposed along a

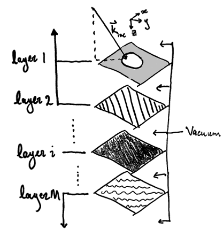

Figure 1: Geometry of the unit cell of a stack of conducting sheets

discrete spectrum, λ_{nm}, of planewaves of the form $\psi_{nm}(z)\exp\big(i(\boldsymbol{k}_{\parallel,nm}\cdot\boldsymbol{r}_{\parallel}-\omega t)\big)$. In particular, in `DeltaRCWA.jl` we assume that the medium through which the fields propagates is a uniform medium (i.e. vacuum) whose eigenmodes are planewaves, so the solutions are of the form $\exp(i(\boldsymbol{k}_{nm}\cdot\boldsymbol{r}-\omega t))$ where due to the dispersion of electromagnetic waves in a uniform medium, $\boldsymbol{k}\cdot\boldsymbol{k}=\omega^2\epsilon\mu$, we determine $k_z^{\pm}=\pm k_z=\pm\sqrt{\omega^2\epsilon\mu-k_{x,n}^2-k_{y,m}^2}$.

We may rewrite the GSTCs in the planewave/Fourier basis by Fourier transforming the position-basis equations in each component of the fields with the operator

$$\boldsymbol{\mathcal{F}}_{\parallel}=\begin{pmatrix}\boldsymbol{\mathcal{F}} & 0\\ 0 & \boldsymbol{\mathcal{F}}\end{pmatrix} \tag{7}$$

where $\boldsymbol{\mathcal{F}}$ is the (Discrete) Fourier transform and $\boldsymbol{\mathcal{F}}^*$ its adjoint/inverse:

$$\boldsymbol{\mathcal{F}}_{\parallel}\boldsymbol{R}\boldsymbol{\mathcal{F}}_{\parallel}^*(\tilde{\boldsymbol{H}}_{\parallel}^2-\tilde{\boldsymbol{H}}_{\parallel}^1)=\boldsymbol{R}(\tilde{\boldsymbol{H}}_{\parallel}^2-\tilde{\boldsymbol{H}}_{\parallel}^1)=\frac{\tilde{\boldsymbol{\sigma}}^e}{2}(\tilde{\boldsymbol{E}}_{\parallel}^1+\tilde{\boldsymbol{E}}_{\parallel}^2) \tag{8}$$

$$\boldsymbol{\mathcal{F}}_{\parallel}\boldsymbol{R}^{-1}\boldsymbol{\mathcal{F}}_{\parallel}^*(\tilde{\boldsymbol{E}}_{\parallel}^2-\tilde{\boldsymbol{E}}_{\parallel}^1)=\boldsymbol{R}^{-1}(\tilde{\boldsymbol{E}}_{\parallel}^2-\tilde{\boldsymbol{E}}_{\parallel}^1)=\frac{\tilde{\boldsymbol{\sigma}}^m}{2}(\tilde{\boldsymbol{H}}_{\parallel}^1+\tilde{\boldsymbol{H}}_{\parallel}^2) \tag{9}$$

Here a quantity with a tilde is in the Fourier basis, i.e. $\tilde{\boldsymbol{H}}=\boldsymbol{\mathcal{F}}_{\parallel}\boldsymbol{H}$ for vectors and $\tilde{\boldsymbol{\sigma}}=\boldsymbol{\mathcal{F}}_{\parallel}\boldsymbol{\sigma}\boldsymbol{\mathcal{F}}_{\parallel}^*$ for matrices.

3 Planewave solutions to Maxwell's equations

Maxwell's curl equations

$$-\mu\partial_t\boldsymbol{H}=\nabla\times\boldsymbol{E} \tag{10}$$

$$\epsilon\partial_t\boldsymbol{E}=\nabla\times\boldsymbol{H}, \tag{11}$$

take the following form for planewave solutions:

$$\mu\omega\tilde{\boldsymbol{H}}=\boldsymbol{k}_{nm}\times\tilde{\boldsymbol{E}} \tag{12}$$

$$-\epsilon\omega\tilde{\boldsymbol{E}}=\boldsymbol{k}_{nm}\times\tilde{\boldsymbol{H}}. \tag{13}$$

For brevity I will drop the nm subscript for the modes, which is implied by the tilde signifying the planewave basis. In particular, the equations for the $\hat{\boldsymbol{z}}$ components imply

$$\omega\mu\tilde{H}_z = k_x\tilde{E}_y - k_y\tilde{E}_x \tag{14}$$
$$-\omega\epsilon\tilde{E}_z = k_x\tilde{H}_y - k_y\tilde{H}_x, \tag{15}$$

which substituted into rescaled equations for the $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ components give

$$-\omega^2\epsilon\mu\tilde{H}_x = -\omega\epsilon(k_y\tilde{E}_z - k_z\tilde{E}_y) \qquad = k_y(k_x\tilde{H}_y - k_y\tilde{H}_x) + \omega\epsilon k_z\tilde{E}_y \tag{16}$$
$$-\omega^2\epsilon\mu\tilde{H}_y = -\omega\epsilon(k_z\tilde{E}_x - k_x\tilde{E}_z) \qquad = -\omega\epsilon k_z\tilde{E}_x - k_x(k_x\tilde{H}_y - k_y\tilde{H}_x) \tag{17}$$
$$-\omega^2\epsilon\mu\tilde{E}_x = \omega\mu(k_y\tilde{H}_z - k_z\tilde{H}_y) \qquad = k_y(k_x\tilde{E}_y - k_y\tilde{E}_x) - \omega\mu k_z\tilde{H}_y \tag{18}$$
$$-\omega^2\epsilon\mu\tilde{E}_y = \omega\mu(k_z\tilde{H}_x - k_x\tilde{H}_z) \qquad = \omega\mu k_z\tilde{H}_x - k_x(k_x\tilde{E}_y - k_y\tilde{E}_x) \tag{19}$$

which can be rewritten in matrix form as

$$(\boldsymbol{R}\boldsymbol{K}\boldsymbol{R}^{-1} - \omega^2\epsilon\mu\boldsymbol{I})\tilde{\boldsymbol{H}}_\parallel = \omega\epsilon\boldsymbol{R}^{-1}k_z\tilde{\boldsymbol{E}}_\parallel \tag{20}$$
$$(\boldsymbol{R}\boldsymbol{K}\boldsymbol{R}^{-1} - \omega^2\epsilon\mu\boldsymbol{I})\tilde{\boldsymbol{E}}_\parallel = \omega\mu\boldsymbol{R}k_z\tilde{\boldsymbol{H}}_\parallel \tag{21}$$

where

$$\boldsymbol{K} = \begin{pmatrix} k_xk_x & k_xk_y \\ k_yk_x & k_yk_y \end{pmatrix}. \tag{22}$$

4 Obtaining the scattering matrix

4.1 Eliminating the electric field

Taking equations (8) and (21), and eliminating the electric field yields

$$\boldsymbol{R}(\tilde{\boldsymbol{H}}_\parallel^2 - \tilde{\boldsymbol{H}}_\parallel^1) = \frac{\tilde{\boldsymbol{\sigma}}^e}{2}(\boldsymbol{R}\boldsymbol{K}\boldsymbol{R}^{-1} - \omega^2\epsilon\mu\boldsymbol{I})^{-1}\omega\mu\boldsymbol{R}k_z(\tilde{\boldsymbol{H}}_\parallel^1 + \tilde{\boldsymbol{H}}_\parallel^2). \tag{23}$$

If we know $\boldsymbol{\rho}^e = (\boldsymbol{\sigma}^e)^{-1}$ we can also rewrite this as

$$(\boldsymbol{R}\boldsymbol{K}\boldsymbol{R}^{-1} - \omega^2\epsilon\mu\boldsymbol{I})\tilde{\boldsymbol{\rho}}^e\boldsymbol{R}(\tilde{\boldsymbol{H}}_\parallel^2 - \tilde{\boldsymbol{H}}_\parallel^1) = \frac{1}{2}\omega\mu\boldsymbol{R}k_z(\tilde{\boldsymbol{H}}_\parallel^1 + \tilde{\boldsymbol{H}}_\parallel^2), \tag{24}$$

which is easier to compute because there is no need for matrix inversions if the data for the resistivity is specified. Also, resistivities allow for easy descriptions of ideal conductors ($\boldsymbol{\rho}^e = 0$), which is a sensible approximation for metallic materials, however it is inconvenient to express a nonconducting vacuum with a infinite resistivity. If we left multiply the preceding equation by $\boldsymbol{R}^{-1}$ it simplifies to

$$(\boldsymbol{K} - \omega^2\epsilon\mu\boldsymbol{I})\boldsymbol{R}^{-1}\tilde{\boldsymbol{\rho}}^e\boldsymbol{R}(\tilde{\boldsymbol{H}}_\parallel^2 - \tilde{\boldsymbol{H}}_\parallel^1) = \frac{1}{2}\omega\mu k_z(\tilde{\boldsymbol{H}}_\parallel^1 + \tilde{\boldsymbol{H}}_\parallel^2). \tag{25}$$

Taking equations (9) and (21), and eliminating the electric field yields

$$2\omega\mu\boldsymbol{R}k_z(\tilde{\boldsymbol{H}}_\parallel^2 - \tilde{\boldsymbol{H}}_\parallel^1) = (\boldsymbol{R}\boldsymbol{K}\boldsymbol{R}^{-1} - \omega^2\epsilon\mu\boldsymbol{I})\boldsymbol{R}\tilde{\boldsymbol{\sigma}}^m(\tilde{\boldsymbol{H}}_\parallel^1 + \tilde{\boldsymbol{H}}_\parallel^2), \tag{26}$$

which can be simplified by multiplying through by R^{-1} to obtain

$$2\omega\mu k_z(\tilde{\boldsymbol{H}}_\parallel^2 - \tilde{\boldsymbol{H}}_\parallel^1) = (K - \omega^2\epsilon\mu\boldsymbol{I})\tilde{\boldsymbol{\sigma}}^m(\tilde{\boldsymbol{H}}_\parallel^1 + \tilde{\boldsymbol{H}}_\parallel^2). \tag{27}$$

On the other hand, equations (9) and (20), and eliminating the electric field yields

$$(\boldsymbol{R}\boldsymbol{K}\boldsymbol{R}^{-1} - \omega^2\epsilon\mu\boldsymbol{I})(\tilde{\boldsymbol{H}}_\parallel^2 - \tilde{\boldsymbol{H}}_\parallel^1) = \frac{1}{2}\omega\epsilon k_z\tilde{\boldsymbol{\sigma}}^m(\tilde{\boldsymbol{H}}_\parallel^1 + \tilde{\boldsymbol{H}}_\parallel^2). \tag{28}$$

A problem arises from this equation: the scattering matrix created from it is singular.

We have obtained the easiest to solve forms of these equations for the H field. If we had instead eliminated the H field, we would have obtained identical equations for the E field except replacing $\epsilon \leftrightarrow \mu$, $\boldsymbol{\rho}^e \to \boldsymbol{\rho}^m$, and $\boldsymbol{\sigma}^m \to \boldsymbol{\sigma}^e$, which makes less sense since it is impossible to have $\boldsymbol{\rho}^m = 0$ since there are no magnetic charges and thus no magnetic conductors.

4.2 Incident and scattered components

Each mode can propagate in either the $+\hat{\boldsymbol{z}}$ or $-\hat{\boldsymbol{z}}$ direction, so their amplitudes combine

$$\tilde{\boldsymbol{H}}_\parallel^i = \tilde{\boldsymbol{H}}_\parallel^{(i,+)} + \tilde{\boldsymbol{H}}_\parallel^{(i,-)} \tag{29}$$

Note that in this notation for planewaves, $\partial_z\tilde{\boldsymbol{H}}_\parallel^{(i,\pm)} = \pm ik_z\tilde{\boldsymbol{H}}_\parallel^{(i,\pm)}$, however, the instances of k_z were assumed to have positive sign regardless so, we reinterpret the previous instances as being $k_z^\pm = \pm k_z$, with sign matching that of $\tilde{\boldsymbol{H}}^{(i,\pm)}$. After substituting the incident and scattered components, both equations (25) and (28) have the form

$$A\Big(\tilde{\boldsymbol{H}}_\parallel^{(2,+)} + \tilde{\boldsymbol{H}}_\parallel^{(2,-)} - \tilde{\boldsymbol{H}}_\parallel^{(1,+)} - \tilde{\boldsymbol{H}}_\parallel^{(1,-)}\Big) \tag{30}$$

$$= B^\pm\Big(\tilde{\boldsymbol{H}}_\parallel^{(2,+)} + \tilde{\boldsymbol{H}}_\parallel^{(2,-)} + \tilde{\boldsymbol{H}}_\parallel^{(1,+)} + \tilde{\boldsymbol{H}}_\parallel^{(1,-)}\Big) \tag{31}$$

$$= B\Big(\tilde{\boldsymbol{H}}_\parallel^{(2,+)} - \tilde{\boldsymbol{H}}_\parallel^{(2,-)} + \tilde{\boldsymbol{H}}_\parallel^{(1,+)} - \tilde{\boldsymbol{H}}_\parallel^{(1,-)}\Big) \tag{32}$$

which after moving the incident components, $(1,+)$ and $(2,-)$ to one side and the scattered components, $(1,-)$ and $(2,+)$ to the other becomes

$$A\Big(\tilde{\boldsymbol{H}}_\parallel^{(2,+)} - \tilde{\boldsymbol{H}}_\parallel^{(1,-)}\Big) - B\Big(\tilde{\boldsymbol{H}}_\parallel^{(2,+)} - \tilde{\boldsymbol{H}}_\parallel^{(1,-)}\Big) \tag{33}$$

$$= -A\Big(\tilde{\boldsymbol{H}}_\parallel^{(2,-)} - \tilde{\boldsymbol{H}}_\parallel^{(1,+)}\Big) + B\Big(-\tilde{\boldsymbol{H}}_\parallel^{(2,-)} + \tilde{\boldsymbol{H}}_\parallel^{(1,+)}\Big) \tag{34}$$

and write it in matrix form

$$\begin{pmatrix}-A+B & A-B\end{pmatrix}\begin{pmatrix}\tilde{\boldsymbol{H}}_\parallel^{(1,-)} \\ \tilde{\boldsymbol{H}}_\parallel^{(2,+)}\end{pmatrix} = \begin{pmatrix}A+B & -A-B\end{pmatrix}\begin{pmatrix}\tilde{\boldsymbol{H}}_\parallel^{(1,+)} \\ \tilde{\boldsymbol{H}}_\parallel^{(2,-)}\end{pmatrix}. \tag{35}$$

Similarly (27) has the form

$$C^{\pm}\Big(\tilde{\boldsymbol{H}}_{\|}^{(2,+)} + \tilde{\boldsymbol{H}}_{\|}^{(2,-)} - \tilde{\boldsymbol{H}}_{\|}^{(1,+)} - \tilde{\boldsymbol{H}}_{\|}^{(1,-)}\Big) \tag{36}$$

$$= C\Big(\tilde{\boldsymbol{H}}_{\|}^{(2,+)} - \tilde{\boldsymbol{H}}_{\|}^{(2,-)} - \tilde{\boldsymbol{H}}_{\|}^{(1,+)} + \tilde{\boldsymbol{H}}_{\|}^{(1,-)}\Big) \tag{37}$$

$$= D\Big(\tilde{\boldsymbol{H}}_{\|}^{(2,+)} + \tilde{\boldsymbol{H}}_{\|}^{(2,-)} + \tilde{\boldsymbol{H}}_{\|}^{(1,+)} + \tilde{\boldsymbol{H}}_{\|}^{(1,-)}\Big) \tag{38}$$

which can be written in matrix form as

$$\begin{pmatrix} C - D & C - D \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{H}}_{\|}^{(1,-)} \\ \tilde{\boldsymbol{H}}_{\|}^{(2,+)} \end{pmatrix} = \begin{pmatrix} C + D & C + D \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{H}}_{\|}^{(1,+)} \\ \tilde{\boldsymbol{H}}_{\|}^{(2,-)} \end{pmatrix}. \tag{39}$$

Hence the full scattering matrix, derived from (25) and (27), is

$$\begin{pmatrix} -(\boldsymbol{K} - \omega^2\epsilon\mu\boldsymbol{I})\boldsymbol{R}^{-1}\tilde{\boldsymbol{\rho}}^e\boldsymbol{R} + \frac{1}{2}\omega\mu k_z & (\boldsymbol{K} - \omega^2\epsilon\mu\boldsymbol{I})\boldsymbol{R}^{-1}\tilde{\boldsymbol{\rho}}^e\boldsymbol{R} - \frac{1}{2}\omega\mu k_z \\ -(\boldsymbol{K} - \omega^2\epsilon\mu\boldsymbol{I})\tilde{\boldsymbol{\sigma}}^m + 2\omega\mu k_z & -(\boldsymbol{K} - \omega^2\epsilon\mu\boldsymbol{I})\tilde{\boldsymbol{\sigma}}^m + 2\omega\mu k_z \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{H}}_{\|}^{(1,-)} \\ \tilde{\boldsymbol{H}}_{\|}^{(2,+)} \end{pmatrix}$$
$$= \begin{pmatrix} (\boldsymbol{K} - \omega^2\epsilon\mu\boldsymbol{I})\boldsymbol{R}^{-1}\tilde{\boldsymbol{\rho}}^e\boldsymbol{R} + \frac{1}{2}\omega\mu k_z & -(\boldsymbol{K} - \omega^2\epsilon\mu\boldsymbol{I})\boldsymbol{R}^{-1}\tilde{\boldsymbol{\rho}}^e\boldsymbol{R} - \frac{1}{2}\omega\mu k_z \\ (\boldsymbol{K} - \omega^2\epsilon\mu\boldsymbol{I})\tilde{\boldsymbol{\sigma}}^m + 2\omega\mu k_z & (\boldsymbol{K} - \omega^2\epsilon\mu\boldsymbol{I})\tilde{\boldsymbol{\sigma}}^m + 2\omega\mu k_z \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{H}}_{\|}^{(1,+)} \\ \tilde{\boldsymbol{H}}_{\|}^{(2,-)} \end{pmatrix} \tag{40}$$

This is a system of the form $\boldsymbol{A}\boldsymbol{H}_{\text{scattered}} = \boldsymbol{B}\boldsymbol{H}_{\text{incident}}$, so the scattering matrix is simply $\boldsymbol{S} = \boldsymbol{A}\backslash\boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

5 Fourier basis

The system of equations for the scattering system, (40), is written so that tensorial quantities with respect to the spatial components are in bold whereas scalars are not bolded. However, these equations are meant to be solved on the spatial or reciprocal lattice, and in this setting the scalar quantities take on the dimensionality of this lattice and the Fourier operators transform this lattice. These dimensions can be viewed in addition to the spatial components. The scalars components of the fields in the equations above become indexed by n and m, the mode numbers for the periodic x and y axes, while the scalar components of the linear operators become linear transformations from the indices nm to $n'm'$.

6 Decoupled case

For completeness, here are the results for when the TE and TM modes decouple.

When the GSTCs have diagonal conductivity matrices that are invariant with respect to y (i.e. $k_y = 0$), then the TE and TM modes decouple and we obtain an equation for the jump condition/scattering matrix of the H field/TM mode:

$$
\begin{pmatrix} I + \frac{\tilde{\sigma}^e_{xx}}{2}\frac{k_{z,n}}{\epsilon\omega} & -I - \frac{\tilde{\sigma}^e_{xx}}{2}\frac{k_{z,n}}{\epsilon\omega} \\ \frac{k_{z,n}}{\epsilon\omega} + \frac{\tilde{\sigma}^m_{yy}}{2} & \frac{k_{z,n}}{\epsilon\omega} + \frac{\tilde{\sigma}^m_{yy}}{2} \end{pmatrix} \begin{pmatrix} \tilde{H}_y^{(1,-)} \\ \tilde{H}_y^{(2,+)} \end{pmatrix}
= \begin{pmatrix} -I + \frac{\tilde{\sigma}^e_{xx}}{2}\frac{k_{z,n}}{\epsilon\omega} & I - \frac{\tilde{\sigma}^e_{xx}}{2}\frac{k_{z,n}}{\epsilon\omega} \\ \frac{k_{z,n}}{\epsilon\omega} - \frac{\tilde{\sigma}^m_{yy}}{2} & \frac{k_{z,n}}{\epsilon\omega} - \frac{\tilde{\sigma}^m_{yy}}{2} \end{pmatrix} \begin{pmatrix} \tilde{H}_y^{(1,+)} \\ \tilde{H}_y^{(2,-)} \end{pmatrix} \tag{41}
$$

An analogous equation exists for the E field/TE mode by substituting $H \to E, \sigma^e_{xx} \to \sigma^m_{xx}, \sigma^m_{yy} \to \sigma^e_{yy}$. Unlike the fully coupled case, the difference between using ρ and σ incurs no additional computational cost.

If the off-diagonal conductivity matrix elements are nonzero, then the TE and TM modes again become coupled and we would find that we need to return to using the fully coupled equations. In this special case we could also derive the following form for the coupled TE and TM polarizations (only when $k_y = 0$)

$$
\begin{pmatrix} I + \frac{\tilde{\sigma}^e_{xx}}{2}\frac{k_{z,n}}{\epsilon\omega} & -\frac{\tilde{\sigma}^e_{xy}}{2} & -I - \frac{\tilde{\sigma}^e_{xx}}{2}\frac{k_{z,n}}{\epsilon\omega} & -\frac{\tilde{\sigma}^e_{xy}}{2} \\ \frac{k_{z,n}}{\epsilon\omega} + \frac{\tilde{\sigma}^m_{yy}}{2} & \frac{\tilde{\sigma}^m_{yx}}{2}\frac{k_z}{\mu\omega} & \frac{k_{z,n}}{\epsilon\omega} + \frac{\tilde{\sigma}^m_{yy}}{2} & -\frac{\tilde{\sigma}^m_{yx}}{2}\frac{k_z}{\mu\omega} \\ -\frac{\tilde{\sigma}^m_{xy}}{2} & I + \frac{\tilde{\sigma}^m_{xx}}{2}\frac{k_{z,n}}{\mu\omega} & -\frac{\tilde{\sigma}^m_{xy}}{2} & -I - \frac{\tilde{\sigma}^m_{xx}}{2}\frac{k_{z,n}}{\mu\omega} \\ \frac{\tilde{\sigma}^e_{yx}}{2}\frac{k_z}{\epsilon\omega} & \frac{k_{z,n}}{\mu\omega} + \frac{\tilde{\sigma}^e_{yy}}{2} & -\frac{\tilde{\sigma}^e_{yx}}{2}\frac{k_z}{\epsilon\omega} & \frac{k_{z,n}}{\mu\omega} + \frac{\tilde{\sigma}^e_{yy}}{2} \end{pmatrix} \begin{pmatrix} \tilde{H}_y^{(1,-)} \\ \tilde{E}_y^{(1,-)} \\ \tilde{H}_y^{(2,+)} \\ \tilde{E}_y^{(2,+)} \end{pmatrix}
= \begin{pmatrix} -I + \frac{\tilde{\sigma}^e_{xx}}{2}\frac{k_{z,n}}{\epsilon\omega} & \frac{\tilde{\sigma}^e_{xy}}{2} & I - \frac{\tilde{\sigma}^e_{xx}}{2}\frac{k_{z,n}}{\epsilon\omega} & \frac{\tilde{\sigma}^e_{xy}}{2} \\ \frac{k_{z,n}}{\epsilon\omega} - \frac{\tilde{\sigma}^m_{yy}}{2} & \frac{\tilde{\sigma}^m_{yx}}{2}\frac{k_z}{\mu\omega} & \frac{k_{z,n}}{\epsilon\omega} - \frac{\tilde{\sigma}^m_{yy}}{2} & -\frac{\tilde{\sigma}^m_{yx}}{2}\frac{k_z}{\mu\omega} \\ \frac{\tilde{\sigma}^m_{xy}}{2} & -I + \frac{\tilde{\sigma}^m_{xx}}{2}\frac{k_{z,n}}{\mu\omega} & \frac{\tilde{\sigma}^m_{xy}}{2} & I - \frac{\tilde{\sigma}^m_{xx}}{2}\frac{k_{z,n}}{\mu\omega} \\ \frac{\tilde{\sigma}^e_{yx}}{2}\frac{k_z}{\epsilon\omega} & \frac{k_{z,n}}{\mu\omega} - \frac{\tilde{\sigma}^e_{yy}}{2} & -\frac{\tilde{\sigma}^e_{yx}}{2}\frac{k_z}{\epsilon\omega} & \frac{k_{z,n}}{\mu\omega} - \frac{\tilde{\sigma}^e_{yy}}{2} \end{pmatrix} \begin{pmatrix} \tilde{H}_y^{(1,+)} \\ \tilde{E}_y^{(1,+)} \\ \tilde{H}_y^{(2,-)} \\ \tilde{E}_y^{(2,-)} \end{pmatrix} \tag{42}
$$

References

[1] Ariel Epstein and George V. Eleftheriades. "Passive Lossless Huygens Metasurfaces for Conversion of Arbitrary Source Field to Directive Radiation". In: *IEEE Transactions on Antennas and Propagation* 62.11 (Nov. 2014). Conference Name: IEEE Transactions on Antennas and Propagation, pp. 5680–5695. ISSN: 1558-2221. DOI: 10.1109/TAP.2014.2354419.