SYNTHESIS OF OPTIMAL POLICIES FOR STOCHASTIC MANUFACTURING SYSTEMS

by

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Abstract

This thesis deals with production control and scheduling in unreliable flexible manufacturing systems. A dynamic programming formulation of this problem requires a solution to the Hamilton-Jacobi-Bellman (HJB) equations which are a set of coupled hyperbolic partial differential equations. An analytical solution is available only for the single-part-type case.

In the first part of the thesis, we study extensions of existing single-part-type results to systems with general distributions of repair times and to systems with operation dependent failure rates. We also describe a methodology to solve the HJB equations for a particular control policy in a two-part-type system. A simple analytical solution is obtained over a subset of the state space. The overall solution includes some regions that require a numerical solution of two one-dimensional integro-differential equations. Relationships between this solution and those of related deterministic systems are investigated. Necessary optimality conditions are derived for this policy.

In the second part of the thesis, we restrict our attention to a class of policies called *hedging point policies* and prove structural properties for an optimal policy belonging to this class. Conditions for stability are derived. A combination of sample path arguments and average cost analysis is used to obtain a partial characterization of the optimal policy for a two-part-type system. A complete characterization is obtained for some special parameter values. Extensions to multiple-part-type systems and insights for designing heuristic scheduling rules are also discussed.

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Chapter 1

Introduction

1.1 Motivation

The term *Manufacturing* refers to a sequence of steps, typicaly carried out by multiple deparments, that take various inputs (such as raw materials, subassemblies) and covert them into desired outputs (other subassemblies or finished products) using limited resources (such as machines, personnel, computers). Each step involves one or more of the following: a *conversion activity* that adds value to the product; a *material-flow activity* that moves material from one location to another; or a *information-flow* activity that helps in coordination of the material flow and conversion activities and also provides feedback for continuous improvement in the factories products, procedures and operating characteristics. Most manufacturing processes are subject to random disruptions caused by machine failures, arrival of demands, changes in product design *etc.* A *flexible manufacting system* or *FMS* is a manufacturing system that can produce many different part types with little time and expense lost in changeover from one product to another. In practice the term FMS is often applied to systems where each of the three types of activities described above is entirely under computer control.

A number of performance objectives exist for a manufacturing system. Commonly used objectives include: maximizing throughput (the number of parts produced per unit time); minimizing in-process inventory (unfinished material in storage) and reduction of cycle times (the amount of time a part spends in system). These objectives can be acheived in principle by a *scheduling policy* that coordinates the three activities such that every activity is performed exactly when it is required. In practice, process complexity, random disruptions and the limited resources make this endeavour difficult In addition, since a FMS is under computer control, these policies must be explicitly stated and be easily computable for real-time software implementation.

A good policy must anticipate future disruptions if it is to satisfy the objectives mentioned above. In addition, the policy must respect capacity limitations. It is therefore imperative that the scheduling policy use feeback information about available capacity in the system to respond to failures.

In this thesis, we consider the design of feedback policies for certain classes of FMS that are subject to random machine failures and repairs. The goal of these policies is to allocate the available capacity to the different part types in such a manner that the the production of each part type is kept close to its demand.

1.2 Non-technical problem statement and related literature

In this thesis, we consider a simple model of the FMS described above that has a single unreliable machine capable of producing many part types. We construct a continous material flow model for this system. This model is appropriate for high volume manufacturing. Each part type has a specified demand rate. The state of such a system is determined by: the machine state that is random and a vector *surplus state* of the part types which at any time represents the difference between cumulative production and the cumulative demand for any product type. There are costs associated with both positive (inventory costs) and negative (backog costs) values of each part type's surplus. The objective is to find a feedback control policy that chooses the production rates of the different part types. These rates must satisfy instantaneous capacity constraints at the machine and must be chosen to minimize either the expected long term average cost or expected total cost from some starting time to some final time.

We now review some of the relevant literature in control theoretic and queueing models of flexible manufacturing systems.

The theoretical insights for the results in this thesis come from Rishel (1975). Rishel describes an abstract dynamic programming problem for systems where the state has both continous and discrete components and the control influences only the continuous component and shows that the solution to the dynamic programming problem divides the continuous state space into regions such that the feedback policy is different in each region.

Oldser and Suri (1980) propose a dynamic programming model to describe the disruptive nature of machine failures. While their solution also divides the surplus space into regions of different policies, their model cannot be used for analytical solutions.

Kimemia (1982) and Kimemia and Gershwin (1983) derived a closed-loop solution to the problem of dispatching parts in a failure-prone FMS by separating long term decision (machine failures) from short term decisions (part loading) by a hierarchical organization of the computations. They formulate sub-optimal strategies for each type of decision that are easy to calculate.

Akella and Kumar (1986) and Bielecki and Kumar (1988) obtain exact analytical solutions for the single part type case of [13]. While the authors in Akella and Kumar (1986) miminize a discounted cost objective function, the authors in Bielecki and Kumar (1988) minimize a long term average cost function. In each case, it is shown that the optimal policy is a *hedging point policy.* which is parametrized by a single parameter. Simplified representations for the differential cost function for the latter case appear in Gershwin (1993).

Sharifnia (1988) extends the Bilecki and Kumar (1988) model to systems with more than two machine states. Hu and Xiang (1991) show the equivelence between the system in Bielecki and Kumar (1988) and a classical $M/M/1$ queueing system. Hu and Vakili (1991) prove some structural properties of the optimal control policy for the Akella and Kumar system but where the failure rate depends on production rate. Numerical performance for different policies for their system is given in Liberopolous (1992).

Caramanis and Sharifnia (1991) design sub-optimal controllers for the multiple part type case of the model in Kimemia and Gershwin (1983). Their approach is to decompose the system into many different systems and solve each one using the method of Bielecki and Kumar (1988). Liberopolous (1992) also reports numerical results on the optimal control policy for a two-part-type system. Connolly (1992) derives an analytical solution for a two-part-type deterministic system.

Several other reseachers have looked at queueing methods for scheduling. Hildebrant (1980) uses queueing network models of FMS's. The effect of failures is represented by using a different queueing network model for each failure state and solving for the steady state performance measure for each of these models.

Whittle (1988) gives an index policy based on a multi-armed bandit formulation that assigns an index to each part type and serves the class with the minimum index.

Wein (1991) develops a dynamic policy for a multiclass queue based on a diffusion approximation. The optimal policy is characterized by a idling curve which defines when the machine is idled and switching curve that defines the dynamic resource allocation to the various part types. The hedging point is the intersection of these curves. Zheng and Zipkin (1990) consider two policies: a non-Markov policy that services demands in a First-Come-First-Serve basis a longest queue policy for identical products and a given hedging point. Veatch (1992) provides a comparison of these different approximations for the hedging point and switching curves.

1.3 Thesis outline

The rest of the thesis is organized as follows. In Chapter 2 we formulate the dynamic programming model for a multiple part type system and derive the Hamilton-Jacobi-Bellman (HJB) equations. The structure of the optimal solution is also described.

Chapter 3 describes the extension of the single part type case of the system in Chapter 2 for systems where the failure rate of the machine is a given function of the production rate The behaviour of surplus for such systems under a class of policies called *hedging point policies* is described. An explicit expression for the differential cost function of the dynamic programming problem is obtained. This expression is used to verify optimality conditions for hedging point policies.

Chapter 4 describes the extension of the single part type case of the system in Chapter 2 for systems where machine repair times have a general probability distribution. Conditions for stability of hedging point policies are derived A method for evaluating the differential cost function in the non-transient region of the surplus space for a hedging point policy with a zero hedging point is described. It is shown that this solution is a quadratic and is related to the differential cost of an equivalent deterministic system.

In Chapter 5, we consider the two-part type case of the system described in Chapter 2 and explain a method for solving for the differential cost function under a particular hedging point policy. First, we divide the state space into regions and subregions and identify a particular sequence for evaluating the differential cost function. Then we consider a one-dimensional surface and use results from Chapter 4 to obtain the solution on that surface. This solution is then propagated successively to other subregions according to the identified sequence using continuity conditions. It is shown that the differential cost function in each subregion is twice continously differentiable. A simple analytical expression is obtained in some subregions. The solution in other subregions is obtained by solving two one dimensional integro-differential equations. Optimality conditions are derived for this policy

In Chapter 6, we restrict our attention to the class of hedging point policies and prove structural properties that an optimal policy belonging to this class has to satisfy. Conditions for stability are derived. A combination of sample path arguments and average cost analysis is used to obtain a partial characterization of the optimal policy. A complete characterization is obtained for some special cases of parameter values.

Finally in Chapter 7, we summarize the contributions of this thesis and point out directions for future research.

Chapter 2

The Production Control Problem

2.1 Introduction

In this chapter, we formulate the basic production control problem in a stochastic manufacturing system. A dynamic programming formulation of this problem and a description of the model assumptions are given in Section 2.2. In Section 2.3, we introduce notations for performance evaluation and derive the optimality conditions for this model. In Section 2.4, we describe the structure of the control policy that satisfies the optimality conditions. Finally in Section 2.5, we describe the behavior of the surplus under the optimal policy.

2.2 Dynamic programming model

Consider a manufacturing system with a single unreliable machine capable of producing N part types. The machine is *flexible* in the sense that change-over times (setup times) between part types is negligible. Machine failure and repair times are random. Part Type i requires an average processing time τ_i and has an average time between orders $\frac{1}{d_i}$. The difference between actual production and demand at any time represents the surplus of a part type (backlog if the difference is negative) at any point. The objective is to find production rates of the various part types such that the total expected surplus and backlog costs are minimized.

We consider a *continuous material flow* model for the system. Let g_i^+ and g_i^- be the unit surplus and backlog costs of Part Type i. $\boldsymbol{d}$ is the vector of demand rates of the various part types. We assume failures and repairs are time dependent (see Buzzacott and Hanifin, 1978; Dallery and Gershwin, 1992). The time between a machine repair and the next machine failure is a machine *uptime*. The time between a machine failure and the next repair is a machine *downtime*. Machine uptimes and downtimes are assumed to be exponentially distributed with rates p and r respectively. Let

$$\mu_i = \frac{1}{\tau_i} \qquad i = 1, \ldots N.$$

Then μ_i represents the maximum possible production rate of Part Type i at the machine.

Since the average time between orders is $\frac{1}{d_i}$, d_i represents the average demand rate of Part Type i.

The continuous flow model can then be represented as follows:

- State: $(\boldsymbol{x}(t), \alpha(t))$

  The state of the system at time t has two components: the machine state, $\alpha(t)$; and the surplus state vector, $\boldsymbol{x}(t) = (x_1(t), \ldots, x_N(t))$.

  <u>State space</u>

$$\boldsymbol{x} \in \mathcal{R}^N, \alpha \in \{0, 1\}$$

- Control: $\boldsymbol{u}(t)$

  The control vector, $\boldsymbol{u}(t) = (u_1(t), \ldots u_N(t))$ is the vector of production rates of the various part types.

  <u>Control space: $\Omega(\alpha(t))$</u>

  The production rates at each time instant have to satisfy the capacity constraint at the machine. This constraint states that the machine cannot be utilized for more than 100% of its time and can be represented as

$$\sum_{i=1}^{N} \tau_i u_i(t) \leq \alpha(t).$$

  In addition, to make physical sense, the production rates must be non negative at all time instants. Therefore

$$u_i(t) \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, N.$$

The capacity set at time t can therefore be defined as

$$\Omega(\alpha(t)) = \left\{ \boldsymbol{u}(t) | \sum_{i=1}^{N} \tau_i u_i(t) \leq \alpha(t), u_i(t) \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, N. \right\}$$

<u>Dynamics</u>

The machine state evolves according to a continuous time Markov process with a generator matrix Q that is given by

$$Q = \begin{vmatrix} -r & r \\ p & -p \end{vmatrix}$$

In other words the evolution of the machine state in the time interval $(t, t+\delta t)$ can be expressed as:

$$\begin{aligned}
\text{prob} \quad (\alpha(t+\delta t) = 1 | \alpha(t) = 0) &= r\delta t + o(\delta t) \\
\text{prob} \quad (\alpha(t+\delta t) = 0 | \alpha(t) = 0) &= 1 - r\delta t + o(\delta t) \\
\text{prob} \quad (\alpha(t+\delta t) = 0 | \alpha(t) = 1) &= p\delta t + o(\delta t) \\
\text{prob} \quad (\alpha(t+\delta t) = 1 | \alpha(t) = 1) &= 1 - p\delta t + o(\delta t)
\end{aligned}$$

where $o(\delta t)$ denotes terms that tend to 0 faster than δt as δt tends to 0.

The rate of change of the surplus vector at time t, denoted by $\dot{\boldsymbol{x}}(t)$ is given by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(t) - \boldsymbol{d}$$

A *policy* is a sequence of control vectors $\boldsymbol{u}(t)$ over time. A policy is *admissible* or *feasible* if at every time the control vector satisfies $\boldsymbol{u}(t) \in \Omega(\alpha(t))$.

- Cost : $g(\boldsymbol{x}(t))$

The instantaneous cost for the system at time t is the sum of instantaneous costs of all part types at time t. The individual part type costs are assumed to be piecewise linear and are as shown below.

$$g_i(x_i(t)) = \begin{cases} g_i^+ x_i(t), & x_i(t) \geq 0 \\ -g_i^- x_i(t), & x_i(t) < 0 \end{cases}$$

where g_i^+ and g_i^- are non-negative scalars. The total instantaneous cost is given by

$$g(\boldsymbol{x}(t)) = \sum_{i=1}^{N} g_i(x_i(t))$$

The production control problem is to find an admissible policy a such that for any other admissible policy b,

$$\liminf_{T\to\infty} \left\{ E_a[\int_0^T g(\boldsymbol{x}(t))dt\,|\boldsymbol{x}(0),\alpha(0)] - E_b[\int_0^T g(\boldsymbol{x}(t))dt\,|\boldsymbol{x}(0),\alpha(0)] \right\} \geq 0$$

This objective function is called *Overtaking Optimality* objective (Haurie, 1992). The control vector $\boldsymbol{u}$ under Policy a is a solution of the continuous time stochastic dynamic program (SDP) as As $T \to \infty$ (Kimemia and Gershwin, 1982):

$$\min_u E_u \left[\int_0^T g(\boldsymbol{x}(t))dt|\boldsymbol{x}(0),\alpha(0) \right] \tag{2.1}$$

such that

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(t) - \boldsymbol{d} \tag{2.2}$$

$$\sum_{i=1}^{N} \tau_i u_i(t) \leq \alpha(t) \tag{2.3}$$

$$\boldsymbol{u}(t) \geq 0 \tag{2.4}$$

where the machine state α evolves according to the Markovian generator Q.

A related objective function is one that minimizes the long term expected average cost. This objective function can be expressed as

$$\min_u \quad \limsup_{T\to\infty} \frac{1}{T} E_u \left[\int_0^T g(\boldsymbol{x}(t))dt\,|\boldsymbol{x}(0),\alpha(0) \right] \tag{2.5}$$

This objective function is referred to as the *ergodic optimality* objective. Ergodic optimality considers only the long term average behavior and does not differentiate between policies that differ in costs over an initial (transient) time interval, as long as they have the same long term average cost. In that sense, the objective function in (2.1), is tighter since it considers the cost over the entire interval $[0, T]$ for the performance evaluation.

2.3 Hamilton-Jacobi-Bellman equations

A *feedback policy* is a set of controls of the form $u(\boldsymbol{x}(t), \alpha(t), t)$. A feedback policy is *stationary* or *time-invariant* if it does not explicitly depend on time, *i.e.*, it is of the form $u(\boldsymbol{x}(t), \alpha(t))$. An *optimal* policy is an admissible policy that minimizes the objective function of the stochastic dynamic program.

Let $\boldsymbol{x}^a(t)$ be the value of the surplus at time t under Policy a. Following the definition in Bielecki and Kumar (1988), Policy a is said to be *stable* for a given manufacturing system if and only if the surplus state of the system under Policy a satisfies:

$$\lim_{t \to \infty} \frac{E\{x_i^a(t)^2\}}{t} = 0$$

This notion of stability is used in Chapters 4 and 6.

The expected cost of policy a in the interval $[s, f]$ (also referred to as to *cost-to-go*) is denoted by

$$J^a(\boldsymbol{x}, \alpha, s, f) = E\{\int_s^f g(\boldsymbol{x}^a(t))\, dt | \boldsymbol{x}(s) = \boldsymbol{x}, \alpha(s) = \alpha\}$$

The difference in the expected costs of two policies a and b in the interval $[s, f]$ is denoted by

$$\Delta J^{ab}(\boldsymbol{x}, \alpha, s, f) = J^a(\boldsymbol{x}, \alpha, s, f) - J^b(\boldsymbol{x}, \alpha, s, f)$$

Let

$$\bar{J}^a = \lim T \to \infty \frac{1}{T}\{J^a(x, \alpha, 0, T)\}.$$

if the limit exists. This limit exists for some policy if the policy leads to a stationary distribution for the surplus (Kimemia, 1982). Then $\bar{J}^a$ denotes the long term average cost of Policy a. Let J^* denote the long term average cost for the optimal policy.

For each state $(\boldsymbol{x}, \alpha)$ let $W^a(\boldsymbol{x}, \alpha)$ be defined as

$$W^a(\boldsymbol{x}, \alpha) = \lim_{T \to \infty} (J^a(\boldsymbol{x}, \alpha, 0, T) - \bar{J}^a T) \tag{2.6}$$

if the limit exists. A sufficient condition for the existence of this limit is that Policy a be feasible and there be a point $(\boldsymbol{x}, \alpha)$ that is visited infinitely often under Policy a (see Tsitsiklis, 1982). In Chapter 6, we show that the condition for the existence of this limit is related to whether the system has enough capacity in the long run to meet demand. $W^a(\boldsymbol{x}, \alpha)$ is called the *differential cost* of Policy a. In the remainder of the thesis, the policy supersripts on the differential cost are suppressed if there is a single policy being considered.

Continuity properties: From Rishel (1975), $W^a(x, \alpha)$ is continuous over the entire surplus space and continuously differentiable at every point that does not belong to the set of discontinuities of the control vector $\boldsymbol{u}$.

For points where $W^a(x, \alpha)$ is continuously differentiable, the Hamilton-Jacobi-Bellman conditions for the optimal policy can be expressed in terms of the functions $W(\boldsymbol{x}, \alpha)$ and the scalar J^*. These conditions constitute the necessary conditions of optimality for the objective function (2.1) (see Gershwin, 1993) and sufficient conditions for optimality for the average cost objective in (2.5) (see Bielecki and Kumar, 1988). Dropping the policy superscript and the time argument of all quantities for notational convenience, these conditions can be written as a system of coupled partial differential equations as in Gershwin (1993).

$$J^* = \min_{u \in \Omega(0)} (g(\boldsymbol{x}) + \sum_{i=1}^{N} (u_i - d_i) \frac{\partial W(\boldsymbol{x}, 0)}{\partial x_i} - r(W(\boldsymbol{x}, 0) - W(\boldsymbol{x}, 1))) \tag{2.7}$$

$$J^* = \min_{u \in \Omega(1)} (g(\boldsymbol{x}) + \sum_{i=1}^{N} (u_i - d_i) \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_i} + p(W(\boldsymbol{x}, 0) - W(\boldsymbol{x}, 1))) \tag{2.8}$$

Note that for any $W(\boldsymbol{x}, \alpha)$ that satisfies these two equations, $W(\boldsymbol{x}, \alpha) + c$ also satisfies the equation, for an arbitrary constant c. Thus any solution for $W(\boldsymbol{x}, \alpha)$ using the HJB equation has one degree of freedom. For $\alpha = 0$, the capacity set in (2.7) is a unitary set. The only feasible control vector is $\boldsymbol{u} = \mathbf{0}$. For $\alpha = 1$, the capacity set is a convex polyhedron defined by

$$\Omega(1) = \left\{ \boldsymbol{u}(t) | \sum_{i=1}^{N} \tau_i u_i(t) \leq 1, u_i(t) \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, N. \right\}$$

From (2.8), it follows that the optimal production rates are solutions to the following linear program.

$$\min_{\boldsymbol{u}} \quad \sum_{i=1}^{N} u_i \frac{\partial W(\boldsymbol{x},1)}{\partial x_i} \tag{2.9}$$

such that

$$\sum_{i=1}^{N} \tau_i u_i \leq 1$$

$$u_i \geq 0 \qquad i = 1, \ldots, N.$$

2.4 Structure of the optimal control policy

The constraint set of the linear program (2.9) is a polyhedron in $\boldsymbol{u}$ space with $N+1$ vertices. Let V^j denote the vertex for which

$$\begin{aligned} u_j = &\quad \mu_j \\ u_i = &\quad 0 \quad i \neq j \end{aligned}$$

and let V^0 denote the vertex $\mathbf{0}$. An optimal solution $\boldsymbol{u}^*(\boldsymbol{x}, 1)$ is always at one of these vertices. The objective function depends on the surplus state. As the surplus state changes, the optimum solution moves from vertex to another. In other words, the optimal control policy divides the N dimensional surplus space into $(N+1)$ distinct regions. Let Region j be the region over which the optimum solution is at vertex V^j.

For each α, it has been shown in Kimemia (1982) that the functions $W(\boldsymbol{x}, \alpha)$ corresponding to the optimal policy, are convex. We further assume that $W(\boldsymbol{x}, \alpha)$ attains its minimum at a unique point $\boldsymbol{z}$. If the system has enough capacity in the long run, the optimal policy for $\alpha = 1$, guides the surplus trajectory towards point $\boldsymbol{z}$. When the surplus trajectory reaches $\boldsymbol{z}$, it is maintained there by producing at $\boldsymbol{u} = \boldsymbol{d}$. The point $\boldsymbol{z}$ is called the *hedging point* for the system since it represents the surplus maintained to hedge against future machine failures.

We assume that the optimal policy for $\alpha = 1$, divides the surplus space into exactly $N+1$ regions, one corresponding to every vertex of the capacity set. This vertex denotes the optimal $\boldsymbol{u}$ for $\alpha = 1$ for all $\boldsymbol{x}$ in the interior of that region. A boundary between two regions therefore is the set of points at which the $\boldsymbol{u}$ changes discontinuously from one vertex solution to another.

2.5 Qualitative description of surplus behavior

Consider the surplus trajectory starting from some point in Region i with $\alpha = 1$. The control guides the surplus in a space of dimension N until the trajectory reaches a boundary between Region i and a neighboring region (say, Region j). There are now two possibilities for the surplus behavior. The control in Region j could either guide the surplus back towards the boundary or guide it away from the boundary. In, the former case the boundary corresponds to an *attractive* boundary and the latter case corresponds to a *deflective* boundary. If the boundary is deflective, the surplus enters Region j. If the boundary is attractive, then the optimal production rate on the boundary is chosen as a convex combination of the production rates of Region i and Region j to prevent the trajectory from *chattering* around the boundary (Gershwin, 1993). This causes the surplus trajectory to move along the boundary (in a space of in dimension $N-1$) until there is a machine failure or an attractive boundary of lower dimension is reached. This behavior continues until the surplus reaches the hedging point. The optimal control then maintains the surplus at the hedging point.

Lemma 1 *For a system that has sufficient capacity to meet demand in the long run, all regions corresponding to machine state $\alpha = 1$ intersect at $\boldsymbol{z}$.*

Proof:
At the hedging point, the control vector for $\alpha = 1$ is given by $\boldsymbol{u}(\boldsymbol{z}, 1) = \boldsymbol{d}$. The system can meet demand in the long run only if the demand vector lies in in the interior of the capacity set when the machine is up, in order to compensate for loss of capacity when the machine is down. Thus

$$\sum_i \tau_i d_i < 1$$

Since $\boldsymbol{u}(\boldsymbol{z}, 1) = d$, therfore

$$\sum_i \tau_i u_i(\boldsymbol{z}, 1) < 1$$

In other words, the vector $\boldsymbol{u}(\boldsymbol{z}, 1)$ is in the interior of the capacity set $\Omega(1)$ (Here we assume both demands are strictly positive. If one of the demands is zero, we can drop that part type and the arguments follow). From Caratheodory's theorem (Luenberger, 1973) it follows that $\boldsymbol{u}(\boldsymbol{z}, 1)$ can be uniquely expressed as

$$\boldsymbol{u}(\boldsymbol{z}, 1) = \sum_{i=0}^{N} \beta_i V^i \tag{2.10}$$

where β_i are numbers such that

$$0 < \beta_i \leq 1, \quad \sum_{i=0}^{N} \beta_i = 1.$$

In words, $\boldsymbol{u}(\boldsymbol{z}, 1)$ is a convex combination (with strictly positive weights β_i) of the control vectors of all regions. Therefore $\boldsymbol{z}$ must be a point of intersection of all regions.

It can be seen that $(\boldsymbol{x}(t), \alpha(t))$ is a $N+1$ dimensional stochastic process (Malhame and Boukas, 1991). The epochs when the state makes a transition into $(\boldsymbol{z}, 1)$ are renewal times for this process. If this process is recurrent, then $(\boldsymbol{x}, \alpha)$ has a stationary distribution. Conditions under which the process is recurrent are derived in Chapter 6 for a two-part-type system.

Let $F(\boldsymbol{x}, \alpha))$ be the joint distribution function of this process. The region over which this function is non-zero is referred to as the *non-transient region* under the hedging point policy.

Then the average cost for the optimal policy can be expressed as

$$J^* = \sum_{\alpha} \int_{\boldsymbol{x}} g(\boldsymbol{x}) dF(\boldsymbol{x}, \alpha) \tag{2.11}$$

Equation (2.11) can be used to calculate the average cost for any policy that leads to a stationary distribution for $\boldsymbol{x}$.

In subsequent chapters, we use (2.11) to determine the average cost in terms of the system parameters and then use (2.7) and (2.8) for determining the functions, $W(x, \alpha)$.

Chapter 3

Single-Part-Type System with Operation Dependent Failures

3.1 Introduction

In this chapter, we consider a manufacturing system with a single unreliable machine producing a single part type. This system is similar to the single part type case of the system considered in Chapter 2 except here, we consider machine failures that are operation dependent. In other words, the probability that a machine that is up at time t will fail during the interval $(t, t + \delta t)$ is given by $\hat{p}(u)\delta t$ where u is the production rate of the machine at time t and the function $\hat{p}(u)$ is assumed to be of the form:

$$\hat{p}(u) = p(\frac{u}{\mu})^{\beta} \qquad 0 \leq \beta \leq 1 \tag{3.1}$$

where p is a positive constant and μ is the maximum production rate of the machine. It can be noted that $\hat{p}(u)$ is concave for $0 < \beta < 1$; convex for $\beta > 1$; linear for $\beta = 1$; and independent of u for $\beta = 0$.

We seek an optimal production policy for this problem in terms of the performance function described in Chapter 2. For $\beta = 0$, this problem is the same as the single part type case of the system explained in Chapter 2. Hu and Vakili (1991) give optimality conditions for the system considered in this chapter but with a discounted cost objective function. Their paper provides insights for many of the results derived here. Numerical experiments for the system considered in Hu and Vakili (1991) are reported in Liberopolous (1992).

In Section 3.2, we formulate the production control problem for this system as a dynamic program and derive the HJB optimality conditions. In Section 3.3, we analyze the behavior of the surplus under a hedging point policy. We also derive an expression for the steady state density function of the surplus under a hedging point policy. In Section 3.4, we explain a method to evaluate the differential cost function of the dynamic program. Section 3.5 describes conditions under which the differential cost of a hegding point policy satisfies the HJB equations for different values of β. Finally we summarize the results in Section 3.6.

3.2 Dynamic programming model

The production control can be formulated as a stochastic dynamic program as in Chapter 2. The only difference in the two models is in the dynamics of machine state α when the machine is up since failure rates are now dependent on the production rates. The dynamics of α for operation dependent failures can be expressed as:

$$\begin{aligned}
\text{prob } (\alpha(t+\delta t) = 0 | \alpha(t) = 1) &= \hat{p}(u(t))\delta t + o(\delta t) \\
\text{prob } (\alpha(t+\delta t) = 1 | \alpha(t) = 1) &= 1 - \hat{p}(u(t))\delta t + o(\delta t) \\
\text{prob } (\alpha(t+\delta t) = 1 | \alpha(t) = 0) &= r\delta t + o(\delta t) \\
\text{prob } (\alpha(t+\delta t) = 0 | \alpha(t) = 0) &= 1 - r\delta t + o(\delta t)
\end{aligned} \tag{3.2}$$

Similar to (2.1) in Chapter 2, the control problem (without the part type subscript since there is only a single part type in this model) can be expressed as

$$\min_u E_u \left[\int_0^T g(x(t))dt | x(0), \alpha(0) \right] \tag{3.3}$$

subject to the following constraints:

$$\dot{x}(t) = u(t) - d \tag{3.4}$$

$$u \leq \mu\alpha(t) \tag{3.5}$$

$$u(t) \geq 0 \tag{3.6}$$

and the machine state α evolves according to (3.2).

The Hamilton-Jacobi-Bellman equation Let the system be in the state $(x, 1)$ at time t. The dynamics of the cost-to-go function in an interval $(t, t+\delta t)$ under the optimal policy can be expressed as:

$$\begin{aligned} J(x,1,t,T) &= \min_{u\in\Omega(1)}[g(x)\delta t + \hat{p}(u)\delta t\, J(x+(u-d)\delta t, 0, t+\delta t, T) \\ &\quad +(1-\hat{p}(u)\delta t)\; J(x+(u-d)\delta t, 1, t+\delta t, T)] + o(\delta t) \end{aligned} \tag{3.7}$$

Here the left side denotes the cost-to-go at time t. The right side is an expression for the same cost obtained by conditioning on the two possible machine state transitions in the interval $(t, t+\delta t)$. The first term is the incremental cost in $(t, t+\delta t)$. The second term is the cost-to-go at time $t+\delta t$ if there is a machine failure in $(t, t+\delta t)$ and the third term is the cost-to-go if there is a no machine failure in $(t, t+\delta t)$. Using Taylor series expansion of the functions $J(x, 1, t, T)$ and $J(x, 0, t, T)$ gives

$$\begin{aligned} -\frac{\partial J(x,1,t,T)}{\partial t}\delta t &= \min_{u\in\Omega(1)}[\, g(x)\delta t - \hat{p}(u)\delta t J(x,1,t,T) + \frac{\partial J(x,1,t,T)}{\partial x}(u-d)\delta t \\ &\quad +\hat{p}(u)\delta t J(x,0,t,T) + o(\delta t)] \end{aligned}$$

Dividing by δt and taking the limit as $\delta t \to 0$ yields

$$-\frac{\partial J(x,1,t,T)}{\partial t} = \min_{u\in\Omega(1)}[\, g(x) - \hat{p}(u)J(x,1,t,T) + -\frac{\partial J(x,1,t,T)}{\partial x}(u-d) + \hat{p}(u)J(x,0,t,T)]$$

Taking limits of both sides as $T \to \infty$, and assuming that the differential cost function is defined by (2.6), we obtain

$$J^* = \min_{u\in\Omega(1)}\left[g(x) + \frac{dW(x,1)}{dx}(u-d) - \hat{p}(u)(W(x,1) - W(x,0))\right] \tag{3.8}$$

Similarly the cost-to-go dynamics starting from state with $\alpha(t) = 0$ can be expressed as:

$$J^* = \min_{u\in\Omega(0)}\left[g(x) + \frac{dW(x,1)}{dx}(u-d) + r(W(x,1) - W(x,0))\right] \tag{3.9}$$

Equations (3.8) and (3.9) are the differential form of the HJB equations for this system. For $\alpha = 0$, the only feasible production rate is $u = 0$. For $\alpha = 1$, the optimal production rate at any time t is a solution of the non linear program:

$$\min \frac{dW(x,1)}{dx}u - \hat{p}(u)\left(W(x,1) - W(x,0)\right) \tag{3.10}$$

subject to

$$0 \leq u \leq \mu$$

The non linear program (3.10), is a generalization of the linear program (2.9) for operation dependent failures.

In Section 3.3, we define a hedging point policy for the system described above and analyze the surplus trajectory under a hedging point policy. Section 3.4 describes a method to evaluate the differential cost for a hedging point policy. In Section 3.5, we give conditions under which a hedging point policy with this differential cost solves (3.10).

3.3 Hedging point policies

A *hedging point policy* for the system described above is a stationary, feedback policy of the form:

$$u(x,1) = \begin{cases} \mu & x < z \\ d & x = z \\ 0 & x > z \end{cases}$$

The parameter z is called the *hedging point*. It represents the level of surplus that is built up to hedge against future failures. This policy is known to be optimal for the case when $\beta = 0$ (Bielecki and Kumar, 1988). When $\beta = 0$, an optimal solution for any x is always at one of the vertices of the capacity set. For $x = z$, all points in the capacity set are optimal but the production rate is chosen to be equal to the demand rate (which is a convex combination of the two vertex solutions, $u = 0$ and $u = \mu$) to prevent the surplus from *chattering* around the hedging point.

It can be recalled from Chapter 2 that a hedging point policy is stable if the surplus trajectory generated by the policy satisfies:

$$\lim_{t\to\infty} \frac{E\{x^2(t)\}}{t} = 0.$$

In Chapter 4, we derive conditions under which a hedging point policy leads to a stable behavior of the surplus for a more general class of systems (those with general repair time distributions) than the systems considered here. It is shown that the hedging point policy is stable if the demand satisfies

$$d \leq \frac{r}{r + p(\mu)} \mu \tag{3.11}$$

For $\beta = 0$, the right hand side of the above equation is the product of the maximum production rate and the fraction of time the machine is up with this production rate. This product can therefore be interpreted as the average capacity of the system when the machine is always utilized at full capacity. A demand that satisfies (3.11) is called a *feasible demand* for the system with operation dependent failures. From any starting surplus state, the hedging point policy guides the surplus to the hedging point. Once the surplus is at the hedging point, it is maintained there until a machine failure causes the surplus to fall below the hedging point. For a feasible demand, the process $(x(t), \alpha)$ is a recurrent stochastic process with a steady state distribution.

The steady state transition equations for this system for the $\beta = 0$ case are given in Gershwin (1993). In this section, we give the steady state transition equations for general values of β. The derivation is not presented here since it parallels that given in Gershwin (1993).

Let $f(x, \alpha)\delta x = \text{prob}(x \leq x(t) \leq x + \delta x)$ and let $P(z, 1)$ denote the probability mass at z in machine state $\alpha = 1$.

The steady state transition equations for any general β lead to the following boundary value problem that is a generalization of the boundary value problem in Gershwin (1993):

$$\begin{aligned}
0 &= -f(x,0)r + \frac{df(x,0)}{dx}d + f(x,1)\hat{p}(\mu) &(3.12)\\
0 &= -f(x,0)r - \frac{df(x,1)}{dx}(\mu - d) - f(x,1)\hat{p}(\mu) &(3.13)\\
0 &= \hat{p}(d)P(z,1) - f(z,0)d &(3.14)\\
0 &= -\hat{p}(d)P(z,1) + f(z,1)(\mu - d) &(3.15)
\end{aligned}$$

Equations (3.12) and (3.13) hold for $x < z$. When $x < z$, the production rate is given by $u = \mu$ and hence the machine failure rate in these equations given by $\hat{p}(\mu)$. On the other hand, equations (3.14) and (3.15) hold for $x = z$. When $x = z$, $u = d$ and hence the machine failure rate in these equations is given by $\hat{p}(d)$. To solve (3.12)–(3.15), we assume a solution of the form

$$f(x, \alpha) = A(\alpha)e^{bx}$$

There are four unknowns, namely $A(0)$, $A(1)$, $P(z, 1)$ and b to be determined. Substituting this in equations (3.12)–(3.15) yields three independent equations in these four unknowns (It can be verified that the fourth equation is redundant). An additional condition comes from the fact that all the probabilities must sum to 1. This condition can be expressed as:

$$\int_{-\infty}^{z} [f(x, 1) + f(x, 0)]\, dx + P(z, 1) = 1.$$

A solution to these equations yields:

$$\begin{aligned} f(x, 0) &= Ae^{bx} \\ f(x, 1) &= A\frac{d}{\mu - d}e^{bx} \\ P(z, 1) &= A\frac{d}{\hat{p}(d)}e^{bz} \end{aligned}$$

where

$$A = \left[\frac{\mu}{b(\mu - d)} + \frac{d}{\hat{p}(d)}\right]^{-1} e^{-bz}$$

$$b = \frac{r(\mu - d) - \hat{p}(\mu)d}{(\mu - d)d}$$

For $\beta = 0$, these expressions reduce those found in Bielecki and Kumar (1988) and Gershwin (1993). From (3.11), it can be noted that for a feasible demand the parameter b is strictly positive. The average cost $\bar{J}$ can be found using (2.11). Thus,

$$\bar{J} = \int_{-\infty}^{0} -g^{-}x\, [f(x, 1) + f(x, 0)]\, dx + \int_{0}^{z} g^{+}x\, [f(x, 1) + f(x, 0)]\, dx + g^{+}zP(z, 1)$$

Plugging in the expressions for $f(x, \alpha)$ and $P(z, 1)$ and simplifying, we get

$$\bar{J}(z) = g^{+}(z - k) + k(g^{+} + g^{-})e^{-bz} \tag{3.16}$$

where

$$k = \frac{\mu \hat{p}(d)}{b(\mu \hat{p}(d) + bd(\mu - d))} \tag{3.17}$$

Again, for $\beta = 0$ this expression reduces to that in Gershwin (1993). Let k^0 denote the value of k for this special case. Then k^0 is given by

$$k^0 = \frac{\mu p}{b(\mu p + bd(\mu - d))} \tag{3.18}$$

Differentiating (3.16) with respect to z, we get

$$\frac{d\bar{J}}{dz} = g^+ - kb(g^+ + g^-)e^{-bz} \tag{3.19}$$

Differentiating (3.19) again with respect to z we get

$$\frac{d^2\bar{J}}{dz^2} = kb^2(g^+ + g^-)e^{-bz} \tag{3.20}$$

The right hand side of (3.20) is non-negative for any non negative value of z. Thus for $z \geq 0$, $\bar{J}$ is convex in z.

Let Z be the value of z that minimizes $\bar{J}$ and let J^* denote the minimum value of $\bar{J}$. A necessary and sufficient condition for this minimization is given by:

$$\left.\frac{d\bar{J}}{dz}\right|_Z = 0$$

Substituting in (3.19) yields

$$Z = \begin{array}{ll} \frac{\ln\left(kb(1+\frac{g^-}{g^+})\right)}{b} & \text{if } \; g^+ - kb(g^+ + g^-) < 0 \\ 0 & \text{if } \; g^+ - kb(g^+ + g^- \geq 0 \end{array} \tag{3.21}$$

Therefore form (3.16), we get

$$J^* = g^+(Z - k) + k(g^+ + g^-)e^{-bz} \qquad Z \geq 0 \tag{3.22}$$

This expression, with k replaced by k^0, is the same that for time dependent failures in Bielecki and Kumar (1988) and Gershwin (1993).

3.4 Evaluation of the differential cost

3.4.1 Dynamics of the cost-to-go

In this section, we evaluate the differential cost function of the dynamic program. The differential cost function under a hedging point policy satisfies the HJB equations (3.8) and (3.9) without the minimization (since the policy is given). Thus the differential cost function satisfies a set of coupled partial differential equations in $W(x,0)$ and $W(x,1)$ that can be expressed as

$$J^* = g(x) + \frac{dW(x,1)}{dx}(u-d) - \hat{p}(u)\left(W(x,1) - W(x,0)\right) \tag{3.23}$$
$$J^* = g(x) + \frac{dW(x,0)}{dx}(-d) + r\left(W(x,1) - W(x,0)\right) \tag{3.24}$$

This equation holds for $x \neq z$. The HJB equation for the state $x = z$ is different because the surplus state is maintained at z under the hedging point policy. The dynamics of the cost-to-go from state $(x,1)$ at time t can be written as

$$J(z,1,t,T) = g(z)\delta t + \hat{p}(d)\delta t J(z,0,t+\delta t,T) + (1 - \hat{p}(d)\delta t)J(z,1,t+\delta t,T)]$$

This equation is similar to (3.7) except that at the hedging point

$$x(t+\delta t) = x(t) = z$$

Following the steps in the derivation of (3.8) leads to the following equation:

$$J^* = g(z) - \hat{p}(d)\left(W(z,1) - W(z,0)\right). \tag{3.25}$$

Equations (3.23), (3.24) and (3.25) completely characterize the dynamics of the cost-to-go function. It can be recalled from Chapter 2 that there is no unique solution to the HJB equations because if $W(x,\alpha)$ solves the HJB equations, then so does $W(x,\alpha) + c$, where c is a constant. Without loss of generality it will be assumed that c is chosen such that

$$W(z,1) = 0 \tag{3.26}$$

In the next section, we give a solution to the HJB equation using (3.26) as an additional condition that the solution must satisfy.

3.4.2 Solution of HJB equations

A solution to the HJB equations for the special case of $\beta = 0$ *i.e*, $\hat{p}(u) = p$, can be found in Bielecki and Kumar (1988) and Gershwin (1993). The solution in this case divides the surplus space into three regions I_1, I_2 and I_3 where

$$\begin{aligned} I_1 &= \{x : x > z\} \\ I_2 &= \{x : 0 < x < z\} \\ I_3 &= \{x : x < 0\} \end{aligned}$$

such that $W(x, 0)$ and $W(x, 1)$ are twice continuously differentiable within each of these regions.

Each boundary between adjacent regions is either a point of discontinuity in u (at $x = z$) or a discontinuity in the derivative of the instantaneous cost function $g(x)$ (at $x = 0$). This suggests the same partitions of the surplus space for the system considered here. Equations (3.23) and (3.24) in each region are a set of linear coupled first order differential equations. The control u takes exactly one value for all x in each region and therefore the differential equations have constant coefficients in each region.

In interval I_1, we have $u = 0$ and $g(x) = g^+x$ Thus (3.23) and (3.24) become

$$\begin{aligned} J^* &= g^+x - d\frac{dW(x,1)}{dx} - \hat{p}(0)\left(W(x,1) - W(x,0)\right) \\ J^* &= g^+x - d\frac{dW(x,0)}{dx} + r\left(W(x,1) - W(x,0)\right) \end{aligned}$$

The solution in I_1 is given by

$$W(x,0) = \frac{g^+x^2}{2d} - \frac{J^*x}{d} + \frac{C\hat{p}(0)}{r + \hat{p}(0)}e^{-ax} + D \tag{3.27}$$

$$W(x,1) = \frac{g^+x^2}{2d} - \frac{J^*x}{d} - \frac{Cr}{r + \hat{p}(0)}e^{-ax} + D \tag{3.28}$$

where

$$a = \frac{r + \hat{p}(0)}{d_1}$$

and C and D are arbitrary constants.

Similarly, in I_2, the solution is given by

$$W(x,0) = Ee^{-bx} + \frac{-g^+x^2}{2\{r(\mu-d)-\hat{p}(\mu)d\}} - \frac{(r+\hat{p}(\mu))J^* - \frac{g^+r\mu}{bd}}{bd(\mu-d)}x + F \tag{3.29}$$

$$\begin{aligned} W(x,1) &= \frac{\hat{p}(\mu)dE}{r(\mu-d)}e^{-bx} - \frac{g^+x^2}{2\{r(\mu-d)-\hat{p}(\mu)d\}} \\ &\quad -[\frac{(r+\hat{p}(\mu))J^* - \frac{g^+r\mu}{bd}}{bd(\mu-d)} - \frac{g^+\mu}{bd(\mu-d)}]\,x + F \\ &\quad +\frac{1}{r}\{J^* + \frac{d(r+\hat{p}(\mu))J^* + \frac{g^+r\mu}{b}}{bd(\mu-d)}\} \end{aligned} \tag{3.30}$$

where

$$b = \frac{r(\mu-d)-\hat{p}(\mu)d}{d(\mu-d)}$$

and E and F are arbitrary constants.

The solution in I_3 is given by

$$W(x,0) = Ge^{-bx} + \frac{g^-(r+\hat{p}(\mu))x^2}{2\{r(\mu-d)-\hat{p}(\mu)d\}} - \frac{(r+\hat{p}(\mu))J^* - \frac{g^-r\mu}{bd}}{bd(\mu-d)}x + H \tag{3.31}$$

$$\begin{aligned} W(x,1) &= \frac{\hat{p}(\mu)dG}{r(\mu-d)}e^{-bx} + \frac{g^-(r+\hat{p}(\mu))x^2}{2\{r(\mu-d)-\hat{p}(\mu)d\}} \\ &\quad -[\frac{(r+\hat{p}(\mu))J^* + \frac{g^-r\mu}{bd}}{bd(\mu-d)} + \frac{g^-\mu}{bd(\mu-d)}]\,x \\ &\quad +H + \frac{1}{r}\{J^* + \frac{d(r+\hat{p}(\mu))J^* - \frac{g^-r\mu}{b}}{bd(\mu-d)}\} \end{aligned} \tag{3.32}$$

where F and H are constants.

3.4.3 Evaluation of constants

The continuity conditions from Chapter 2 yield boundary conditions on the differential cost functions $W(x,0)$ and $W(x,1)$ at the boundary between different regions. Specifically for this system, these conditions imply that $W(x,0)$ and $W(x,1)$ are continuous for all x and differentiable at all points in the surplus space except the hedging point.

This leads to a set of boundary conditions at the region boundaries, namely at

$x = 0$ and $x = z$. Specifically, at $x = 0$

$$\lim_{x \to 0, x < 0} W(x, 1) = \lim_{x \to 0, x > 0} W(x, 1) \tag{3.33}$$

$$\lim_{x \to 0, x < 0} W(x, 0) = \lim_{x \to 0, x > 0} W(x, 0) \tag{3.34}$$

The solutions in both regions I_1 and I_2 satisfy (3.23). As $x \to 0$, a comparison of the right hand side of (3.23) for the two regions shows that since $g(x)$ and u are continuous at $x = 0$, the derivative $\frac{dW(x,1)}{dx}$ must be continuous at $x = 0$. Similarly it can be shown that condition (3.34) is equivalent to the continuity of $\frac{dW(x,0)}{dx}$ at $x = 0$.

At $x = z$, the continuity conditions can be expressed as

$$\lim_{x \to z, x < z} W(x, 1) = \lim_{x \to z, x > z} W(x, 1) = W(z, 1) \tag{3.35}$$

$$\lim_{x \to z, x < z} W(x, 0) = \lim_{x \to z, x > z} W(x, 0) \tag{3.36}$$

$W(x, \alpha)$ on either side of z satisfies the HJB equations (3.23) and (3.24). If in addition, (3.35) is satisfied then

$$\begin{aligned} & \lim_{x \to z, x < z} \left[(\mu - d) \frac{dW(x, 1)}{dx} - \hat{p}(\mu)(W(x, 1) - W(x, 0)) \right] \\ = & \lim_{x \to z, x > z} \left[-d \frac{dW(x, 1)}{dx} - \hat{p}(0)(W(x, 1) - W(x, 0)) \right] \end{aligned} \tag{3.37}$$

Condition (3.37) is not an additional condition since it follows from the other conditions.

The continuity condition at $x = z$ also implies that

$$\lim_{x \to z, x < z} J(x, 1, t) = J(z, 1, t)$$

The left hand side of this equation can be expressed in terms dynamics of the $J(x, 1, t)$ using (3.23). The right hand side of the equation can be expressed in terms of the dynamics of $J(z, 1, t)$ using (3.7). Thus if (3.36) is satisfied, then a comparison of the terms of (3.7) and (3.23) yields,

$$\lim_{x \to z, x < z} \frac{dW(x, 1)}{dx} = \frac{(\hat{p}(\mu) - \hat{p}(d))}{\mu - d} [W(z, 1) - W(z, 0)] \tag{3.38}$$

A similar analysis for $x > z$ leads to the following condition:

$$\lim_{x \to z, x < z} \frac{dW(x,1)}{dx} = \frac{(\hat{p}(0) - \hat{p}(d))}{-d} [W(z,1) - W(z,0)] \tag{3.39}$$

Condition (3.39) is redundant since it follows from conditions (3.37) and (3.38).

It can be verified that equations (3.26), (3.33), (3.34), (3.35), (3.36) and (3.38) are satisfied by the expressions given in (3.28)-(3.32) for the following values of the constants.

$$\begin{aligned}
C &= 0 \\
E &= \frac{(g^+ + g^-)\mu(\mu - d)r}{b^3 d^2 (\mu - d)^2} \\
G &= 0 \\
D &= -\frac{g^+ z^2}{2d} + \frac{J^* z}{d} \\
F &= -\frac{\hat{p}(\mu) d E}{r(\mu - d)} e^{-bz} - \frac{g^+ z^2}{2\{r(\mu - d) - \hat{p}(\mu) d\}} \\
&\quad + [\frac{(r + \hat{p}(\mu))J^* - \frac{g^+ r \mu}{bd}}{bd(\mu - d)} - \frac{g^+ \mu}{bd(\mu - d)}] z \\
&\quad - \frac{1}{r}\{J^* + \frac{d(r + \hat{p}(\mu))J^* + \frac{g^+ r \mu}{b}}{bd(\mu - d)}\} \\
H &= F + \frac{(g^+ + g^-)\mu}{b^2 d(\mu - d)}
\end{aligned} \tag{3.40}$$

It can be noted that $G = 0$ implies that the differential cost is a quadratic function for $x < 0$. This is consistent with the argument given in Gershwin (1993) for the limiting behavior of $W(x,1)$ as $x \to -\infty$. Gershwin (1993) points out that as $x \to -\infty$, there are many machine failures and repairs before the surplus reaches the hedging point and therefore argues that the differential cost for such x can be interpreted as the cost incurred by an average trajectory which is known to be a quadratic function.

It is interesting to observe that for the special case of $\beta = 0$ considered in Gershwin (1993), conditions (3.37) and (3.38) imply that

$$\lim_{x \to z, x < z} \frac{dW(x,1)}{dx} = \lim_{x \to z, x > z} \frac{dW(x,1)}{dx} = 0$$

and thus in this case, the differential cost, W, is continuously differentiable at all points in the surplus space. This is not necessarily true for systems with other values of β.

3.5 Optimality of the hedging point policy

In this section, we analyze the differential cost function for a hedging point policy with a hedging point chosen according to (3.21). We refer to such a hedging point as an *optimal hedging point* and denote it by Z in the following sections. We show that the differential cost function of a hedging point policy with an optimal hedging point is convex. Then we find conditions under which the optimality conditions for the non linear program (3.10) hold for such a policy.

3.5.1 Results on the differential cost function

Theorem 1 *The function $W(x,1)$ for an optimal hedging point policy is convex.*

For $x \in I_1$, $W(x,1)$ is given by (3.28). Differentiating (3.28) twice with respect to x, and substituting the value of C from (3.40), we get

$$\frac{d^2W(x,1)}{dx^2} = \frac{g^+}{d} \geq 0$$

Thus $W(x,1)$ is convex in I_1. For $x \in I_2$, the differential cost function $W(x,1)$, is given by (3.30). Differentiating (3.30) twice with respect to x, we get

$$\frac{d^2W(x,1)}{dx^2} = \frac{pdb^2E}{r(\mu-d)}e^{-bx} - \frac{g^+(r+p)}{bd(\mu-d)}$$

In I_2, $x < Z$. Therefore

$$\frac{d^2W(x,1)}{dx^2} \geq \frac{pdb^2E}{r(\mu-d)}e^{-bZ} - \frac{g^+(r+p)}{bd(\mu-d)}$$

Substituting for Z from (3.21),

$$\frac{d^2W(x,1)}{dx^2} \geq \frac{pdbEg^+}{kr(\mu-d)(g^++g^-)} - \frac{g^+(r+p)}{bd(\mu-d)} \tag{3.41}$$

We now use the fact that $W(x,1)$ for the special case where $\beta = 0$ is known to be convex (Kimemia and Gershwin, 1983). Let Z^0 denote the value of the optimal hedging point, I_2^0 the region $0 < x < Z^0$ and $W^0(x, \alpha)$ the differential cost function corresponding to $\beta = 0$. Thus we have

$$\lim_{x \to z^0,\ x \in I_2^0} \frac{d^2 W^0(x,1)}{dx^2} \geq 0.$$

From (3.30), this is

$$\frac{pdbEg^+}{k^0 r(\mu - d)(g^+ + g^-)} - \frac{g^+(r+p)}{bd(\mu - d)} \geq 0 \tag{3.42}$$

It can be recalled that k and k^0 are defined by equations (3.17) and (3.18) respectively. For the failure rate function given in (3.1), we have

$$k^0 > k \tag{3.43}$$

From (3.41), (3.42) and(3.43) we get,

$$\frac{d^2 W(x,1)}{dx^2} \geq 0$$

Thus $W(x, 1)$ is convex for $x \in I_2$.

For $x \in I_3$, the differential cost function is given by (3.32). Substituting for G from (3.40) and differentiating (3.32) twice with respect to x, we get

$$\frac{d^2 W(x,1)}{dx^2} = \frac{g^-(r + \hat{p}(\mu))}{r(\mu - d) - \hat{p}(\mu)d} \geq 0$$

Thus, for any x,

$$\frac{d^2\, W(x,1)}{dx} \geq 0$$

This implies that $W(x, 1)$ is convex for $x \in I_3$.

This proves that $W(x, 1)$ for the optimal hedging point policy is convex.

Let $L(u, x)$ denote the objective function in (3.10). Thus

$$L(u,x) = \frac{dW(x,1)}{dx} u - p(\frac{u}{\mu})^{\beta} \left(W(x,1) - W(x,0)\right) \tag{3.44}$$

We now use the W and u corresponding to a hedging point policy in (3.44) to find ranges of β for which the optimality conditions of (3.10) are satisfied.

We first consider the case where $\beta = 1$ and prove that the hedging point policy for this case satisfies the necessary and sufficient conditions for optimality of (3.10).

Theorem 2 *A hedging point policy with an optimal hedging point satisfies the optimality conditions for the non linear program (3.10) for $\beta = 0$ and $\beta = 1$.*

Proof:
For $\beta = 0$, the proof can be found in Bielecki and Kumar (1988).

Consider $\beta = 1$. Thus (3.1) gives

$$\hat{p}(u) = p\frac{u}{\mu}$$

Specifically this implies that $\hat{p}(0) = 0$, $\hat{p}(d) = p\frac{d}{\mu}$, and $\hat{p}(\mu) = p$.

From (3.44),

$$L(u, x) = \frac{dW(x, 1)}{dx}u - p\frac{u}{\mu}\left(W(x, 1) - W(x, 0)\right)$$

Thus, (3.10) is a linear program in this case. A vertex of the capacity set is always an optimal solution. For any x, a necessary and sufficient condition for $u = \mu$ to be the optimal solution to (3.10) is given by:

$$L(\mu, x) \leq L(0, x) = 0$$

This condition can be expressed as

$$\frac{dW(x, 1)}{dx}\mu - p\left(W(x, 1) - W(x, 0)\right) \leq 0 \tag{3.45}$$

It can be verified that the differential cost functions given in (3.28) –(3.32) satisfy this condition for $x \in I_2$ or $x \in I_3$ for a hedging point Z that is given by (3.21).

The solution $u = 0$ is the optimal solution to (3.10) iff

$$L(\mu, x) \geq L(0, x) = 0$$

which can be expressed as

$$\frac{dW(x, 1)}{dx}\mu - p\left(W(x, 1) - W(x, 0)\right) \geq 0 \tag{3.46}$$

It can be verified that the differential cost functions given in (3.28) –(3.32) satisfy this condition for $x \in I_1$ for a hedging point Z that is given by (3.21).

Thus, we have shown that for $\beta = 1$, the differential cost function of a hedging point policy with an optimal hedging point satisfies the optimality criterion for the non linear program (3.10).

We now consider values of β such that $0 < \beta < 1$. It is useful in this case to define a feedback policy that is closely related to the hedging point policy. Let $\pi^y(z)$ denote a policy that is the same as a hedging point policy with a hedging point z except over some interval $(z, z+y)$. Thus for $x < z$ and $x > z + y$, $\pi^y(z)$ is defined as

$$\pi^y(z)(x,1) = \begin{cases} \mu & x < z \\ 0 & x > z + y \end{cases}$$

For $x \in (z, z+y)$, Policy $\pi^y(z)$ is non-Markovian and allows the surplus to move between z and $z + y$ Whenever the surplus trajectory reaches z from below, the machine is operated at the maximum production rate until x reaches the level $z + y$. At that instant, the production rate is changed to 0. This causes the surplus level to decrease at the demand rate. When $x = z$, the machine is again operated at the maximum rate. Thus this control policy guides the surplus towards the hedging point from any initial state and then maintains the surplus in the interval $(z, z+y)$. The surplus behavior under Policy $\pi^y(z)$ near the hedging point z is illustrated in Figure 3-1

Let t_1 denote the amount of time it takes for the surplus trajectory to move from $x = z$ to $x = z + y$, if the machine is operated at $u = \mu$. and there is no machine failure. Therefore,

$$t_1 = \frac{y}{\mu_1 - d_1}$$

Let t_2 denote the amount of time it takes for the surplus trajectory to move from $x = z + y$ to $x = z$, if the machine is operated at $u = 0$ and there is no machine failure. Thus

$$t_2 = \frac{y}{d_1}$$

It can be seen that a surplus trajectory that originates at the hedging point at some time t with the machine being up, returns to the hedging point after a time period $t_1 + t_2$ if are no machine failures in $(t, t + t_1 + t_2)$. Let θ denote the fraction of the time the machine is operated at full capacity, during the time when the machine is

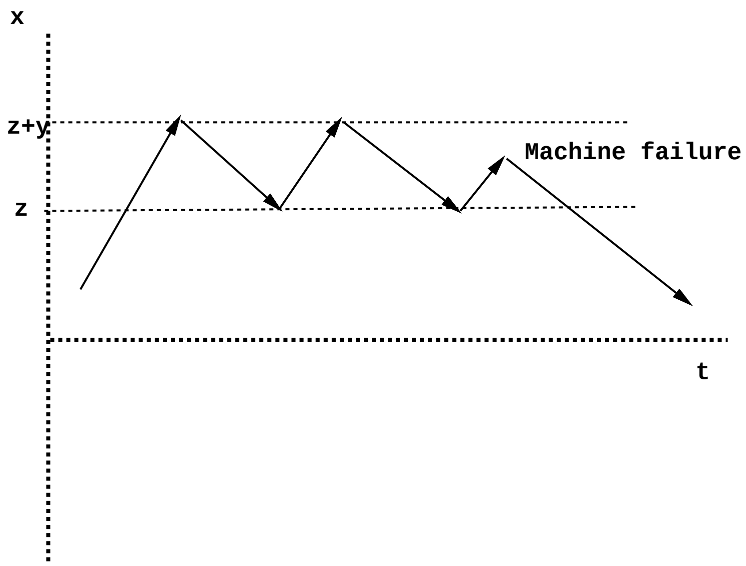

Figure 3-1: Chattering policy

up and the surplus is in the interval $(z, z + y)$. Therefore θ is given by

$$\theta = \frac{t_1}{t_1 + t_2}$$

Thus,

$$\theta = \frac{\frac{y}{\mu - d}}{\frac{y}{\mu - d} + \frac{y}{d}}$$

which on simplification yields

$$\theta = \frac{d}{\mu} \tag{3.47}$$

Let $u(z, 1)$ denote the average production rate of the machine under the limiting hedging policy over the duration when the machine is up and the surplus is in $(z, z+y)$. Therefore,

$$u(z, 1) = \theta\,\mu + (1 - \theta)0$$

Substituting for θ from (3.47), we get

$$u(z, 1) = d \tag{3.48}$$

Let p^y denote the average failure rate over the period of time when the machine is up and the surplus is in $(z, z+y)$ which is defined as

$$p^y = \theta\hat{p}(\mu) + (1-\theta)\hat{p}(0)$$

which gives

$$p^y = p\frac{d}{\mu} \tag{3.49}$$

Let X^y denote the trajectory generated by policy $\pi^y(z)$ and let X denote the trajectory generated by a hedging point policy with a hedging point z and a failure rate function given by

$$\begin{aligned} \bar{p}(u) \quad &= \hat{p}(u) \quad u \neq d \\ &= p\tfrac{d}{\mu} \quad\ \ u = d \end{aligned}$$

Let Policy $\pi^y(z)$ for $y \to 0$, be called the *limiting hedging point policy.* We make the following conjectures about trajectory generated by the limiting hedging policy.

Conjecture 1 *As $y \to 0$, X^y converges to X*

Conjecture 2 *As $y \to 0$, $W(x^y, \alpha)$ converges to the differential cost $W(x, \alpha)$.*

A justification for these conjectures is the following. As y decreases, the time between instants when the machine switches from $u = \mu$ to $u = 0$ and from $u = 0$ to $u = \mu$ decreases. As $y \to 0$, the frequency of these switches goes to infinity. Over periods of time for which X^y is in $(z, z+y)$ and the machine is up, X^y converges to X as $y \to 0$. In addition, at any instant during such a time period, the machine production rate approaches the average production rate given by (3.48) and the its failure rate approaches the average failure rate given by (3.49). This implies that the two trajectories X and X^y exhibit identical behavior over such a period. By definition of Policy $\pi^y(z)$, behavior of the two trajectories is also identical for $x < z$. Therefore, as $y \to 0$, X^y converges to X The definition of $W(x, \alpha)$ in (2.6) therefore implies that $W(x^y, \alpha)$ converges to $W(x^y, \alpha)$.

Theorem 3 *For $0 < \beta < 1$, a limiting hedging point policy with a an optimal hedging point satisfies the optimality conditions for the non linear program (3.10).*

Proof:

For $0 < \beta < 1$, $\hat{p}(u)$ and the function $L(x, u)$ given in (3.44) are concave functions of u. Thus, (3.10) minimizes a concave objective function over a convex set. For any x, a vertex of the capacity set is always an optimum solution. A necessary and sufficient condition for $u = \mu$ to be the optimal solution to (3.10), for any x is given by:

$$L(\mu, x) \leq L(0, x) = 0 \tag{3.50}$$

Similarly the solution $u = 0$ is optimal at some x if and only if;

$$L(\mu, x) \geq L(0, x) = 0 \tag{3.51}$$

From Conjectures 1 and 2, the failure rate of the limiting hedging policy converges to $\bar{p}(u)$ and the differential cost of the limiting hedging policy converges to that of a hedging point policy with an average failure rate of $\bar{p}(u)$.

It can be verified from (3.28) – (3.32) that this this differential cost function satisfies condition (3.50) for $x > Z$ and the condition (3.51) for $x < Z$ where Z is chosen according to (3.21).

Thus, we have shown that for $0 < \beta < 1$, a limiting hedging policy with an optimal hedging point is an optimal solution to (3.10).

We now provide an intuitive reason for this result. For $0 < \beta < 1$, $\hat{p}(d)$ is greater than both $\hat{p}(0)$ and $\hat{p}(\mu)$. Instead of operating the machine at the demand rate when the surplus is at the hedging point, one can operate the machine at full capacity for some time and then let it idle for some time and thus reduce the failure rate to $\hat{p}(\mu)$. This result implies that one can come arbitrarily close to the cost of a hedging point policy by chattering around the hedging point but the hedging point policy itself is not optimal. This means that the problem is not well formulated for $\beta < 1$.

We now consider the case where $\beta > 1$ and show that no hedging point policy would satisfy the optimality criteria for (3.10).

Let $\hat{p}'(u)$ denote the derivative of $\hat{p}(u)$ Therefore

$$\hat{p}'(u) = \frac{d\hat{p}(u)}{du}$$

It can be seen that

$$\hat{p}'(\mu) = \frac{\beta}{\mu}\hat{p}(\mu) \tag{3.52}$$

Theorem 4 *For $\beta > 1$, there exists no hedging point policy that satisfies the optimality conditions for the non linear program (3.10).*

Proof:(By contradiction)

Let us assume that there exists a hedging point policy with a hedging point z that satisfies the optimality conditions of (3.10).

For $\beta > 1$, the function (3.44) is convex. Thus (3.10) is a convex programming problem with a convex objective function and a convex capacity set. The Kuhn-Tucker conditions for (3.10) imply that $u = \mu$ is an optimum solution at x if and only if

$$\frac{d}{du} L(x, u)|u = \mu \leq 0 \tag{3.53}$$

It can be noted that the above expression is equivalent to the condition $\frac{dW(x,\alpha}{dx} \leq 0$ for $x < z$ in the case when $\beta = 0$) Equation (3.53) implies that

$$\frac{dW(x,1)}{dx} - p^{'}(\mu)\left(W(x,1) - W(x,0)\right) \leq 0, \quad x < z \tag{3.54}$$

Taking the limit of both sides of (3.5.1) as $x \to z$ gives

$$\lim_{x \to z, x < z} \frac{dW(x,1)}{dx} - \hat{p}^{'}(\mu)\left(W(x,1) - W(x,0)\right) \leq 0$$

Using condition (3.38) to substitute for the first term on the left side leads to

$$\frac{(\hat{p}(\mu) - \hat{p}(d))}{\mu - d}\left[W(z,1) - W(z,0)\right] - \hat{p}^{'}(\mu)\left(W(z,1) - W(z,0)\right) \leq 0 \tag{3.55}$$

It can be noted from (3.30)-(3.32) that

$$W(x,1) \leq W(x,0) \quad \text{for all } x$$

Therefore (3.55) implies that

$$\frac{\hat{p}(\mu) - \hat{p}(d)}{\mu - d} \geq \hat{p}^{'}(\mu)$$

Substituting for the right hand side of the above expression using (3.52), we get

$$\frac{\hat{p}(\mu) - \hat{p}(d)}{\mu - d} \geq \frac{\beta}{\mu}\hat{p}(\mu)$$

Using (3.1) and rearranging terms, the above expression becomes

$$\beta(1 - \frac{d}{\mu}) + (\frac{d}{\mu})^{\beta} \leq 1 \tag{3.56}$$

where $\beta > 1$ and $\frac{d}{\mu} < 1$ (for the demand to be feasible) We now show that the left hand side of (3.56) is strictly greater than 1. For this purpose, let us define a function $b(y)$ as

$$b(y) = \beta(1 - y) + y^{\beta} \quad 0 \leq y \leq 1, \ \beta > 1$$

Therefore

$$\frac{db(y)}{dy} = -\beta + \beta y^{\beta - 1} > 0$$

Also

$$b(0) = \beta > 1$$

This implies that

$$b(y) > 1 0 < y < 1$$

Taking $y = \frac{d}{\mu}$ we get

$$\beta(1 - \frac{d}{\mu}) + (\frac{d}{\mu})^{\beta} > 1 \tag{3.57}$$

which is contradictory to (3.56). Thus our assumption that there exists a hedging point policy that satisfies the optimality criterion of (3.10) is not true. Consequently there exists no hedging point policy that satisfies the optimality criterion of (3.10).

3.5.2 Optimality properties

In the previous section, we proved some properties related to the differential cost function of a hedging point policy. In this section, we summarize these results and point out directions for further research.

Theorems 2 and 3 imply that for $0 \leq \beta \leq 1$, the differential cost function for a hedging point policy given by (3.28)–(3.32) satisfies the optimality condition for the non linear program (3.10). It can be recalled from Chapter 2 that this optimality criterion is a necessary condition of optimality for the overtaking optimality problem. Therefore Theorem 4 implies that for $\beta > 1$, there does not exist a hedging point policy or a limiting hedging policy that is overtaking optimal. Bielecki and Kumar (1988) show that the optimality condition of (3.10) is also sufficient for the optimality for the dynamic programming problem with an ergodic cost objective for the special case when $\beta = 0$. Sufficiency of the optimality conditions for $0 < \beta \leq 1$ are open research questions.

It can be seen that the functions $W(x, 1)$ and $W(x, 0)$ for $0 < \beta < 1$ are the same as the corresponding functions for $\beta = 1$. Thus the value of the optimal hedging point , Z, in Theorems 2 and 3 is the same. For $0 < \beta < 1$, the failure rate p^y under the limiting hedging policy is less than the failure rate $\hat{p}(d)$ obtained from (3.1) for a corresponding hedging point policy at the hedging point.

Hu and Vakili (1991) show that for a discounted cost objective function, there exists a hedging point policy satisfying the optimality conditions iff $\beta = 0$ or $\beta = 1$. Liberopolous (1992) reports numerical results for the same objective and observes that the hedging point policy seems to be optimal for $0 \leq \beta \leq 1$. Our results, though derived for the overtaking or ergodic optimality objective function, provides an explanation for the apparent discrepancy between these two results. For $0 \leq \beta \leq 1$, there exists no hedging point policy that satisfies (3.10) but as shown in Theorem 3, there exists a limiting hedging policy that does satisfy the optimality conditions. In the limit as $y \to 0$, the behavior of the surplus under this policy approaches that under a hedging point policy as observed in the numerical experiments in Liberopolous (1992).

3.6 Summary

In this chapter, we have focussed on a single part-type system with operation dependent failures. The behavior of the surplus in such a system when it is operated under a hedging point policy was described and optimality conditions were derived based on an analytical solution to the HJB equation. It was shown that for some special failure rate functions, the hedging point policy satisfies the optimality conditions. This is true in particular for the case where $\beta = 1$, which is of practical interest in many real-world systems.

There are many extensions of this problem to be explored. Further analytical research is required to rigorously prove the convergence properties of the differential cost in Conjectures 1 and 2 for $0 < \beta < 1$. Another direction for research is the extension of these results is to multiple-part type systems like the one described in Chapter 5.

In the following chapters, we only consider the case of time dependent failures, *i.e.*, $\beta = 0$. The analytical solution of the HJB equations for $\beta = 0$ is used in Chapter 5 in the solution of the HJB equation in a two-part-type system and again in Chapter 6 to prove results on the comparison of costs of trajectories generated by two different policies.

Chapter 4

Single-Part-Type System with General Repair Times

4.1 Introduction

In this chapter, we generalize the single-part-type case of the system considered in Chapter 2. Here machine repair times are not restricted to have exponential distributions. Exponential repair times are not appropriate models of the repair behavior in many real world systems where repair times are often deterministic or have a multi-modal distribution corresponding to different failure modes of the machine. In addition, the general repair time distribution is also useful in evaluating the performance of the particular policy in Chapter 5 for a two-part-type system.

In Section 4.2, we formulate the dynamic programming problem for optimal production control in such systems and derive the HJB optimality conditions. In Section 4.3, we discuss conditions for stability of hedging point policies. We also derive conditions under which a hedging point policy with a zero hedging point is optimal among the class of hedging point policies. In Section 4.4, we give an analytical solution for the differential cost function for the dynamic program over a subset of the surplus space for the zero hedging point case. The motivation for this is two-fold: (a) the differential cost function can be used to verify the HJB optimality conditions; and (b) the explicit solution developed in this chapter is used as a boundary condition in the evaluation of the differential cost function under a specific policy for a two-part-type system in Chapter 5.

4.2 Dynamic programming model

Let t^r denote the random length of time between the instant the machine fails and the instant it is repaired. Let F_r and f_r denote the distribution and density functions of random variable t^r.

The production control can be formulated as a stochastic dynamic program as in Chapter 2. The two models differ only in the dynamics of machine state α when the machine is down since machine repair times are no longer restricted to have exponential distributions. The dynamics for α when the machine is up can be written as

$$\begin{aligned} \text{prob } \ (\alpha(t+\delta t) = 0|\alpha(t) = 1) &= p\delta t + o(\delta t) \\ \text{prob } \ (\alpha(t+\delta t) = 1|\alpha(t) = 1) &= 1 - p\delta t + o(\delta t) \end{aligned} \tag{4.1}$$

On the other hand, when the machine is down, the time until the machine gets repaired depends, in general, on the time since the last machine failure. A machine that fails at time t_1 gets repaired at a random time $t_1 + t^r$ where

$$\text{prob } \ (\alpha(t_1{+}t^r{+}\delta t) = 1 \,|\, \alpha(t_1{+}t^r) = 0, \alpha(t_1^-) = 1, \alpha(t_1^+) = 0) = F_r(t^r{+}\delta t){-}F(t^r){+}o(\delta t) \tag{4.2}$$

The production control problem can thus be expressed as

$$\min_u E_u \left[\int_0^T g(x(t))dt|x(0), \alpha(0) \right] \tag{4.3}$$

subject to the following constraints,

$$\dot{x}(t) = u(t) - d \tag{4.4}$$

$$u \leq \mu\alpha(t) \tag{4.5}$$

$$u(t) \geq 0 \tag{4.6}$$

and the machine state α evolves according to (4.1) and (4.2).

The Hamilton-Jacobi-Bellman equation When the machine is down, the only feasible control is $u = 0$ until the machine is repaired. Let t be the time instant immediately after a machine failure and let x be the value of the surplus at time t. The machine next gets repaired at some random time, $t + t^r$. The cost-to-go under any feasible policy at time t can be written as a sum of the expected cost until the machine is repaired and the expected cost-to-go from $t + t^r$ with the machine state being up:

$$J(x, 0, t, T) = E\{\int_t^{t+t^r} g(x(s))ds + J(x(t + t^r), 1, t + t^r, T)\}.$$

Here the expectation is over the random variable t^r. During the interval $(t, t + t^r)$, $u = 0$ and therefore $\dot{x} = -d$. Thus the value of the surplus at $t+t^r$ is given by $x - dt^r$. Consequently we have,

$$J(x, 0, t, T) = E\{\int_t^{t+t^r} g(x(s))ds + J(x - dt^r, 1, t + t^r, T)\}. \tag{4.7}$$

When the machine is up, the dynamics of the cost-to-go for the optimal policy can be written as

$$\begin{aligned} J(x, 1, t, T) \quad = \quad & \min_u \Bigg[g(x)\delta t + (1 - p\delta t)J(x(t + \delta t), 1, t + \delta t, T) \\ & + p\delta t J(x(t + \delta t), 0, t, T) + o(\delta t) \Bigg]. \end{aligned} \tag{4.8}$$

Here the left side denotes the cost-to-go at time t. The right side is an expression for the same cost obtained by conditioning on the two possible machine state transitions in the interval $(t, t+\delta t)$. The first term is the incremental cost in $(t, t+\delta t)$. The second term is the cost-to-go at time $t + \delta t$ if there is no machine failure in $(t, t + \delta t)$ and the third term is the cost-to-go if there is a machine failure in $(t, t + \delta t)$. Using Taylor series expansion of the functions $J(x, 1, t, T)$ and $J(x, 0, t, T)$, neglecting second order terms in δt, and canceling common terms from both sides we get,

$$\begin{aligned} -\frac{\partial J(x, 1, t, T)}{\partial t}\delta t \quad = \quad & \min_u \Bigg[g(x)\delta t - p\delta t J(x, 1, t, T) + \frac{\partial J(x, 1, t, T)}{\partial x}(u - d)\delta t \\ & + p\delta t J(x, 0, t, T) + o(\delta t) \Bigg] \end{aligned}$$

Dividing by δt and taking the limit as $\delta t \to 0$ yields

$$-\frac{\partial J(x,1,t,T)}{\partial t} = \min_u \left[g(x) - pJ(x,1,t,T) + \frac{\partial J(x,1,t,T)}{\partial x}(u-d) + pJ(x,0,t,T) \right]$$

Substituting for $J(x,0,t,T)$ from (4.7) gives

$$\begin{aligned} -\frac{\partial J(x,1,t,T)}{\partial t} &= \min_u \Big[g(x) - pJ(x,1,t,T) + -\frac{\partial J(x,1,t,T)}{\partial x}(u-d) \\ &\quad + pE\{\int_t^{t+t^r} g(x(s))\delta s + J(x - dt^r, 1, t+t^r, T)\} \Big] \end{aligned} \quad (4.9)$$

Finally, we take limits of both sides as $T \to \infty$ and define

$$W(x,\alpha) = \lim_{T\to\infty} J(x,\alpha,t,T) - J^*(T-t) \quad (4.10)$$

if the limit exists. As explained in Chapter 2, $W(x,\alpha)$ can be interpreted as the differential cost of state (x,α). For a general repair time distribution, the machine state when the machine is down is not completely represented by $\alpha = 0$. since the time until the next repair depends on the duration of time since the last machine failure. To avoid ambiguity, we use $W(x,0)$ to denote the differential cost at state $(x,0)$ immediately after a machine failure. J^* is the average cost of the optimal policy. Limit (4.10 exists if the optimal policy is such that there exists at least one point in the surplus space that is visited infinitely often. In Section 4.3.1 we show that this condition is related to the demand being feasible for the system.

Equations (4.9) and (4.10) imply that

$$\begin{aligned} J^*(1+pE\{t^r\}) &= g(x) - pW(x,1) + pE\{\int_t^{t+t^r} g(x(s))ds\} + E\{W(x-dt^r,1)\} \\ &\quad + \min_u \frac{dW(x,1)}{dx}(u-d) \end{aligned} \quad (4.11)$$

Equation (4.11) is the integro-differential form of the HJB equation in terms of a single function $W(x,1)$. This equation is satisfied for all x at which the function $W(x,1)$ is differentiable. For any $W(x,1)$ that satisfies the HJB equation $W(x,1)+c$ also satisfies the equation, for an arbitrary constant c.

Similarly, taking the limits of both sides of (4.7) as $T \to \infty$ and using (4.10) we get

$$W(x,0) = -J^* E\{t^r\} + E\{\int_t^{t+t^r} g(x(s))ds\} + E\{W(x - dt^r, 1)\} \tag{4.12}$$

Equation (4.12) holds for all x and is an expression for $W(x,0)$ in terms of the function $W(x,1)$.

4.3 Analysis of a hedging point policy

The optimal control policy is a solution to the minimization in (4.11). Thus the optimal policy is a solution to the following linear program

$$\min \frac{dW(x,1)}{dx} u$$

subject to

$$0 \leq u \leq \mu$$

Using the same arguments as in Kimemia (1982), we can show that $W(x,1)$ for the optimal policy is convex. This means that $W(x,\alpha)$ can attain its minimum at $x = \infty$; $x = -\infty$; or $x = Z$, where Z is a finite number. We consider the third case since it is of most interest in real-world systems that operate within capacity (Gershwin, 1993). The optimal policy in this case is given by:

$$\begin{aligned} u &= \mu \quad x < Z \\ &= 0 \quad x > Z \end{aligned}$$

As in Bielecki and Kumar (1988) and Gershwin (1993), we select $u = d$ when $x = Z$. Thus the optimal policy is of the form

$$\begin{aligned} u &= \mu \quad x < Z \\ &= d \quad x = Z \\ &= 0 \quad x > Z \end{aligned}$$

The above policy is referred to as a *hedging point policy* for this system with a hedging point Z. In the following section, we analyze stability properties of this policy.

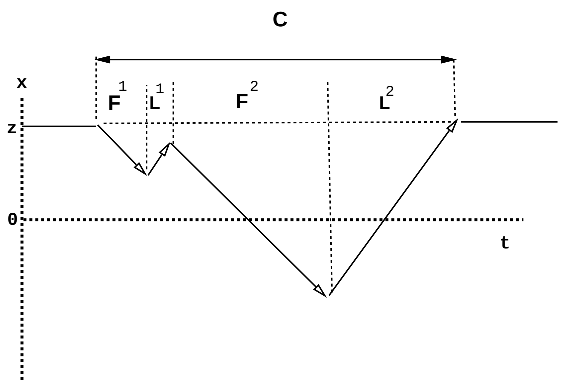

Figure 4-1: Typical renewal cycle for general repair times

4.3.1 Stability of the hedging point policy

In this section, we derive conditions under which the hedging point policy is stable, in the sense of the stability definition given in Section 2.2. Similar results have been previously obtained by Bielecki and Kumar (1988). However, our approach is more general since it applies to general repair time distributions. Besides being of interest on its own, this generality enables us to prove stability in two-part-type systems in Chapter 5.

Consider the single-part-type system described in Section 4.2 in which machine repair times have a general distribution. The random variable t^r is the repair time of the machine and f_r is its probability density function. Thus $E\{t^r\}$ is the mean time to repair (MTTR). Let the system be operated under a hedging point policy with a hedging point z. Then $(x(t), \alpha(t))$ is a controlled stochastic process. The epochs of transitions into state $(z, 1)$ constitute renewal times for this stochastic process (see Malhame and Boukas, 1991). A typical renewal cycle under the hedging point policy is shown in Figure 4-1. If the machine is up and the surplus is at the hedging point, the surplus is maintained at that level until a machine failure causes it to decrease at a rate equal to the demand rate d. When the machine is next repaired, it produces at its maximum rate causing the surplus to increase at a rate $\mu - d$. If the machine fails again before the surplus reaches the hedging point, then the surplus again decreases at the demand rate until the next repair after which it increases at a rate $\mu - d$. This continues until the the surplus reaches the hedging point.

Consider the time interval between the instant the surplus leaves state $(z, 1)$ to the instant when it next makes a transition into state $(z, 1)$. Let us denote this interval by $\mathcal{I}$ and define the following random variables over this interval (some of these notations are illustrated in Figure 4-1):

- C: Length of the time interval $\mathcal{I}$;
- M: Number of repairs during the time interval $\mathcal{I}$.
- F^m: Length of the mth period during which the machine is down in $\mathcal{I}$, $m = 1, ..., M$.
- L^m: Length of the mth period during which the machine is up in $\mathcal{I}$, $m = 1, ..., M$.
- F: Total duration of time during which the machine is down during $\mathcal{I}$;
- L: Total duration of time during which the machine is up during $\mathcal{I}$;

Then

$$F = \sum_{m=1}^{M} F^m,$$

$$L = \sum_{m=1}^{M} L^m,$$

and

$$C = F + L. \tag{4.13}$$

When the machine is down, the surplus decreases at a rate d whereas when the machine is up, the surplus increases at a rate $\mu - d$. The surplus level at the beginning of interval $\mathcal{I}$ is the same as the surplus level at the end of this interval ($= z$). Thus,

$$L(\mu - d) = Fd$$

From (4.13),

$$C = \frac{\mu}{d} L \tag{4.14}$$

Hu and Xiang (1991) show the equivalence between measures in the system in Bielecki and Kumar (1988) and a corresponding $M/M/1$ queueing system. Their methods can be extended to the system considered in this section and imply an equivalence between the measures in this system and a corresponding $M/GI/1$ queueing system. Let $w(t)$ denote the workload process of a $M/GI/1$ queueing system (Kleinrock, 1975a). $w(t)$ represents the amount of time the server needs to be busy to clear all outstanding requests in the queue. Let f_w denote its steady state density function. The result of Hu and Xiang (1991) generalizes to the following theorem:

Theorem 5 *The random variable L has the same distribution as the busy period of an $M/GI/1$ queue with a Poisson arrival process with rate λ and a service time random variable s with a density function $f_s(.)$ given by*

$$\lambda = p$$

and

$$f_s(s) = f_r(\frac{d}{\mu - d}s), \ \ s \geq 0$$

If the $M/GI/1$ queue is stable, the process $y(t) = z - x(t)$ has a steady state density function given by

$$f_y(y) = \frac{\mu}{d(1 + pE\{t^r\})} f_w(y), \ \ y \geq 0$$

From Theorem 5 and equation (4.14), we get the following corollary.

Corollary 1 *The random variable C has the same distribution as the busy period of an $M/GI/1$ queue with a Poisson arrival process with rate λ and a service time random variable y with a density function $f_s(.)$ where:*

$$\lambda = \frac{d}{\mu}p$$

and

$$f_s(y) = f_r(\frac{\mu}{\mu - d}y), \ \ y \geq 0$$

Note that the $M/GI/1$ queues in Theorem 5 and Corollary 1 have the same load intensity, ρ, given by:

$$\rho = \frac{d}{\mu - d}\frac{p}{E\{t^r\}} \tag{4.15}$$

The parameters of the two queues differ only by a scaling factor (namely $\frac{d}{\mu}$). This difference is due to the scaling factor in (4.14).

Corollary 2 *A hedging point policy with a finite hedging point is stable for the manufacturing system if the demand satisfies:*

$$d < \frac{\mu}{1 + pE\{t^r\}} \tag{4.16}$$

Proof:
It follows from equation (4.15) that the condition (4.16) is equivalent to the condition $\rho < 1$ for the $M/GI/1$ queueing system in Theorem 5. If $\rho < 1$, this $M/GI/1$ queue has a steady state workload distribution with a finite second moment (Kleinrock, 1975b). Therefore, it follows from the equivalence stated in Theorem 5 that

$$\lim_{t\to\infty} \frac{E\{x^2(t)\}}{t} = 0.$$

Thus the theorem is proved.

The right hand side of (4.16) represents the average capacity of the machine. A demand that satisfies (4.16) is called a *feasible demand*. For a feasible demand, the hedging point is positive recurrent *i.e.*, if the surplus leaves the hedging point due to a machine failure, the optimal policy guides it back to the hedging point after a finite time with probability 1. For $\rho = 1$, the hedging point is null recurrent. That is, the surplus on leaving the hedging point, returns to the hedging point with probability 1 after some time, but the expected length of this time is infinite. For $\rho > 1$, the hedging point is transient. In other words, there is a non-zero probability that the surplus never reaches the hedging point.

Note that if t^r is exponentially distributed with mean $\frac{1}{r}$ then the stability criterion becomes

$$d < \frac{\mu r}{r+p}$$

which is the condition given for the exponential repair time distribution in Bielecki and Kumar (1988).

Using Corollary 1 and standard results pertaining to the busy period of an $M/GI/1$ queue (Kleinrock, 1975a), we get the following result:

Corollary 3 *For a feasible demand, the random variable C has finite first and second moments given by*

$$E\{C\} = \frac{\mu}{\frac{\mu-d}{E\{t^r\}} - pd}$$

$$E\{C^2\} = \frac{2\mu^2(\mu-d)}{(\frac{\mu-d}{E\{t^r\}} - pd)^3}$$

Malhame and Boukas (1991) define the following stability criterion for their system:

$$E\{C^2\} < \infty$$

It is interesting to note that this stability criterion is equivalent to that given in Chapter 2. Indeed, both criteria are satisfied if the demand satisfies (4.16).

From Theorem 5, it follows that

$$E\{y\} = \frac{\mu}{d(1 + pE\{t^r\})} E\{w\}$$

where $E\{w\}$ denotes the expected value of the workload in the $M/GI/1$ queue. Using the formula for the first moment of the workload of a $M/GI/1$ queue (Kleinrock, 1975b) leads to the following result

Corollary 4 *For a feasible demand, the expected value of the surplus for a hedging point policy with hedging point z is given by*

$$E\{x\} = z - \frac{p\mu d E\{(t^r)^2\}}{2((\mu - d) - pdE\{t^r\})(1 + pE\{t^r\})}$$

This result is used in Section 4.4 to find the expected average cost of a hedging point policy. It is interesting to note that the expected value of the surplus depends only on the first two moments of the repair time distribution.

Since the hedging point is a recurrent point for a stable system, this implies that the limit in (4.10) exists for a feasible demand.

Consider a typical renewal cycle of the surplus that is illustrated in Figure 4-1. Once the surplus reaches the hedging point, it remains at the hedging point until the next failure. Using the memoryless property of the exponential failure time distribution, the average length of the interval between the instant the surplus reaches the hedging point and the next time when the machine fails is $\frac{1}{p}$. Once the surplus leaves the hedging point, the average amount of time it takes for the surplus to return to the hedging point is $E\{C\}$. Let P denote the probability mass at the hedging point. Thus P is the fraction of the time that the surplus spends at the hedging point. Therefore

$$P = \text{Prob}(x = z) = \frac{\frac{1}{p}}{\frac{1}{p} + E\{C\}} = \frac{1}{1 + pE\{C\}} \tag{4.17}$$

Note that P is independent of z. Theorem 5 implies that the density function $f_w(.)$ is independent of z and depends only on the parameters of the machine. Thus z is a location parameter for the distribution of the surplus x.

Extension for systems with operation dependent failures Consider the system analyzed in Chapter 3 operated under a hedging point policy with a hedging point z. This system is similar to the one in Section 4.2 except repair times are exponentially distributed and the failure rate is given by (3.1). The non-transient region under this policy is given by $x \leq z$. The behavior of the surplus in this case is as illustrated in Figure 4-1. A machine failure when the surplus is at z causes the trajectory to fall below the hedging point. Under the hedging point policy, the machine is utilized at full capacity from that instant until the surplus again reaches the hedging point. Thus the production rate over this interval is equal to μ whenever the machine is up and the surplus behavior of this system over this interval is similar to that of the single-part type system with exponential repair times and exponential failure with a rate $p(\mu)$. Consequently, the queueing system in Corollary 1 for this case has a Poisson arrival rate $p(\mu)$. Its stability condition, (given by $\rho < 1$) yields:

$$d < \frac{\mu r}{r + p(\mu)}$$

4.3.2 Optimality of zero hedging point

In this section, we restrict our attention to the class of hedging point policies. A policy that belongs to this class and that minimizes the objective function among all other policies in this class is referred to as an *optimal hedging point policy* and its hedging point is referred to as the *optimal hedging point* and denoted by Z.

We define some notation for a hedging point policy with a hedging point z. Let $\bar{J}^z$ denote the average cost, X^z denote the surplus trajectory and $x^z(t)$ denote the value of Trajectory X^z at time t. Let F^z and f^z denote the stationary distribution and density functions of x.

By definition of Z, we have

$$\bar{J}^Z \leq \bar{J}^z, \quad z \neq Z. \tag{4.18}$$

Lemma 2 *The optimal hedging point Z is non-negative.*

Proof: Consider two surplus trajectories X^Z and $X^{Z+\epsilon}$ generated by hedging point policies with hedging points Z and $Z+\epsilon$ respectively, such that

$$x^{z+\epsilon}(0) = x^{z}(0) + \epsilon$$

The behavior of these trajectories is illustrated in Figure 4-2 for $Z > 0$. Let us assume that $Z < 0$. Let ϵ be a small positive number such that

$$Z + \epsilon < 0.$$

Therefore both trajectories X^{Z} and $X^{z+\epsilon}$ have negative hedging points. At any time t, the two trajectories are such that

$$x^{z}(t) + \epsilon = x^{z+\epsilon}(t) \leq 0$$

Therefore

$$x^{z}(t) < x^{z+\epsilon}(t) \leq 0$$

which yields

$$g(x^{z}(t)) > g(x^{z+\epsilon}(t))$$

This implies that

$$\bar{J}^{Z} > \bar{J}^{Z+\epsilon}$$

which contradicts (4.18). Thus $Z < 0$ is not possible and the lemma is proved.

Theorem 6 *The optimal hedging point policy satisfies*

$$Z > 0 \implies F(0) = \frac{g^{+}}{g^{-} + g^{+}} \tag{4.19}$$

$$Z = 0 \implies F(0) \leq \frac{g^{+}}{g^{-} + g^{+}} \tag{4.20}$$

Proof: This proof follows the same steps as those given in Bielecki and Kumar (1988) for exponential repair time distributions. For $z \geq 0$, the average cost for the hedging point policy with hedging point z is given by

$$\bar{J}^{z} = -g^{-} \int_{-\infty}^{0} x f_{x}^{z}(x) dx + g^{+} \int_{0}^{z} x f_{x}^{z}(x) dx + g^{+} z P.$$

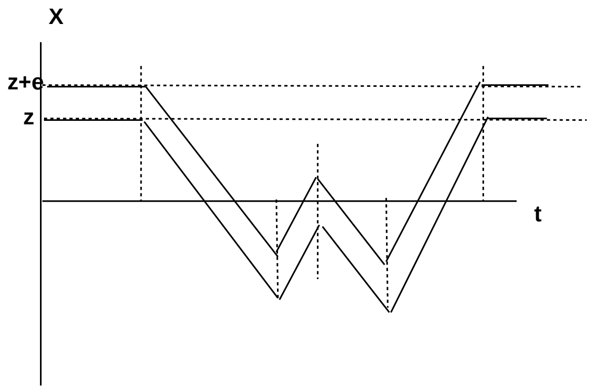

Figure 4-2: Trajectories with different hedging points

Similarly,

$$\bar{J}^{z+\epsilon} = -g^{-}\int_{-\infty}^{0} x f_x^{z+\epsilon}(x)dx + g^{+}\int_{0}^{z} x f_x^{z+\epsilon}(x)dx + g^{+}(z+\epsilon)P.$$

Consider two trajectories X^z and $X^{z+\epsilon}$ that are such that

$$x^z = z$$

and

$$x^{z+\epsilon} = z + \epsilon$$

These trajectories are illustrated in Figure 4-2. It can be observed that the trajectory $X^{z+\epsilon}$ is obtained by translating X^z by ϵ. Therefore,

$$x^{z+\epsilon}(t) = x^z(t) + \epsilon, \quad \text{for all} \ \ t$$

Therefore

$$f_x^{z+\epsilon}(x) = f_x^z(x-\epsilon)$$

Thus

$$\bar{J}^{z+\epsilon} - \bar{J}^z = -g^{-}\int_{-\infty}^{0} \epsilon f_x^z(x)dx + g^{+}\int_{0}^{z} \epsilon f_x^z(x)dx + \epsilon g^{+}P + o(\epsilon)$$

where the $o(\epsilon)$ term corresponds to the value of the integral over x such that $x > 0$

but $x - \epsilon < 0$. Dividing by ϵ and taking the limit as $\epsilon \to 0$ we get,

$$\frac{d\bar{J}^z}{dz} = -g^- \int_{-\infty}^{0} f_x^z(x)dx + g^+ \int_0^z f_x^z(x)dx + g^+ P$$

Therefore

$$\frac{d\bar{J}^z}{dz} = -g^- \text{Prob}(x \leq 0) + g^+ \text{ Prob } (x > 0) \tag{4.21}$$

Condition 4.21 is the hedging point estimator used for larger systems in Yan *et al.* (1992). For $Z > 0$ to be an optimal hedging point, the necessary condition for optimality of the function J^z at Z can be expressed as

$$\frac{d\bar{J}}{dz}|_{z=Z} = 0$$

This condition applied to (4.21) yields (4.19).

For $Z = 0$ to be an optimal hedging point, the necessary condition for optimality can be expressed as

$$\frac{d\bar{J}}{dz}|_{z=0} \geq 0$$

This condition applied to (4.21) yields (4.20). Thus the theorem is proved.

Lemma 3 *The optimal hedging point is equal to zero if and only if*

$$\frac{g^+}{g^-} \geq \frac{\mu p}{\frac{\mu - d}{E\{t^r\}} - pd} \tag{4.22}$$

Proof: To prove the 'only if' part, let $Z > 0$. Thus,

$$F^Z(0) < 1 - P$$

Substituting for P from (4.17) yields

$$F^Z(0) < 1 - \frac{\frac{1}{p}}{\frac{1}{p} + E\{C\}}$$

Using (4.19) from Theorem 6, we get

$$\frac{g^-}{g^- + g^+} > \frac{\frac{1}{p}}{\frac{1}{p} + E\{C\}}$$

Substituting for $E\{C\}$ from Corollary 3 and rearranging terms, we get

$$\frac{g^+}{g^-} < \frac{\mu p}{\frac{\mu - d}{E\{t^r\}} - pd}$$

To prove the 'if' part, consider $Z = 0$. Thus

$$F^0(0) = 1 - P$$

Substituting for P from (4.17), and for $F^0(0)$ from (4.20) in Theorem 6, and rearranging terms

$$\frac{g^-}{g^- + g^+} \leq \frac{\frac{1}{p}}{\frac{1}{p} + E\{C\}}$$

Substituting for $E\{C\}$ from Corollary 3 and rearranging terms, we get

$$\frac{g^+}{g^-} \geq \frac{\mu p}{\frac{\mu - d}{E\{t^r\}} - pd}$$

Thus the lemma is proved.

Equation (4.22) is an extension of the condition given by Bielecki and Kumar (1988) for the optimality of a zero hedging point in a system with exponential repair time distributions. The ratio in (4.22) depends only on the first moment of the repair time distribution and not on higher order moments.

For systems for which condition (4.22) is not satisfied, the optimal hedging point is strictly positive. For such systems, the optimal hedging point Z is chosen such that equation (4.19) in Theorem 6 is satisfied. An analytical solution for Z requires an analytical expression for $F^z(0)$. Theorem 5 yields the distribution function F_x^z for any repair time distribution in terms of the distribution of the workload process (f_w) of a $M/GI/1$ queue. In general, there is no closed form solution for this distribution but its Laplace transform can be easily calculated. An analytical calculation of the optimal hedging point is possible for repair time distributions where this Laplace transform can be inverted to find $F^z(0)$. In cases where the Laplace transform is not easily invertible, one can use Chernoff bounds (Kleinrock, 1975a) to get a good estimate of $F(0)$ and then use (4.19) to estimate Z.

4.4 Analysis of the differential cost for zero hedging point

In this section, we derive an exact analytical solution for the function $W(x,1)$ for $x \leq 0$. We assume that the system is operated under a hedging point policy with a hedging value of zero. As mentioned in Section 4.1, this analysis leads to a boundary condition for the partial differential equation for the differential cost of a two-part type system operated under the particular policy described in Chapter 5. Finally, we suggest an approach for evaluating $W(x,1)$ for $x > 0$.

4.4.1 Solution for $x < 0$

Consider a system operated under a hedging point policy with a zero hedging point. The non-transient region for this system is the subset $x \leq 0$. For any $x < 0$, the control is given by $u = \mu$. Since the policy is specified, the HJB equation, (4.11, without the minimization, represents the dynamics of the cost-to-go under the given policy. For any time t such that $x(t) < 0$, the cost-to-go dynamics in terms of the differential cost function $W(x,\alpha)$ and the average cost $\bar{J}$ can be expressed as

$$\begin{aligned} \bar{J}(1 + pE\{t^r\}) &= -g^{-}x - pW(x,1) + pE\left\{-\int_t^{t^r} g^{-}x(s)ds\right\} + E\{W(x - dt^r, 1)\} \\ &\quad + \frac{dW(x,1)}{dx}(\mu - d) \end{aligned} \tag{4.23}$$

In the interval $(t, t+t^r)$, the machine is down. Thus for any $s \in (t, t+t^r)$, we have

$$x(s) = x - d(s - t)$$

The integral on the right side of (4.23) is therefore

$$-pE\left\{\int_t^{t^r} g^{-}x(s)\underset{\cdot}{s}\right\} = -pg^{-}\left(xE\{t^r\} - \frac{dE\{(t^r)^2\}}{2}\right)$$

Plugging this back into (4.23),

$$(\bar{J} + g^{-}x)(1 + pE\{t^r\}) - pg^{-}\frac{dE\{(t^r)^2\}}{2}$$

$$= \quad -pW(x,1) + pE\{W(x - dt^r, 1)\} + \frac{dW(x,1)}{dx}(\mu - d) \tag{4.24}$$

Boundary condition: Equation(4.24) holds for all $x < 0$. At $x = 0$, the control changes discontinuously from $u = \mu$ to $u = 0$. To derive a boundary condition at $x = 0$, we consider the dynamics of the cost-to-go at the hedging point. Using the notation in Section 4.2, the dynamics of the cost-to-go at the hedging point can be expressed as

$$J(0,1,t,T) = (1 - p\delta t)J(0,1,t+\delta t,T) + p\delta t J(x(t+\delta t),0,t,T) + o(\delta t). \tag{4.25}$$

This equation is similar to equation (4.8). Here the left side denotes the cost-to-go at time t with the machine state $\alpha(t) = 1$ and $x(t) = 0$. The right side is an expression for the same cost obtained by conditioning on the two possible machine state transitions in the interval $(t, t+\delta t)$. The first term is the cost-to-go at time $t+\delta t$ if there is no machine failure in $(t, t+\delta t)$ and the second term is the cost-to-go if there is a machine failure in $(t, t+\delta t)$. Using Taylor series expansion of the functions $J(0,1,t,T)$ and $J(0,0,t,T)$ and neglecting second order terms in δt, we get,

$$\begin{aligned} J(0,1,t,T) - \frac{\partial J(0,1,t,T)}{\partial t}\delta t \quad &= \quad J(0,1,t,T) - p\delta t J(0,1,t,T) \\ &\quad\; + p\delta t J(x,0,t,T) + o(\delta t) \end{aligned}$$

Following the same steps as in the derivation of equation (4.11) in Section 4.2, the above expression can be written in terms of $W(x,1)$ and $\bar{J}$ as

$$\bar{J}(1 + pE\{t^r\}) = -pW(x,1) + pE\left\{-\int_t^{t^r} g^- x(s)ds\right\} + E\{W(x - dt^r, 1)\} \tag{4.26}$$

At $x = 0$, the functions $g(x)$ and $W(x,1)$ are continuous. Therefore a comparison of (4.23) and (4.26) yields the following condition:

$$\lim_{x \to 0^-} \frac{dW(x,1)}{dx} = 0. \tag{4.27}$$

Equation (4.27) is the boundary condition for the integro-differential equation

(4.24). Let us assume a solution to (4.24) of the form

$$W(x,1) = \frac{1}{2}ax^2 + bx + c \tag{4.28}$$

An intuitive reason for a quadratic form of the differential cost for $x_2 < 0$ is discussed in Chapter 3.

Plugging the quadratic form into (4.24) gives

$$\begin{aligned} & (\bar{J} + g^- x)(1 + pE\{t^r\}) - pg^- \frac{dE\{(t^r)^2\}}{2} \\ = \; & -p(\frac{1}{2}ax^2 + bx + c) + p(\frac{1}{2}aE\{(x - dt^r)^2\} + bE\{(x - dt^r)\} + c) \\ & +(ax + b)(u - d). \end{aligned}$$

Expanding the expectations, and equating terms in x^2, x and constant terms on both sides, we find

$$a = \frac{g^-(1 + pE\{t^r\})}{(\mu - d) - pdE\{t^r\}} \tag{4.29}$$

$$b = \frac{\bar{J}(1 + pt^r) - g^- pd\frac{E\{(t^r)^2\}}{2} - apE\{(dt^r)^2\}}{(\mu - d) - pdt^r} \tag{4.30}$$

For a zero hedging point the surplus is always non-negative. The average cost can therefore be expressed as

$$\bar{J} = -g^- \int_{-\infty}^{0} x f_x dx = -g^- E\{x\}.$$

Substituting for $E\{x\}$ from Corollary 4 and plugging the resulting expression into the equation for b yields

$$b = 0.$$

The constant c is undetermined because of the one degree of freedom in the HJB equation that was explained in Section 4.2.

The solution given in (4.28) with a determined by (4.29), $b = 0$ and any value of c satisfies the HJB equation, (4.24), and the boundary condition (4.27). It is therefore the complete solution for the differential cost $W(x,1)$ for $x < 0$. Using equation (4.12), one can evaluate $W(x,0)$.

Equation (4.28) is an extension of the solution in Bielecki and Kumar (1988), in the region $x_2 < 0$ for general repair time distribution. It is surprising that $W(x,1)$ is quadratic for $x < 0$ for any repair time distribution. This provides a validation for the use of quadratic approximations for $W(x, 1)$ in larger systems where it is difficult to evaluate it analytically. $W(x, 1)$ for $x < 0$ depends only on the the first moment of the repair time distribution. We now give an interpretation foe the quadratic coefficient in $W(x, 1)$. This interpretation may be useful for easily calculating the quadratic coefficient in $W(x, 1)$ for larger systems.

A related deterministic system Consider a system similar to the one considered thus far in this chapter, except it consists of a reliable machine whose capacity set is given by

$$0 \leq u \leq \bar{\mu} \tag{4.31}$$

where

$$\bar{\mu} = \mu \frac{\frac{1}{p}}{\frac{1}{p} + E\{t^r\}}. \tag{4.32}$$

Since the machine does not fail, this system is deterministic. Its state is represented by the surplus x, which evolves according to the differential equation

$$\dot{x} = u - d.$$

We now explain the relation between this deterministic system and the stochastic system studied in this chapter. Consider a typical failure-repair cycle of the machine in the stochastic system. The average length of a period when the machine is up is given by $\frac{1}{p}$. Once the machine fails, the average length of time for which it is down is given by $E\{t^r\}$. The fraction of time for which the machine is up is given by $\frac{\frac{1}{p}}{\frac{1}{p}+E\{t^r\}}$. Thus the capacity of the deterministic system, $\bar{\mu}$ is the product of μ and the fraction of the time the unreliable machine is up on average. In other words, the capacity set for the deterministic system is the average capacity set for the stochastic system. We refer to this deterministic system as the *AD* (for Average Deterministic) system corresponding to the stochastic system considered in this chapter.

Consider the production control problem for the AD system with a demand that satisfies (4.16). Since the machine is reliable, it follows that the optimum production control is given by

$$\begin{aligned} u &= 0 \quad x > 0 \\ u &= d \quad x = 0 \end{aligned}$$

$$u = \bar{\mu} \quad x < 0$$

That is, the hedging point is at the origin. From any starting point, this control guides the surplus towards the origin. Once the surplus reaches the origin, it can be maintained there since there are no machine failures and the system does not incur any further cost. Let $W^0(x)$ denote the differential cost for this system that is defined similar to equation (2.6) for the stochastic system. The long term average cost (J^*) for the AD system is 0. Consider some x such that $x < 0$. The trajectory from x reaches the origin after a time $\frac{x}{\mu - d}$. Thus

$$W^0(x) = \int_0^{(\frac{x}{\mu} - d)} g^- x(s) ds$$

Therefore

$$W^0(x) = \frac{g^-}{2(\bar{\mu} - d)} x^2.$$

Equations (4.29) and (4.32) imply that

$$a = \frac{g^-}{\bar{\mu} - d}.$$

From (4.28), the differential cost of state $(x, 1)$ for the stochastic system for $x < 0$ is given by

$$W(x,1) = \frac{g^-}{2(\bar{\mu} - d)} x^2 + c = W^0(x) + c. \tag{4.33}$$

For the stochastic system, the region $x < 0$ corresponds to the non-transient region of the surplus space for a zero hedging point. Since c is an arbitrary constant, (4.33) states that in the non-transient region of the surplus space, the differential cost function is equal to that of the AD system for any repair time distribution.

This justifies the argument in Gershwin (1993) that as $x \to -\infty$, there are many machine failures and repairs before the surplus reaches 0 and therefore the surplus trajectory on an average moves towards 0 with a rate $\bar{\mu} - d$ which is exactly the rate in the AD system.

We have shown that the linear term in the differential cost is zero for a zero hedging point. For systems with a non-zero hedging point, the boundary condition (4.27) is no longer valid at $x = 0$. We conjecture that the linear term is non-zero for this case. The quadratic part of the differential cost still has the same interpretation as before and should therefore be equal to the differential cost of the AD system.

Exponential repair times Consider a system where the machine repair times are exponentially distributed. In this case, $E\{t^r\} = \frac{1}{r}$ and the capacity set given by (4.31) is

$$0 \leq \bar{u} \leq \mu \frac{r}{r+p}$$

If the hedging point is zero, the expression for $W(x, 1)$ given by (4.33) for this case reduces to

$$W(x, 1) = \frac{g^-}{2(r(\mu - d) - pd)} x^2 + c, \text{ for } x < 0$$

which is the expression given in Gershwin (1993) for $x < 0$.

4.4.2 Solution for postive surplus

Since the system considered here is one that has a hedging point of zero, the control for $x > 0$ is given by $u = 0$. Consider a trajectory that originates at x at time $t = 0$ with $\alpha = 1$. This trajectory reaches reaches 0 at time t_1 where

$$t_1 = \frac{x}{d}$$

Using (4.11) without the minimization, the dynamics of the cost-to-go at x can be expressed as

$$\begin{aligned} \bar{J}(1 + pE\{t^r\}) \quad &= \quad g^+ x - pW(x, 1) + pE\left\{ \int_t^{t^r} g(x(s))ds \right\} + E\{W(x - dt^r, 1)\} \\ &\quad + \frac{dW(x, 1)}{dx}(\mu - d). \end{aligned} \tag{4.34}$$

Using similar arguments as in the derivation of (4.27) it can be shown that the boundary condition at the origin is given by:

$$\lim_{x \to 0^+} \frac{dW(x, 1)}{dx} = 0. \tag{4.35}$$

The right side of the integro-differential equation (4.34) depends on the realization of t^r. If $t^r < t_1$, then the surplus is always in the positive region in $(0, t^r)$ whereas for $t^r \geq t_1$, the surplus goes from a positive value to a negative value in $(0, t^r)$. This suggests that a simple analytical form for the solution to (4.34) is unlikely because the evaluation of the second and third terms on the right side in (4.34) requires knowledge of the entire distribution of t^r. It is however possible to solve (4.34) subject to the boundary condition (4.35) numerically, since the initial conditions are completely specified by the solution given by (4.33) for all x belonging to the interval $[-\infty, 0]$.

4.5 Summary

In this chapter, we have considered a single-part type system with general repair time distribution. The surplus behavior for such a system under any hedging point policy was analyzed and conditions were derived under which the best hedging point policy is one that has a zero hedging point.

A methodology for solving the HJB equation was derived for the special case of zero hedging point and an analytical solution was obtained over the non-transient region (*i.e.*, for $x < 0$). Besides being of interest in the context of a single-part-type system this solution also forms a boundary condition on the HJB equation for a two-part-type system under the policy described in Chapter 5. It was shown that this differential cost is quadratic for any repair time distribution. This result is important because it validates the use of quadratic approximations for systems where it is difficult to evaluate the differential cost. Moreover, the quadratic term was shown to be equal to the differential cost of a related deterministic system that is easy to evaluate. For $x > 0$, the differential cost was shown to be the solution of an integro-differential equation whose initial conditions are given by the solution for $x < 0$. We believe that the hedging point policy is optimal for general repair time distributions but since there is no closed-form solution for $W(x, 1)$ for $x > 0$, verification of the optimality conditions can only be done numerically. This is a direction for future research. Another area of research is in developing approximate procedures for estimating the optimal hedging point for a given repair time distribution.

Chapter 5

Two-part-type system: Analytical solution of the HJB equations for a specific policy

In this chapter, we describe a method for finding an analytical solution for the HJB equations for a particular control policy in a two-part-type system. The policy is the optimal overtaking policy if the machine were perfectly reliable. In general, this policy will be overtaking optimal for some but not all ranges of parameters in an unreliable-machine system. There are many motivations for seeking analytical solutions for such systems

- We expect that the optimal policy for a system with a reliable machine is also optimal for a system with an unreliable machine if failures are infrequent and machine repair times are small or if backlog costs are small and surplus costs are high. This is indeed the case for single-part-type systems (Bielecki and Kumar, 1988; Gershwin, 1993, Chapters 3, 4). A solution to the HJB equations for some policy enables one to verify whether that policy satisfies the optimality conditions given by (2.9).

- For systems involving several machines, a controller based on a quadratic approximation to the differential cost has been proposed in Kimemia (1982). Exact expressions for the differential cost for the simple system considered here provide a benchmark for testing the the validity of a quadratic approximation in stochastic systems.

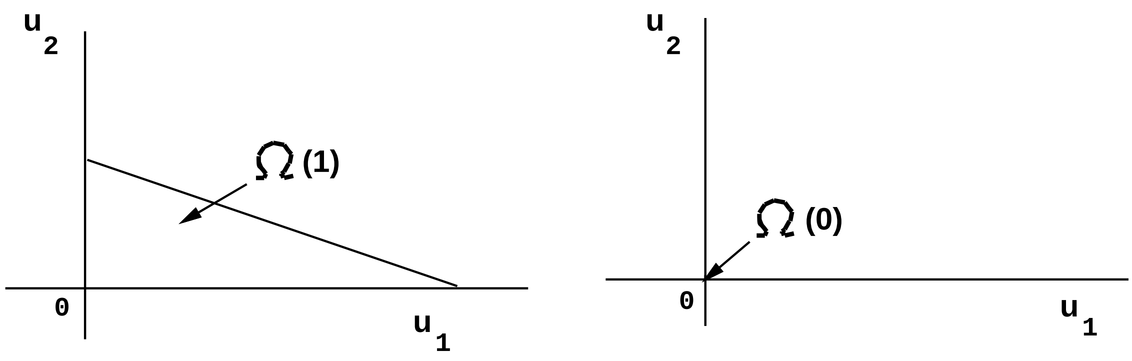

Figure 5-1: Capacity set for the two machine states

- Liberopolous (1992) has considered numerical solutions for the kind of systems considered here. Though numerical solutions remain the only method for finding the optimal policy in larger systems, they involve a great deal of computational effort. A good understanding of analytical results for the system considered here can lead to better formulations of boundary conditions and more efficient solution procedures for finding numerical solutions.

5.1 Description of the two-part-type system

In this section, we describe the two-part-type case of the system given in Section 2.2. The capacity set is specified by equations (2.3) and (2.4). For $\alpha = 0$, the the capacity set consists of the single point, $\boldsymbol{u} = (0,0)$. For $\alpha = 1$, the system has a triangular capacity set as shown in Figure 5-1. In other words, when $\alpha = 1$ any production rate vector that satisfies the following condition is feasible

$$\begin{aligned} \frac{u_1}{\mu_1} + \frac{u_2}{\mu_2} &\leq 1, \\ u_1 &\geq 0 \\ , u_2 &\geq 0 \end{aligned} \tag{5.1}$$

From (2.7), the optimality condition for $\alpha = 0$ can be expressed as

$$J^* = \min_{u \in \Omega(0)} \left\{ g(\boldsymbol{x}) + \sum_{i=1}^{2} \left[(u_i - d_i) \frac{\partial W(\boldsymbol{x},0)}{\partial x_i} - r(W(\boldsymbol{x},0) - W(\boldsymbol{x},1)) \right] \right\}. \tag{5.2}$$

Similarly from (2.8), the optimality condition for machine state, $\alpha = 1$, can be expressed as

$$J^* = \min_{u \in \Omega(1)} \left\{ g(\boldsymbol{x}) + \sum_{i=1}^{2} \left[(u_i - d_i) \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_i} + p(W(\boldsymbol{x}, 0) - W(\boldsymbol{x}, 1)) \right] \right\}. \quad (5.3)$$

Let $\boldsymbol{z} = (z_1, z_2)$ denote the hedging point for this system. From Chapter 2, it follows that the optimal policy divides the surplus space into exactly three regions that intersect at the hedging point $\boldsymbol{z}$. The optimal control vector in each region is one of the three vertices of the capacity set $\Omega(1)$.

In Section 5.2 , we study two related deterministic two-part-type systems. The following sections will again focus on the stochastic two-part-type system described here but will use insights from Section 5.2. Without loss of generality, it is assumed for the rest of this chapter that the two part types are numbered such that

$$g_1^- \mu_1 \geq g_2^- \mu_2.$$

5.2 Two related deterministic systems

In this section, we describe two deterministic systems that is closely related to the two-part-type system described in Section 5.1 and give expressions for the differential cost function in each system.

5.2.1 Reliable deterministic system

Consider a two-part type system that is similar in all respects to the system described in Section 5.1 except that the machine is reliable. In other words, the machine state is always $\alpha = 1$ and the change of the surplus state is deterministic. The capacity set is the set of production rates of the two part types that satisfy

$$\begin{aligned} \frac{u_1}{\mu_1} + \frac{u_2}{\mu_2} &\leq 1, \\ u_1 &\geq 0 \\ u_2 &\geq 0 \end{aligned} \quad (5.4)$$

We refer to this system as the *Reliable Deterministic* (RD) system.

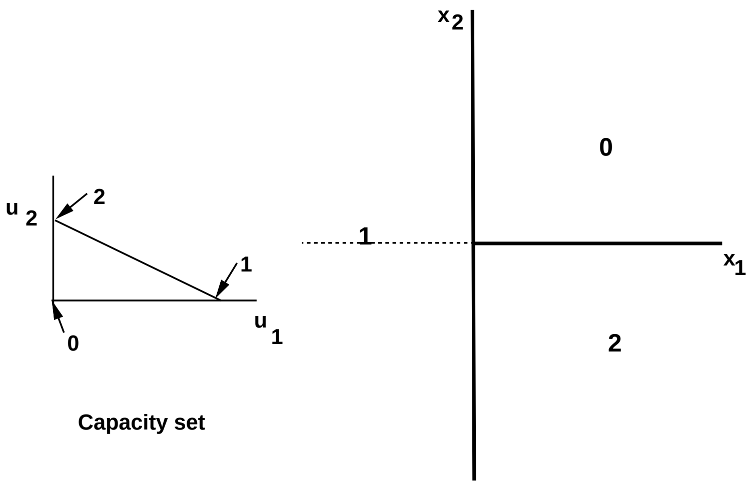

Figure 5-2: The OODS policy

Label	Region definition	Control
0	$x_1 > 0, x_2 > 0$	$u = (0, 0)$
1	$x_1 < 0$	$u = (\mu_1, 0)$
2	$x_1 > 0, x_2 < 0$	$u = (0, \mu_2)$
Label	Boundary definition	Control
$0 - 2$	$x_1 > 0, x_2 = 0$	$u = (0, d_2)$
$0 - 1$	$x_1 = 0, x_2 > 0$	$u = (d_1, 0)$
$1 - 2$	$x_1 = 0, x_2 < 0$	$u = (d_1, \mu_2(1 - \frac{d_1}{\mu_1}))$
$\boldsymbol{z}$	$x_1 = 0, x_2 = 0$	$u = (d_1, d_2)$

Table 5.1: OODS policy

Connolly (1992) has studied the production control problem in a deterministic two-part-type system. She shows that the overtaking optimal policy guides the surplus from any given starting state to a hedging point of $(0, 0)$. Once the surplus reaches $(0, 0)$ it is maintained there. This policy divides the surplus space into three regions each corresponding to one vertex of the capacity set (5.4). We refer to this policy as the *Overtaking Optimal Deterministic System* (OODS) policy.

Description of the OODS policy The OODS policy is graphically depicted in Figure 5-2. The hedging point is at $\boldsymbol{z} = \boldsymbol{x} = 0$. The policy divides the surplus space into three regions each corresponding to a different control vector. The set of

Boundary	Characterization
$0a$-$0b$	$d_1 x_2 + d_2 x_1 = 0, x_1, x_2 \geq 0$
$1a$-$1b$	$(\mu_1 - d_1)x_2 - d_2 x_1 = 0, x_2 \geq 0, x_1 \leq 0$
$2a$-$2b$	$-d_1 x_2 + (\mu_2 - d_2)x_1 = 0, x_1 \geq 0, x_2 \leq 0$
$1b$-$1c$	$x_2 = 0, x_1 \leq 0$

Table 5.2: Subregion boundaries

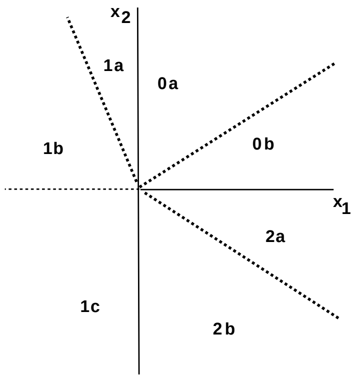

Figure 5-3: Regions and subregions of OODS policy

points at the boundary between adjacent regions is referred to as a *region boundary*. The control $\boldsymbol{u}$ changes discontinuously at region boundaries and all boundaries are attractive. The control vector $u(\boldsymbol{x}, 1)$ in each region and along the region boundaries is defined in Table 5.1. $u(\boldsymbol{x}, 0) = 0$ for all $\boldsymbol{x}$.

Connolly (1992) shows that the differential cost function is piecewise quadratic and divides the surplus space into different *subregions*. Each subregion is a subset of the surplus space over which the differential cost function is twice continuously differentiable. The boundary between adjacent subregions is referred to as a *subregion boundary*. The regions and subregions for the OODS policy are shown in Figure 5-3. Numbers correspond to regions $(0, 1, 2)$ and letters (a, b, c) correspond to subregions within a given region. For example, $1c$ refers to Subregion c in Region 1. The equations for the subregion boundaries are given in Table 5.2.

The differential cost function in each subregion has the form

$$W(\boldsymbol{x}) = \frac{1}{2}a_{11}x_1^2 + a_{12}x_1 x_2 + \frac{1}{2}a_{22}x_2^2.$$

The quadratic coefficients in the various subregions are as given in Table 5.3. The

Subregion	a_{11}	a_{12}	a_{22}
$0a$	$\frac{g_1^+}{d_1}$	0	$\frac{g_2^+}{d_2}$
$0b$	$\frac{g_1^+}{d_1}$	0	$\frac{g_2^+}{d_2}$
$1a$	$\frac{g_1^-}{\mu_1 - d_1}$	0	$\frac{g_2^+}{d_2}$
$1b$	$\frac{g_1^-}{\mu_1 - d_1} + \frac{g_2^- d_2 u_2^b}{(\mu_1 - d_1)^2 (u_2^b - d_2)}$	$\frac{g_2^- u_2^b}{(\mu_1 - d_1)(u_2^b - d_2)}$	$\frac{g_2^- u_2^b}{d_2 (u_2^b - d_2)} + \frac{g_2^+}{d_2}$
$1c$	$\frac{g_1^-}{\mu_1 - d_1} + \frac{g_2^- d_2 u_2^b}{(\mu_1 - d_1)^2 (u_2^b - d_2)}$	$\frac{g_2^- u_2^b}{(\mu_1 - d_1)(u_2^b - d_2)}$	$\frac{g_2^-}{u_2^b - d_2}$
$2a$	$\frac{g_1^+}{d_1}$	0	$\frac{g_2^-}{\mu_2 - d_2}$
$2b$	$\frac{g_1^+}{d_1} + \frac{g_2^- (\mu_2 - d_2)(\mu_2 - u_2^b)}{d_1^2 (u_2^b - d_2)}$	$\frac{-g_2^- (\mu_2 - u_2^b)}{d_1 (\mu_2^b - d_2)}$	$\frac{g_2^-}{(u_2^b - d_2}$

Table 5.3: Quadratic coefficients for RD system

symbol u_2^b denotes the production rate of Part Type 2 on the $1c - 2b$ boundary when the machine is up. Therefore u_b^2 is given by

$$u_2^b = \mu_2 (1 - \frac{d_1}{\mu_1}). \tag{5.5}$$

Notice that the differential cost function has different quadratic coefficients in Subregions $2a$ and $2b$ even though the control vector given by OODS policy is the same. The intuitive reason for this difference is a trajectory that originates in $2a$ hits the $0b - 2a$ boundary whereas a trajectory that originates in $2b$ hits the $1c - 2b$ boundary. A similar argument applies to Subregions $1a$ and $1b$. There is a different reason for the discontinuity in the second derivative of the differential cost at the $1b - 1c$ boundary. This discontinuity is caused by the discontinuity in the derivative of the instantaneous cost function g along $x_2 = 0$.

The results from this section are used in Section 5.2.2 for another deterministic system and in Section 5.4 for the stochastic system.

5.2.2 Average deterministic system

In this section, we describe another deterministic system that is related to the two-part-type system in Section 5.1. We motivate this study using some results for single-part-type systems.

Motivation In Chapter 3, it is shown that the optimal policy for the single-part-type case of the system described in Chapter 2 divides the surplus space into regions over which the control is constant. These partitions are determined by the location of the hedging point z. For some ranges of parameter values, the optimal value of z is zero. The partitions in this case are the same as those for the optimal overtaking policy for a special deterministic system. This deterministic system is identical in all respects to the single part-type stochastic system except that it has a reliable machine with a maximum production rate $\hat{\mu}$ where

$$\hat{\mu} = \mu \frac{r}{r+p}$$

The symbol $\hat{\mu}$ represents the average of the capacity sets over the two machine states of the unreliable machine. This deterministic system was called the *Average Deterministic* (AD) system in Chapter 4. The differential cost of the AD is closely related to that of the unreliable-machine system (Gershwin, 1993). In Chapter 4, it is shown that this relationship holds for single-part-type systems with general repair times. It therefore seems reasonable to expect that this relationship hold for a two-part-type system as well.

Definition of the AD system Consider a deterministic two-part-type system that is similar in all respects to the system in Section 5.1 except that it has a single reliable machine whose capacity set is given by

$$\begin{aligned} \frac{u_1}{\mu_1} + \frac{u_2}{\mu_2} &\leq \frac{r}{r+p}, \\ u_1 &\geq 0 \\ u_2 &\geq 0 \end{aligned} \tag{5.6}$$

That is, the capacity set of the reliable machine is the average of the capacity sets of the two states of the unreliable machine described in Section 5.1. Using the same terminology as in Chapter 4, this deterministic system is referred to as the Average Deterministic (AD) system corresponding to the two-part-type stochastic system.

The capacity set in (5.6) can also be expressed as

$$\frac{u_1}{\frac{r}{r+p}\mu_1} + \frac{u_2}{\frac{r}{r+p}\mu_2} \leq 1$$

Let us define

$$\bar{\mu}_i = \mu_i \left(\frac{r}{r+p} \right) \quad i = 1, 2. \tag{5.7}$$

Region	Control
$x_1 < 0$	$u = (\bar{\mu}_1, 0)$
$x_1 > 0, x_2 > 0$	$u = (0, 0)$
$x_1 > 0, x_2 < 0$	$u = (0, \bar{\mu}_2)$
$x_1 > 0, x_2 = 0$	$u = (0, d_2)$
$x_1 = 0, x_2 > 0$	$u = (d_1, 0)$
$x_1 = 0, x_2 < 0$	$u = (d_1, \bar{\mu}_2(1 - \frac{d_1}{\bar{\mu}_1}))$
$x_1 = 0, x_2 = 0$	$u = (d_1, d_2)$

Table 5.4: Optimal policy for AD system

Boundary	Characterization
$0a$-$0b$	$d_1x_2 + d_2x_1 = 0, x_1, x_2 \geq 0$
$1a$-$1b$	$(\bar{\mu}_1 - d_1)x_2 - d_2x_1 = 0, x_2 \geq 0, x_1 \leq 0$
$2a$-$2b$	$-d_1x_2 + (\bar{\mu}_2 - d_2)x_1 = 0, x_1 \geq 0, x_2 \leq 0$
$1b$-$1c$	$x_2 = 0, x_1 \leq 0$

Table 5.5: Subregion boundaries

Therefore the capacity set of the AD system becomes

$$\frac{u_1}{\bar{\mu}_1} + \frac{u_2}{\bar{\mu}_2} \leq 1 \tag{5.8}$$

A comparison of (5.4) and (5.8) shows that the AD system is same as the RD system in Section 5.2.1 but with a reduced capacity set. Consequently all from the RD system can be translated into those for the AD system with μ_1 replaced by $\bar{\mu}_1$ and μ_2 replaced by $\bar{\mu}_2$. Let

$$\bar{u}_2^b = \bar{\mu}_2(1 - \frac{d_1}{\bar{\mu}_1}). \tag{5.9}$$

Using equation (5.7), $\bar{\mu}_i$ for $i = 1, 2$ can be intepreted as the maximum possible production rate of Part Type i in the AD system. A comparison of (5.5) and (5.9) shows that $\bar{u}_2^b$ is the production rate of Part Type 2 along the $1c - 2b$ boundary in the AD system.

The results from Section 5.2.1 imply that the optimal policy for this system is as shown in The equations for the subregion boundaries are as given in Table 5.5. The differential cost function in each subregion has the same expression as that in the RD system with the modified production rates. Thus the differential cost function is of

Subregion	a_{11}	a_{12}	a_{22}
$0a$	$\frac{g_1^+}{d_1}$	0	$\frac{g_2^+}{d_2}$
$0b$	$\frac{g_1^+}{d_1}$	0	$\frac{g_2^+}{d_2}$
$1a$	$\frac{g_1^-}{\bar{\mu}_1 - d_1}$	0	$\frac{g_2^+}{d_2}$
$1b$	$\frac{g_1^-}{\bar{\mu}_1 - d_1} + \frac{g_2^- d_2 \bar{\mu}_2^b}{(\bar{\mu}_1 - d_1)^2 (\bar{u}_2^b - d_2)}$	$\frac{g_2^- \bar{\mu}_2^b}{(\bar{\mu}_1 - d_1)(\bar{\mu}_2^b - d_2)}$	$\frac{g_2^- \bar{\mu}_2^b}{d_2 (\bar{u}_2^b - d_2)} + \frac{g_2^+}{d_2}$
$1c$	$\frac{g_1^-}{\bar{\mu}_1 - d_1} + \frac{g_2^- d_2 \bar{\mu}_2^b}{(\bar{\mu}_1 - d_1)^2 (\bar{u}_2^b - d_2)}$	$\frac{g_2^- \bar{\mu}_2^b}{(\bar{\mu}_1 - d_1)(\bar{\mu}_2^b - d_2}$	$\frac{g_2^-}{\bar{u}_2^b - d_2)}$
$2a$	$\frac{g_1^+}{d_1}$	0	$\frac{g_2^-}{\bar{\mu}_2 - d_2}$
$2b$	$\frac{g_1^+}{d_1} + \frac{g_2^- (\bar{\mu}_2 - d_2)(\bar{\mu}_2 - \bar{\mu}_2^b)}{d_1^2 (\bar{\mu}_2^b - d_2)}$	$\frac{-g_2^- (\bar{\mu}_2 - \bar{\mu}_2^b)}{d_1 (\bar{\mu}_2^b - d_2)}$	$\frac{g_2^-}{\bar{\mu}_2^b - d_2}$

Table 5.6: Quadratic coefficients for AD system

the form

$$W(\boldsymbol{x}) = \frac{1}{2} a_{11} x_1^2 + a_{12} x_1 x_2 + \frac{1}{2} a_{22} x_2^2.$$

where the coefficients in the various subregions of the AD system are as given in Table 5.6. In the remainder of this chapter, we analyze the two-part-type system in Section 5.1. Througout this analysis, we use insights from the results for the RD and AD systems and look for relationships between the stochastic system and the AD system similar to those observed in the single-part-type system.

5.3 Characterization of boundaries in the stochastic system

In this section, we provide partial characterizations of the optimal policy to the two-part-type system of Section 5.1 based on the results for the deterministic systems in Section 5.2.

It is reasonable to assume that for points that are far away from the hedging point, the behavior of the surplus trajectory under the optimal policy is similar to that in the AD system. This is because for such points, there are typically many machine failures and repairs before the trajectory reaches the hedging point. The average production rate over many machine state changes tends to that given by the AD system. Similarly for points that are close to the hedging point, the behavior must be similar to that of the RD system since the probability of a machine failure before the trajectory reaches the hedging point is very low.

As seen in Section 5.2.2, the region boundaries that characterize the optimal overtaking policy of the AD and the RD systems are all attractive. It is therefore reasonable to assume that all boundaries of the stochastic system are also attractive. This assumption allows for a partial characterization of the boundaries in the stochastic system.

Characterization of boundaries Let u^i be the control vector in Region i. From Chapter 2, the control vector on the boundary between Regions i and j, denoted by $\boldsymbol{u}^{ij}$, can be represented as

$$\boldsymbol{u}^{ij}(\boldsymbol{x}) = \lambda(\boldsymbol{x})\boldsymbol{u}^i + (1 - \lambda(\boldsymbol{x}))\boldsymbol{u}^j \tag{5.10}$$

where $\lambda(\boldsymbol{x})$ is such that $0 \leq \lambda(\boldsymbol{x}) \leq 1$

Consider the boundary between Region 0 and Region 1. This boundary is shown in Figure 5-4. In Region 0, $\boldsymbol{u}^0 = \mathbf{0}$ and thus the rate of change of surplus is given by:

$$\dot{x}_1 = -d_1, \;\; \dot{x}_2 = -d_2 \tag{5.11}$$

Similarly in Region 1, $\boldsymbol{u}^0 = (\mu_1, 0)$ and thus the rate of change of surplus is given by:

$$\dot{x}_1 = \mu_1 - d_1, \;\; \dot{x}_2 = -d_2 \tag{5.12}$$

From (5.10), (5.11) and (5.12), the rate of change of surplus on the boundary between these regions satisfies

$$-d_1 \leq \dot{x}_1 \leq \mu_1 - d_1, \;\; \dot{x}_2 = -d_2 \tag{5.13}$$

Condition (5.13) holds for each point on the boundary. If the boundary is attractive, then the surplus trajectory on reaching the boundary, stays on it, until the next machine failure. This implies that (5.13) is a constraint on the shape of the boundaries. Specifically, constraint (5.13) implies that the boundary between Region 0 and Region 1 lies within a cone with vertex at the hedging point and having the vectors $(-d_1, -d_2)$ and $(\mu_1 - d_1, -d_2)$ as its two extreme rays.

A similar argument for the other regions specifies cones that contain their boundaries. The extreme rays for the different cones are given in Table 5.7. The three cones, one corresponding to the location of each boundary, are illustrated in Figure 5-4.

Boundary	Extreme Rays
$0-1$	$(-d_1, -d_2),\ (\mu_1 - d_1, -d_2)$
$1-2$	$(\mu_1 - d_1, -d_2),\ (-d_1, \mu_2 - d_2)$
$0-2$	$(-d_1, -d_2),\ \ (-d_1, \mu_2 - d_2)$

Table 5.7: Cones for region boundaries

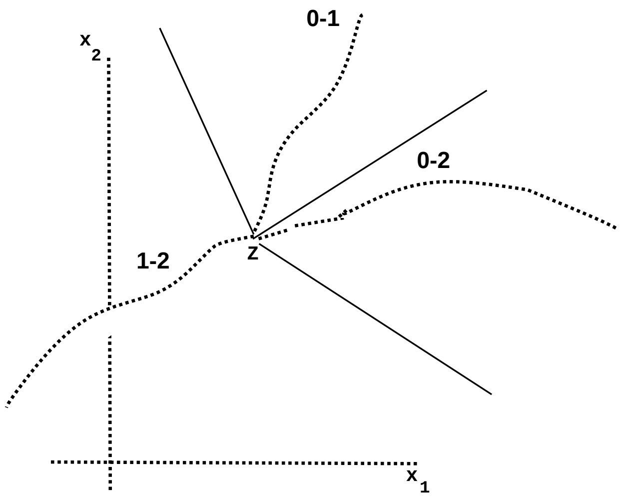

Figure 5-4: Location of boundaries

5.4 Analysis of the OODS policy

In this section, we consider the stochastic system of Sections 5.1 and 5.3 operated under the OODS policy of Section 5.2. This policy is specified in Table 5.1 and illustrated in Figure 5-2. In a deterministic system, the OODS policy drives the surplus to zero at the minimum possible cost and maintains it at zero. In the stochastic system, the policy still drives the surplus to zero from any starting solution but there may be machine failures and repairs before the surplus reaches the hedging point. Also, once the surplus reaches the hedging point, a machine failure would cause the policy to move away from the hedging point. We therefore expect this policy to be overtaking optimal for some but not all parameter values in the stochastic system. Based on the discussion in Section 5.2.2, we also expect some relationship between the differential costs of the stochastic system and those of the AD system.

The HJB equations, (5.2) and (5.3), for the OODS policy (without the minimization since the control is known) represent the dynamics of the cost-to-go under this policy. The coefficients of equations depend on the control vector and are therefore different in different regions. Let $\bar{J}$ be the average cost of the OODS policy as defined in Section 2.3. The HJB equations in Region j become the following set of coupled partial differential equations in $W(\boldsymbol{x}, 0)$ and $W(\boldsymbol{x}, 1)$:

$$\bar{J} = g(\boldsymbol{x}) + \sum_{i=1}^{2} \left(-d_i \frac{\partial W(\boldsymbol{x}, 0)}{\partial x_i} \right) - r(W(\boldsymbol{x}, 0) - W(\boldsymbol{x}, 1)) \tag{5.14}$$

$$\bar{J} = g(\boldsymbol{x}) + \sum_{i=1}^{2} \left((u_i^j - d_i) \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_i} \right) + p(W(\boldsymbol{x}, 0) - W(\boldsymbol{x}, 1)) \tag{5.15}$$

where u_i^j is the production rate of Part Type i in Region j chosen according to Table 5.1.

Based on the insight provided by the partition of the surplus space into subregions under the OODS policy from Section 5.2), we again subdivide the surplus space into subregions such that the differential cost in each subregion is twice continuously differentiable. In other words, as in the deterministic systems of Section 5.2, the second derivatives of the differential cost function may be discontinuous at subregion boundaries. As in Section 5.2, there are two reasons for these discontinuities: discontinuities in the derivative of the instantaneous cost function $g(x)$ at a subregion boundary; and the difference in boundary conditions for the HJB equations in the two subregions on either side of a subregion boundary.

This leads to the same subregion boundaries as in the RD system given in Table

5.2. The regions and subregions are as shown in Figure 5-3. Sub-region boundary $1b-1c$ in Region 1 corresponds to the set of discontinuities in the derivative of $g(\boldsymbol{x})$. On the other hand, the subregion boundaries $0a-0b$, $1a-1b$ and $2a-2b$ correspond to discontinuities in boundary conditions. As seen in Section 5.2, the discontinuities in the latter case arise from the difference in the qualitative behavior between two trajectories that originate on different sides of such a boundary. A trajectory starting in Region $2b$ never crosses into Region $2a$ and always hits the $1c-2b$ boundary before reaching the hedging point. A trajectory starting in Region $2a$ could remain in Region $2a$ until it reaches the hedging point (for example, if the initial machine state is up and the machine remains until the surplus reaches the hedging point); or it could move into Region $2b$ (for example, if the machine fails and remains down until after the trajectory hits the $2a-2b$ boundary).

HJB equations in subregions We now combine the set of two HJB equations (5.14) and (5.15) in each region to obtain a single second order partial differential equation involving only $W(\boldsymbol{x},1)$. In the following sections, the boundary conditions are also expressed in terms of $W(\boldsymbol{x},1)$ and its partial derivatives.

Within each subregion, equations (5.14) and (5.15) are satisfied. Rearranging terms in (5.14) gives

$$pW(\boldsymbol{x},0) = pW(\boldsymbol{x},1) + \bar{J} - g(\boldsymbol{x}) - (u_1-d_1)\frac{\partial W(\boldsymbol{x},0)}{\partial x_1} - (u_2-d_2)\frac{\partial W(\boldsymbol{x},0)}{\partial x_2} \quad (5.16)$$

Let g_1 and g_2 denote the derivatives of $g(x)$ with respect to x_1 and x_2 . Since $W(\boldsymbol{x},0)$ and $W(\boldsymbol{x},1)$ are twice continuously differentiable in any subregion, differentiation of (5.16) with respect to x_1 yields

$$p\frac{\partial W(\boldsymbol{x},0)}{\partial x_1} = p\frac{\partial W(\boldsymbol{x},1)}{\partial x_1} + g_1 - (u_1-d_1)\frac{\partial^2 W(\boldsymbol{x},0)}{\partial x_1^2} - (u_2-d_2)\frac{\partial^2 W(\boldsymbol{x},0)}{\partial x_2 \partial x_1} \quad (5.17)$$

Differentiation of (5.16) with respect to x_2 yields,

$$p\frac{\partial W(\boldsymbol{x},0)}{\partial x_2} = p\frac{\partial W(\boldsymbol{x},1)}{\partial x_2} + g_2 - (u_1-d_1)\frac{\partial^2 W(\boldsymbol{x},0)}{\partial x_1 \partial x_2} - (u_2-d_2)\frac{\partial^2 W(\boldsymbol{x},0)}{\partial x_2^2} \quad (5.18)$$

Substituting equations (5.16), (5.17) and (5.18) into (5.14), we can rewrite the HJB equation within each subregion in terms of a single function $W(\boldsymbol{x},1)$ as

$$\begin{array}{rc} & d_1(u_1-d_1)\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_1^2}+d_2(u_2-d_2)\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_2^2} \\ + & (d_1(u_2-d_2)+d_2(u_1-d_1))\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_1 \partial x_2} \\ + & (-pd_1+r(u_1-d_1))\frac{\partial W(\boldsymbol{x},1)}{\partial x_1} \\ + & (-pd_2+r(u_2-d_2))\frac{\partial W(\boldsymbol{x},1)}{\partial x_2} \\ = & \bar{J}(r+p)-g(x)(r+p)-d_1g_1-d_2g_2 \end{array} \tag{5.19}$$

For example, in subregions $2a$ and $2b$ of Region 2, the control vector is $\boldsymbol{u}=(0,\mu_2)$ and the HJB equation can therefore be written as

$$\begin{array}{rc} & -d_1^2\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_1^2}+d_2(\mu_2-d_2)\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_2^2} \\ + & d_1(u_2-2d_2)\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_1 \partial x_2} \\ - & (r+p)d_1\frac{\partial W(\boldsymbol{x},1)}{\partial x_1} \\ + & (-pd_2+r(\mu_2-d_2))\frac{\partial W(\boldsymbol{x},1)}{\partial x_2} \\ = & \bar{J}(r+p)+(-g_1^+x_1+g_2^-x_2)(r+p)-d_1g_1-d_2g_2 \end{array}$$

This is a hyperbolic second order partial differential equation in $W(\boldsymbol{x},1)$. The two families of *characteristic directions* of this hyperbolic second order partial differential equation are given by:

$$(\mu_2-d_2)x_1+d_1x_2=c_1$$

and

$$d_2x_1+d_1x_2=c_2$$

These two characteristic directions correspond to the directions of movement of the surplus trajectory in the two machine states $\alpha=1$ and $\alpha=0$. For $c_1=0$, the characteristic $(\mu_2-d_2)x_1+d_1x_2=0$, corresponds to the subregion boundary between subregions $2a$ and $2b$. This is not surprising since this subregion boundary corresponds to a discontinuity in the boundary conditions and discontinuities in boundary conditions are propagated along characteristic directions of hyperbolic partial differential equations (Zauderer, 1983).

Boundary conditions The continuity properties of $W(x,\alpha)$ and its derivatives explained in Chapter 2 provide the boundary conditions for the partial differential equations. The boundary conditions can be summarized as follows:

1. $W(\boldsymbol{x}, \alpha)$ is continuous at all region and subregion boundaries.

2. $W(\boldsymbol{x}, \alpha)$ is continuously differentiable at all subregion boundaries.

In the following section, we analyze the $1c - 2b$ boundary and derive a solution for $W(\boldsymbol{x}, 1)$ along the boundary. The later sections focus on each subregion in detail and describe solution methodologies appropriate for each subregion.

5.5 Analysis of the $1c - 2b$ boundary

Outline In this section, we solve for the differential cost function $W(\boldsymbol{x}, 1)$ along the $1c - 2b$ boundary. We first describe the behavior of the surplus along the boundary. The HJB equation along the boundary is then derived. It is shown to be the same as that in the single-part-type system with general repair times in Chapter 4. We then use the results from Chapter 4 to obtain the solution along the $1c - 2b$ boundary.

5.5.1 Surplus behavior

Consider a trajectory that is at some point $(0, x_2)$ on $1c - 2b$ boundary at time t, with $\alpha(t) = 1$ and $x_2 < 0$. The behavior of such a trajectory is illustrated in Figure 5-5. The control vector at any point on the boundary except the hedging point is given by

$$u^{1c-2b}((0, x_2), 1) = (d_1, \mu_2(1 - \frac{d_1}{\mu_1})).$$

This control keeps the trajectory on the boundary until the next machine failure. When the machine fails, the surplus is depleted at the demand rate and trajectory moves into Region $1c$, where the control is given by

$$\begin{aligned} u^{1c}(\boldsymbol{x}, 1) = &\quad (\mu_1 - d_1, 0) \quad \alpha = 1, \\ u^{1c}(\boldsymbol{x}, 0) = &\qquad (0, 0) \qquad \alpha = 0. \end{aligned}$$

When the machine gets repaired, the control vector guides the surplus trajectory from Region $1c$ back towards the boundary. There may be subsequent failures and repairs of the machine when the trajectory is in Region $1c$ but if the system has enough capacity to meet demand in the long-run, the trajectory eventually gets back to the boundary.

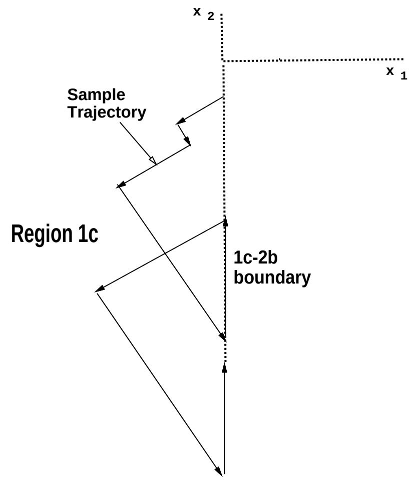

Figure 5-5: Surplus behavior on $1c - 2b$ boundary

Let $t + t^r$ be the random time when the trajectory reaches the boundary again for the first time given that the trajectory is on the boundary at time t and a failure has just occurred. This random variable is independent of the value of the Part Type 2 surplus, x_2, at the instant of the machine failure.

For the time interval when the trajectory is in Region $1c$, the machine devotes all of its available capacity to Part Type 1. Thus Part Type 2 'sees' a machine failure whenever the machine is down or all of its capacity is devoted to Part Type 1 and t^r denotes the 'repair' time of the machine as seen by Part Type 2. The behavior of Part Type 2 on the $1c - 2b$ boundary is therefore equivalent to that of a single-part-type system in Chapter 4 with t^r as the repair time. From Chapter 4, we know that if the distribution of t^r were to be specified, then the functions $W(\boldsymbol{x}, 0)$ and $W(\boldsymbol{x}, 1)$ along the boundary could be evaluated. It is important to note that this evaluation *does not depend* on the values of $W(\boldsymbol{x}, 1)$ and $W(\boldsymbol{x}, 0)$ in other sub-regions or boundaries. This is important because it shows that the HJB equations in the different sub-regions need not all be solved simulatenously but can be solved in a sequential manner. In the following sections, we use this sequential solution procedure to propagate the solution from the $0b - 1c$ boundary to other sub-regions.

5.5.2 Solution along $1c - 2b$ boundary

Let $\dot{\boldsymbol{x}}^b = (\dot{x}_1^b, \dot{x}_2^b)$ denote the velocity of the surplus along the boundary when the machine is up. Therefore,

$$\dot{\boldsymbol{x}}^b = (0, u_2^b - d_2)$$

where u_2^b is given by (5.5). Let t be the time instant immediately after a machine failure. In the interval $(t, t+t^r)$, $u_2 = 0$ and therefore $\dot{x_2} = -d_2$. Thus the surplus vector at $t + t^r$ is given by $(0, x_2 - dt^r)$. Also since the surplus vector from Region $1c$ can only reach the boundary when the machine is up, therefore $\alpha(t + t^r) = 1$. Consequently the dynamics of the cost-to-go along the boundary at time t can be written as

$$J((0,x_2),0,t,T) = E\{\int_t^{t+t^r} g(\boldsymbol{x}(s))ds + J((0,x_2 - d_2t^r),1,t+t^r,T)\} \tag{5.20}$$

For any t such that $\alpha(t) = 1$, the dynamics of the cost-to-go in the interval $(t, t+\delta t)$ can be expressed as

$$J((0,x_2),1,t,T) = E\{\int_t^{t+\delta t} g(\boldsymbol{x}(s))ds + J((0,x_2(t+\delta t)),\alpha(t+\delta t),t+\delta t,T)\}$$

Expanding the right side yields

$$\begin{aligned} J((0,x_2),1,t,T) &= g(\boldsymbol{x})\delta t + (1-p\delta t)J(\boldsymbol{x}(t+\delta t),1,t+\delta t,T) \\ &\quad + p\delta t J(\boldsymbol{x}(t+\delta t),0,t,T) + o(\delta t) \end{aligned} \tag{5.21}$$

Here the left side denotes the cost-to-go at time t. The right side is an expression for the same cost obtained by conditioning on the two possible machine state transitions in the interval $(t, t+\delta t)$. The first term is the incremental cost in $(t, t+\delta t)$. The second term is the cost-to-go at time $t + \delta t$ if there is no machine failure in $(t, t+\delta t)$ and the third term is the cost-to-go if there is a machine failure in $(t, t+\delta t)$. Using a Taylor series expansion of the functions $J(\boldsymbol{x}, 1, t, T)$ and $J(\boldsymbol{x}, 0, t, T)$ along the boundary and neglecting second order terms in δt, we get

$$\begin{aligned} -\frac{\partial J((0,x_2),1,t,T)}{\partial t}\delta t &= g(\boldsymbol{x})\delta t - p\delta t J((0,x_2),1,t,T) + \frac{\partial J((0,x_2),1,t,T)}{\partial x_2}\dot{x}_2^b \delta t \\ &\quad + p\delta t J((0,x_2),0,t,T) + o(\delta t) \end{aligned}$$

Dividing by δt and taking the limit as $\delta t \to 0$, we have

$$-\frac{\partial J((0,x_2),1,t,T)}{\partial t} = g(\boldsymbol{x}) - pJ((0,x_2),1,t,T) + -\frac{\partial J((0,x_2),1,t,T)}{\partial x_2}\dot{x}_2^b + pJ((0,x_2),0,t,T)$$

Substituting for $J((0,x_2),0,t,T)$ from (5.20) we get

$$
\begin{aligned}
-\frac{\partial J((0,x_2),1,t,T)}{\partial t} &= g(\boldsymbol{x}) - pJ((0,x_2),1,t,T) + -\frac{\partial J((0,x_2),1,t,T)}{\partial x_2}\dot{x}_2^b \\
&\quad + pE\{\int_t^{t+t^r} g(\boldsymbol{x}(s))\text{s} + J((0,x_2-d_2t^r),1,t+t^r,T)\}
\end{aligned}
\tag{5.22}
$$

Finally, we take limits of both sides as $T \to \infty$ and assume as in (2.6), that the following limit exists

$$
W^{1c-2b}(x_2,1) = \lim_{T\to\infty} J((0,x_2),1,t,T) - \bar{J}(T-t) \tag{5.23}
$$

Thus W^{1c-2b} denotes the differential cost along the boundary and $\bar{J}$ is the average cost of the OODS policy which has to be calculated. We can express (5.22) in terms of W^{1c-2b} as

$$
\begin{aligned}
\bar{J}(1+pE\{t^r\}) &= -g_2^- x_2 - pW^{1c-2b}(x_2,1) + \frac{dW^{1c-2b}(x_2,1)}{dx_2}\dot{x}_2^b \\
&\quad + pE\{\int_t^{t+t^r} g(x(s))ds + W^{1c-2b}(x_2-d_2t^r,1)\}.
\end{aligned}
\tag{5.24}
$$

Equation (5.24) is the HJB equation along the $1c-2b$ boundary. We now show its relationship with the HJB equation for the single-part-type system in Chapter 4.

Let $\bar{J}_i$ be the average cost of Part Type i defined as

$$
\bar{J}_i = \lim_{T\to\infty} \frac{1}{T} E\left\{\int_0^T g_i(x_i(t))dt\right\}
$$

if this limit exists. Since the instantaneous costs of the two part types are additive, we can express $\bar{J}$ as the sum of the average costs of Part Types 1 and 2. Thus,

$$
\bar{J} = \bar{J}_1 + \bar{J}_2.
$$

Under the OODS policy, the control for Part Type 1 in the interval $[0,T]$ is independent of the surplus level of Part Type 2. Consequently the surplus behavior of Part Type 1 is the same as that in the single-part-type system in Chapter 4 with exponential repairs with rate r operated under a hedging point policy with a hedging point of 0. A typical renewal cycle for Part Type 1 is therefore as shown in Figure

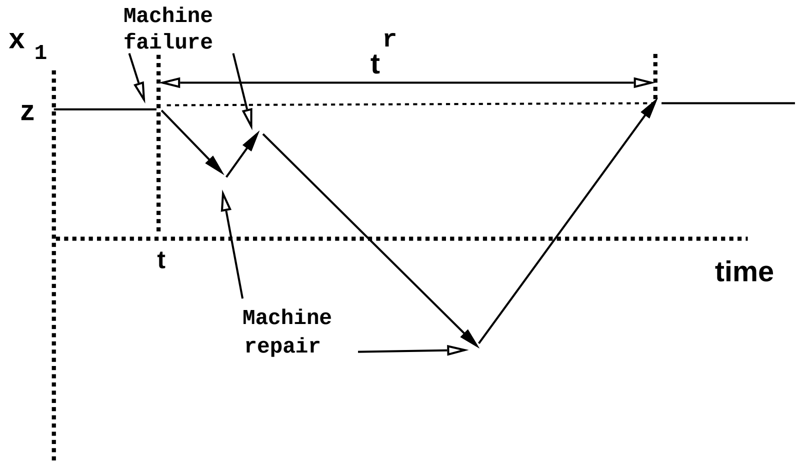

Figure 5-6: Renewal cycle for Part Type 1.

5-6.

Once the surplus reaches 0, it remains there until the next machine failure. The average time from some instant when the machine is up to the next time when the machine fails is equal to $\frac{1}{p}$. After the machine failure, the trajectory returns to 0 after a time t^r. The average length of this cycle is therefore $\frac{1}{p} + E\{t^r\}$.

When the surplus is at 0, there is no cost incurred but when the surplus is negative the system incurs a backlog cost. Using results from renewal theory (Medhi, 1982), the average cost of a renewal cycle (which is equal to $\bar{J}_1$) can be expressed as

$$\bar{J}_1 = \frac{1}{\frac{1}{p} + E\{t^r\}} E\{\int_t^{t+t^r} g_1(x_1(s))ds\}.$$

Therefore

$$\bar{J}_1(1 + pE\{t^r\}) = pE\{\int_t^{t+t^r} g_1(x_1(s))ds\}.$$

Substituting in (5.24), we get

$$\begin{aligned}\bar{J}_2(1 + pE\{t^r\}) \quad = \quad & -g_2^- x_2 \; - \; pW^{1c-2b}(x_2, 1) + \frac{dW^{1c-2b}(x_2, 1)}{dx_2}\dot{x}_2^b \\ & +pE\{\int_t^{t+t^r} g_2(x_2(s))ds + W^{1c-2b}(x_2 - d_2 t^r, 1)\}\end{aligned}$$

Under the OODS policy, the machine does not produce Part Type 2 in the interval $(t, t + t^r)$. Thus

$$\dot{x}_2(s) = -d_2, \qquad t \leq s \leq t + t^r$$

Consequently,

$$E\{\int_t^{t+t^r} g_2(x_2(s))ds\} = -g_2^- x_2 E\{t^r\} - \frac{d_2}{2}E\{(t^r)^2\}.$$

Thus we have

$$\begin{aligned}\bar{J}_2(1+pE\{t^r\}) &= -g_2^- x_2 - pW^{1c-2b}(x_2,1) + \frac{dW^{1c-2b}(x_2,1)}{dx_2}\dot{x}_2^b \\ &\quad -pg_2^-(x_2E\{t^r\} - \frac{d_2}{2}E\{(t^r)^2\}) \\ &\quad +pE\{W^{1c-2b}(x_2 - d_2t^r, 1)\}. \end{aligned} \tag{5.25}$$

Equation (5.25) is the HJB equation under the OODS policy, along the $1c-2b$ boundary. This is exactly the same as equation (4.23) in Chapter 4 which is a result of the equivalence in the surplus behavior of Part Type 2 along the $1c-2b$ boundary and that of the single-part-type system in Chapter 4. Therefore we can use the results from Chapter 4 to find the solution to (5.25) along the $1c-2b$ boundary.

Boundary condition at the hedging point Similar to the boundary condition in Chapter 4, the continuity conditions at the origin imply that

$$\lim_{x_2\to 0^-} \frac{dW^{1c-2b}(x_2,1)}{dx_2} = 0. \tag{5.26}$$

Using (4.33) in Chapter 4, the differential cost along the $1c-2b$ boundary for the two-part-type system is given by

$$W^{1c-2b}(x_2,1) = \frac{g_2^-}{2(\bar{u}_2^b - d_2)}x_2^2 + c^{1c-2b}, \tag{5.27}$$

where $\bar{u}_2^b$ is defined as

$$\bar{u}_2^b = u_2^b \frac{\frac{1}{p}}{\frac{1}{p} + E\{t^r\}} \tag{5.28}$$

This equation follows from (4.32) for a single-part-type system. The behavior of Part Type 1 in the two-part-type system is equivalent to that in the single-part-type system of Chapter 4 for exponential repairs with rate r. Since t^r represents the amount of time it takes for Part Type 1 to return to its hedging point from the instant it leaves the hedging point, the distribution of t^r is exactly the same as that of random variable

C in Chapter 4. Using Corollary 3 we get

$$E\{t^r\} = \frac{\mu_1}{-pd_1 + r(\mu_1 - d_1)} \tag{5.29}$$

Substituting for $E\{t^r\}$ from (5.29) and for u_2^b from (5.5) into (5.28) we get,

$$\bar{u}_2^b = \mu_2[\frac{r}{r+p}](1 - \frac{d_1}{\bar{\mu}_1}) \tag{5.30}$$

The right side is the production rate on the $1c - 2b$ boundary for the AD system. The expression for the differential cost along the $1c-2b$ boundary for the AD system in Section 5.2.2 is exactly the same as that given by (5.27) for the stochastic system. This is an extension of the relationship between the stochastic and AD systems for a single-part-type system with a zero hedging point described in Chapter 4 .

We strongly believe that this relationship between the stochastic system under the OODS policy and the corresponding AD system holds for systems producing more than two part types. This is significant because the differential cost for an AD system with multiple part types can be calculated easily using methods from Section 5.2.2.

5.6 Analysis of Region $1c$

5.6.1 Surplus behavior

Consider a sample surplus trajectory that starts at a point $\boldsymbol{x} = (x_1, x_2)$ in Region $1c$ at time $t = t_0$ with $\alpha(t_0) = 1$. The behavior of this trajectory is illustrated in Figure 5-7. The control vectors in this region are given by

$$\begin{aligned} u^{1c}(\boldsymbol{x}, 1) &= (\mu_1 - d_1, 0), \\ u^{1c}(\boldsymbol{x}, 0) &= (0, 0). \end{aligned}$$

This control guides the trajectory towards the $1c - 2b$ boundary when the machine is up. The point where the trajectory hits the boundary depends on the sample realization of failures and repairs of the machine. Let $t^{'}$ denote the length of time from t_0 until the instant when the trajectory hits the $1c - 2b$ boundary for the first

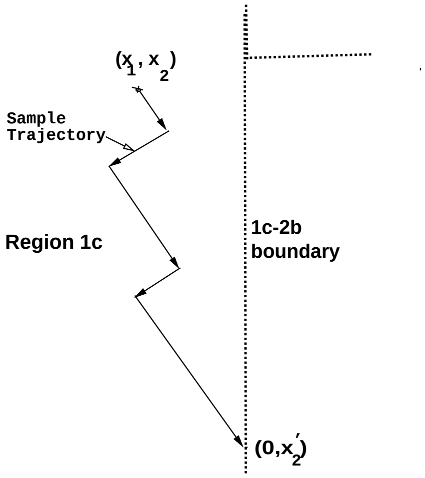

Figure 5-7: Surplus behavior in Region $1c$

time. In Region $1c$, Part Type 2 has a zero production rate. Let us define

$$x_2' = x_2 - d_2 t' \tag{5.31}$$

Thus x_2' is the level of Part Type 2 surplus when the trajectory reaches the boundary. Since the surplus trajectory can only reach the boundary with the machine being up,

$$\alpha(t_0 + t') = 1.$$

As discussed in Section 5.5, the surplus behavior of Part Type 1 over the interval $[t_0, t_0 + t']$ is the same as that in the single-part-type problem in Chapter 4 with exponential failures and repairs operated under a hedging point policy with a hedging point equal to 0.

5.6.2 Solution in Region $1c$

The cost-to-go function from the point $\boldsymbol{x} = (x_1, x_2)$ in Region $1c$ can be written as the sum of two costs: the expected cost until the surplus reaches the boundary and the expected cost-to-go from that instant to the final time. Therefore,

$$J((x_1, x_2), 1, t_0, T) = E\left\{\int_{t_0}^{t_0+t'} g(x(s))ds + J((0, x_2'), 1, t_0 + t', T)\right\}.$$

Taking limits as $T \to \infty$ and using the definition of $W(x,\alpha)$ from (2.6), we get

$$\bar{J}(T-t_0) + W^{1c}(\boldsymbol{x},1) = E\left\{\int_{t_0}^{t_0+t'} g(x(s))ds + W^{1c}(0,x_2',1) + \bar{J}(T-t_0-t')\right\}$$

Since $W(\boldsymbol{x},1)$ is continuous at the $1c-2b$ boundary, we have

$$W^{1c}(0,x_2',1) = W^{1c-2b}(x_2',1)$$

Therefore,

$$W^{1c}(\boldsymbol{x},1) = -\bar{J}E\{t'\} + E\left\{\int_{t_0}^{t_0+t'} g(x(s))ds\} + E\{W^{1c-2b}(x_2',1)\right\}$$

The average cost can be split into its constituent components due to the two part types. Therefore,

$$\bar{J} = \bar{J}_1 + \bar{J}_2$$

Substituting into the expression for $W(x,\alpha)$ we get

$$W^{1c}(\boldsymbol{x},1) = -(\bar{J}_1+\bar{J}_2)E\{t'\}+E\left\{\int_{t_0}^{t_0+t'} (g_1(x_1(s)) + g_2(x_2(s)))ds\right\}+E\{W^{1c-2b}(x_2',1)\} \tag{5.32}$$

In the above expression t' is a random variable. In addition, the entire trajectory over $[t_0, t_0+t']$ is also random because it depends on the machine up and down time realizations. Since Part Type 1 follows a single-part-type hedging point policy over the interval $[t_0, t_0+t']$, one can use Dynkin's equation (in Appendix A) to compute the cost of its trajectory in $[t_0, t_0+t']$. Dynkin's equation, (A.12), yields:

$$E\{\int_{t_0}^{t_0+t'} g_1(x_1(s))ds\} = W^{bk}(x_1,1) - W^{bk}(0,1) + \bar{J}_1 E\{t'\}$$

where $W^{bk}(x,1)$ denotes the differential cost function in the region $x \leq 0$ for a single-part-type system in Chapter 4 with exponential repairs of rate r.
Substituting in (5.32),

$$W^{1c}(\boldsymbol{x},1) = -\bar{J}_2 E\{t'\}+E\left\{\int_{t_0}^{t_0+t'} g_2(x_2(s))ds\right\}+W^{bk}(x_1,1)-W^{bk}(0,1)+E\{W^{1c-2b}(x_2',1)\} \tag{5.33}$$

For any t belonging to $[t_0, t_0 + t']$,

$$\dot{x}_2(t) = -d_2$$

which yields

$$E\int_{t_0}^{t_0+t'} g_2(x_2(s))ds = -g_2^- x_2 E\{t'\} + \frac{1}{2} g_2^- d_2 E\{(t')^2\}. \tag{5.34}$$

From (4.33),

$$W^{bk}(x_1, 1) - W^{bk}(0, 1) = \frac{g_1^-}{2(\bar{\mu}_1 - d_1)} x_1^2, \tag{5.35}$$

where

$$\bar{\mu}_1 = \frac{r}{r+p}\mu_1.$$

The last term on the right side of (5.33) can be evaluated by substituting for x_2' from (5.31) and using the expression for $W(x)$ for $x < 0$ for the single-part-type system in Chapter 4 with the average repair times are given by (5.29). Let $\bar{u}_2$ denote the average production rate of Part Type 2 along the boundary. Equation (4.32) in Chapter 4 implies that

$$\bar{u}_2 = \frac{u_2^b}{1 + pE\{t^r\}}$$

Substituting for u_2^b from (5.5) and for $E\{t^r\}$ from (5.29) and simplifying, we get

$$\bar{u}_2 = \frac{r\mu_2}{r+p}(1 - \frac{d_1}{\bar{\mu}_1})$$

The right side is the maximum possible production rate of Part Type 2 in the AD system. Using (4.33) in Chapter 4 to evaluate W^{1c-2b}, the last term on the right side of (5.33) becomes

$$E\{W^{1c-2b}(x_2', 1)\} = \frac{g_2^-}{2(\bar{\mu}_2 - d_2)}\left[x_2^2 - 2d_2x_2E\{t'\} + d_2^2E\{(t')^2\}\right] + c^{1c-2b} \tag{5.36}$$

Substituting (5.34) - (5.36) in (5.33) we get

$$W^{1c}(\boldsymbol{x}, 1) \;\; = \;\; -\bar{J}_2E\{t'\} - g_2^- x_2 E\{t'\} + \frac{g_2^-}{2(\bar{\mu}_2 - d_2)}\left[x_2^2 - 2d_2x_2E\{t'\}\right]$$

$$+\frac{g_2^- d_2^2 + g_2^- d_2}{2} E\{(t^{'})^2\} \frac{g_1^-}{2(\bar{\mu}_1 - d_1)} x_1^2 + c^{1c-2b} \tag{5.37}$$

Appendix A contains expressions for $E\{t^{'}\}$ $E\{(t^{'})^2\}$ and $\bar{J}_2$. After some manipulation, the substitution of these expressions into (5.37) yields a differential cost function of the form

$$W^{1c}(\boldsymbol{x}, 1) = \frac{1}{2} a_{11}^{1c} x_1^2 + \frac{1}{2} a_{22}^{1c} x_2^2 + a_{12}^{1c} x_1 x_2 + c^{1c}, \tag{5.38}$$

where

$$\begin{aligned}
a_{11}^{1c} &= \frac{g_1^-}{\bar{\mu}_1 - d_1} + \frac{g_2^- d_2}{(\bar{\mu}_1 - d_1)^2} + \frac{g_2^- d_2^2}{(\bar{u}_2^b - d_2)(\bar{\mu}_1 - d_1)^2} \\
a_{22}^{1c} &= \frac{g_2^-}{\bar{u}_2^b - d_2} \\
a_{12}^{1c} &= \frac{g_2^-}{(\bar{\mu}_1 - d_1)} + \frac{g_2^- d_2}{(\bar{u}_2^b - d_2)(\bar{\mu}_1 - d_1)} \\
c^{1c} &= c^{1c-2b}.
\end{aligned} \tag{5.39}$$

It can be verified that this expression satisfies the second-order form of the HJB equation given by (5.19).

It is interesting and surprising to note that the above expression for $W(x, 1)$ involves only quadratic terms even though the system is stochastic. Moreover the expression for $W(x, 1)$ is exactly the same as that in Region $1c$ for the corresponding AD system. The non-transient region under the OODS policy is defined by a cone with its vertex at the origin, the negative demand vector as one of its edges and the $1c - 2b$ boundary as the other. This is because a trajectory that enters this region remains there forever . Also, a trajectory originating at some point outside this region eventually ends up in this region. The results of Section 5.5 and this section mean that in the non-transient region, the differential cost of the stochastic system is exactly the same as that of a deterministic system whose capacity is the average of the capacity over the two machine states of the stochastic system. We have only considered the OODS policy which has a zero hedging point. The nature of the solution for a non-zero hedging point case is an open research question.

We strongly believe that this relationship between the differential costs of a stochastic system and its corresponding AD system in the non-transient region of the OODS policy holds for systems with more than two part types.

5.7 Analysis of Regions $2b$ and Region $1b$

In this section, we first describe the surplus behavior in Region $2b$ and describe a method to obtain the differential cost function. The relations between surplus behaviors in Regions $2b$ and $1b$ are then used to obtain the differential cost function in Region $1b$.

5.7.1 Surplus behavior in Region $2b$

The control vectors in Region $2b$ under the OODS policy is given by:

$$\begin{aligned} u^{1c}(\boldsymbol{x}, 1) &= (\mu_1, 0), \\ u^{1c}(\boldsymbol{x}, 0) &= (0, 0). \end{aligned}$$

Consider a surplus trajectory that starts at a point $P = (x_1, x_2)$ in Region $2b$ at time t with the machine up. The behavior of such a trajectory is illustrated in Figure 5-8. While the machine is up, the surplus trajectory is guided towards the hedging point along a direction parallel to the $2a - 2b$ boundary. While the machine is down, the trajectory moves in a direction parallel to the -$\boldsymbol{d}$ direction, until the machine is repaired. It can be seen from Figure 5-8 that for any starting point in Region $2b$, the surplus always hits the $1c - 2b$ boundary. Let t_1 denote the time at which the surplus trajectory reaches the boundary. Thus,

$$t_1 = \frac{x_1}{d_1} \tag{5.40}$$

The value of x_2 at time t_1 is a random variable that depends on the sample realization of failures and repairs. If there is no machine failure in $[t, t_1]$, then the value of x_2 at time t_1 is maximum and is given by

$$x_2^M(t_1) = x_2 + (\mu_2 - d_2)\frac{x_1}{d_1}$$

On the other hand, if machine fails immediately after time t and there is no machine repair until the trajectory hits the boundary, then the value of x_2 is minimum and is

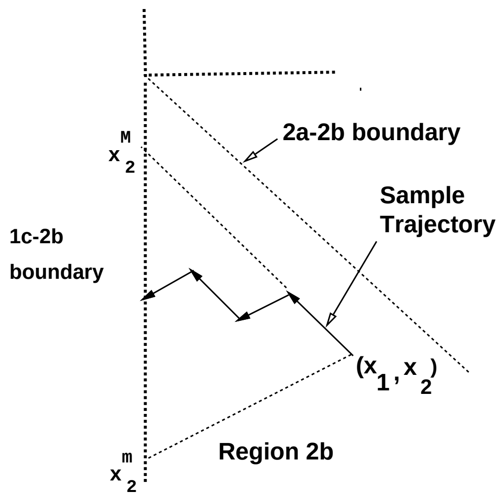

Figure 5-8: Surplus behavior in Region $2b$

given by

$$x_2^m(t_1) = x_2 - d_2\frac{x_1}{d_1}$$

In general, the value of x_2 is in between these two bounds. The machine state at time t_1 is also a random variable. Since α evolves according to a Markov process, is possible to use techniques from transient analysis of Markov processes with rewards to determine the joint distribution for $(x_2(t), \alpha(t))$ by considering the surplus dynamics in each machine state as the rate at which a reward is accrued in that state of the Markov process (Viswanadham *et al.*, 1992).

5.7.2 Solution in Region $2b$

There are two approaches to evaluating the cost-to-go function in Region $2b$. The first is to derive and solve the integral form of the cost-to-go function in Region $2b$ while the second involves solving the differential form of the dynamics of the cost-to-go function given by (5.19).

The derivation of integral form of the cost-to-go function in Region $2a$ is analogous to that for Region $2b$ described in Section 5.6 and leads to an equation similar to (5.33) that is given by

$$\begin{aligned} W^{1c}(\boldsymbol{x}, 1) &= -\bar{J}_1 E\{t_1 - t\} + E\{\int_{t_1}^{t} g_1(x_1(s))ds\} + W^{bk}(x_2, 1) \\ &\quad -E\{W^{bk}(x_2(t_1), \alpha(t_1))\} + E\{W^{1c-2b}(x_2(t_1), \alpha(t_1))\} \end{aligned} \quad (5.41)$$

The difference between (5.41) and (5.33) is that in Region $1c$, the time when the trajectory reaches the x_2 axis (t_1) is deterministic and is given by (5.40) whereas this time is stochastic in Region $1c$.

As mentioned above, a transient analysis of a Markov process with rewards can be used to get the joint distribution for $(x_2(t_1), \alpha(t_1))$. The function $W^{1c-2b}(x_2, 1)$ and $W^{1c-2b}(x_2, 0)$ along the $1c-2b$ boundary are given by (4.33) and (4.12) for the system in Chapter 4 with repair time t^r. Similarly, the functions $W^0(x_2, 1)$ and $W^0(x_2, 0)$ for $x_2 < 0$ are known from the system in Chapter 4 with exponential repairs of rate r. Thus the left hand side of (5.41) can be evaluated for any $\boldsymbol{x}$.

In the remainder of this section, we describe a second approach that uses the differential form of the cost-to-go dynamics given by (5.19). For Region $2b$, the cost-to-go dynamics can be expressed as

$$\begin{aligned}
& -d_1^2 \tfrac{\partial^2 W(\boldsymbol{x},1)}{\partial x_1^2} + d_2(\mu_2 - d_2) \tfrac{\partial^2 W(\boldsymbol{x},1)}{\partial x_2^2} \\
+ \quad & (d_1(\mu_2 - d_2) - d_2 d_1) \tfrac{\partial^2 W(\boldsymbol{x},1)}{\partial x_1 \partial x_2} \\
- \quad & (r+p) d_1 \tfrac{\partial W(\boldsymbol{x},1)}{\partial x_1} \\
+ \quad & (-p d_2 + r(\mu_2 - d_2)) \tfrac{\partial W(\boldsymbol{x},1)}{\partial x_2} \\
= \quad & \bar{J}(r+p) - (g_1^+ x_1 - g_2^- x_2)(r+p) - d_1 g_1^+ + d_2 g_2^- \qquad (5.42)
\end{aligned}$$

Equation (5.42) is a linear partial differential equation with linear forcing terms. This suggests that a solution is the sum of exponential, quadratic, linear and constant terms. Let us assume a solution to (5.42) of the form [1]

$$\begin{aligned}
W((x_1, x_2), 1) \;=\; & (k_1^{2b} x_1 + k_2^{2b} x_2 + k_0^{2b}) e^{-\frac{r+p}{d_1} x_1} \\
& + \frac{1}{2} a_{11}^{2b} x_1^2 + a_{12}^{2b} x_1 x_2 + \frac{1}{2} a_{22}^{2b} x_2^2 + b_1^{2b} x_1 + b_2^{2b} x_2 + c^{2b} \qquad (5.43)
\end{aligned}$$

There are nine unknown constants in this equation. If these constants can be chosen such that the solution satisfies both the HJB equation (5.42) and the boundary conditions on the $1c - 2b$ boundary, then the form that we have assumed is indeed the correct form of the differential cost function.

[1] It can be easily verified by the first approach that the solution includes only exponential terms of the form $e^{-\frac{r+p}{d_1} x_1}$, $x_1 e^{-\frac{r+p}{d_1} x_1}$ and $x_2 e^{-\frac{r+p}{d_1} x_1}$, although calculating the coefficients of these terms by that method is quite cumbersome.

Differentiation of (5.43) yields the following:

$$\begin{aligned}\frac{\partial W(\boldsymbol{x},1)}{\partial x_1} &= -\frac{(r+p)}{d_1}(k_1^{2b}x_1 + k_2^{2b}x_2 + k_0^{2b})e^{-\frac{(r+p)}{d_1}x_1} \\ &\quad + k_1^{2b}e^{-\frac{(r+p)}{d_1}x_1} + a_{11}^{2b}x_1 + a_{12}^{2b}x_2 + b_1^{2b}\end{aligned}$$

$$\frac{\partial W(\boldsymbol{x},1)}{\partial x_2} = k_2^{2b}e^{-\frac{(r+p)}{d_1}x_1} + a_{12}^{2b}x_1 + a_{22}^{2b}x_2 + b_2^{2b}$$

$$\begin{aligned}\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_1^2} &= \frac{(r+p)^2}{d_1^2}(k_1^{2b}x_1 + k_2^{2b}x_2 + k_0^{2b})e^{-\frac{(r+p)}{d_1}x_1} \\ &\quad -2k_1^{2b}\left(\frac{r+p}{d_1}\right)e^{-\frac{(r+p)}{d_1}x_1} + a_{11}^{2b}\end{aligned}$$

$$\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_1 \partial x_2} = -k_2^{2b}\left(\frac{r+p}{d_1}\right)e^{-\frac{(r+p)}{d_1}x_1} + a_{12}^{2b}$$

$$\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_2^2} = a_{22}^{2b}$$

HJB conditions These expressions must satisfy the HJB equation (5.42). Plugging them in (5.42) and equating similar terms we find

- $e^{-\frac{(r+p)}{d_1}x_1}$ terms:

$$k_1^{2b}(r+p)d_1 + k_2^{2b}(rd_2 - p(\mu_2 - d_2)) = 0 \tag{5.44}$$

- x_1 terms:

$$-d_1 a_{11}^{2b} + \left(\frac{-d_2 p + r(\mu_2 - d_2)}{r+p}\right)a_{12}^{2b} = -g_1^{+} \tag{5.45}$$

- x_2 terms:

$$-d_1 a_{12}^{2b} + \left(\frac{-d_2 p + r(\mu_2 - d_2)}{r+p}\right)a_{22}^{2b} = g_2^{-} \tag{5.46}$$

- Constant terms:

$$\begin{aligned}\bar{J}(r+p) - d_1 g_1^+ + d_2 g_2^- &= -d_1^2 a_{11}^{2b} + d_2(\mu_2 - d_2) a_{22}^{2b} + d_1(\mu_2 - 2d_2) a_{12}^{2b} \\ &\quad -(r+p) d_1 b_1^{2b} + (-d_2 p + r(\mu_2 - d_2)) b_2^{2b} \quad (5.47)\end{aligned}$$

The equations for terms in $x_1 e^{-\frac{(r+p)}{d_1}x_1}$ and $x_2 e^{-\frac{(r+p)}{d_1}x_1}$ are automatically satisfied for any choice of constants. Equations (5.44)–(5.47) give four conditions on the nine constants.

Boundary conditions The solution in Region $2b$ must satisfy the boundary conditions along the $1c - 2b$ boundary. These boundary conditions can be expressed as

$$\lim_{x_1 \to 0,\ \boldsymbol{x} \in \text{Region } 2b} W((x_1, x_2), 1) = W^{1c-2b}(x_2, 1) \qquad (5.48)$$

$$\lim_{x_1 \to 0,\ \boldsymbol{x} \in \text{Region } 2b} W((x_1, x_2), 0) = W^{1c-2b}(x_2, 0) \qquad (5.49)$$

Recall from Section 5.5 that W^{1c-2b} is known and is given by (5.27). We can express the boundary conditions in a different form by using the cost-to-go dynamics on the $1c - 2b$ boundary. From equation (5.3), the differential form of the HJB equation for $\alpha = 1$ in Region $2b$ is given by:

$$\bar{J} = g(\boldsymbol{x}) - d_1 \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_1} + (\mu_2 - d_2) \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_2} + p(W(\boldsymbol{x}, 0) - W(\boldsymbol{x}, 1))) \qquad (5.50)$$

Equation (5.24) in Section 5.5 is the integral form of the dynamics of the cost-to-go along the $1c - 2b$ boundary. The differential form of the cost-to-go dynamics yields

$$\bar{J} = g(\boldsymbol{x}) + (u_2^b - d_2) \frac{dW^{1c-2b}(x_2, 1)}{dx_2} + p\left[W^{1c-2b}(x_2, 0) - W^{1c-2b}(x_2, 1)\right] \qquad (5.51)$$

Taking the limit as $x_1 \to 0$ in equation (5.50) and using (5.49) and (5.51) we get

$$(u_2^b - d_2) \frac{dW^{1c-2b}(x_2, 1)}{dx_2} = \lim_{x_1 \to 0} \left[-d_1 \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_1} + (\mu_2 - d_2) \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_2} \right] \qquad (5.52)$$

Equation (5.52) implies that even though the differential cost function may not be differentiable at the $1c - 2b$ boundary, the discontinuities in the derivatives at the boundary are not arbitrary. Equation (5.52) specifies the relation that the partial derivatives in Region $2b$ must satisfy for points in the neigbourhood of the boundary. In particular, this relation states that $(u_1 - d_1)\frac{\partial W(\boldsymbol{x},1)}{\partial x_1} + (u_2 - d_2)\frac{\partial W(\boldsymbol{x},1)}{\partial x_2}$ is continuous at any region boundary.

Only two of the three equations, (5.48), (5.49) and (5.52) are linearly independent. We use (5.48) and (5.52) because these conditions are in terms of $W(x,1)$ alone. The form for $W(\boldsymbol{x},1)$ in Region $2b$ given in (5.43) yields

$$\lim_{x_1 \to 0} W(\boldsymbol{x},1) = \frac{1}{2} a_{22}^{2b} x_2^2 + (b_2^{2b} + k_2^{2b}) x_2 + c^{2b} + k_0^{2b}$$

and

$$\begin{aligned}
&\lim_{x_1 \to 0} \left[-d_1 \frac{\partial W(\boldsymbol{x},0)}{\partial x_1} + (\mu_2 - d_2) \frac{\partial W(\boldsymbol{x},0)}{\partial x_2} \right] = \\
&(\mu_2 - d_2)(a_{22} x_2 + k_2^{2b} + b_2^{2b}) \\
&-d_1 \left((a_{12}^{2b} - \frac{(r+p)k_2^{2b}}{d_1}) x_2 + k_1^{2b} + b_1^{2b} - \frac{(r+p)k_0^{2b}}{d_1} \right)
\end{aligned}$$

Using Boundary Condition (5.48) and the expression for $W^{1c-2b}(x,1)$ from (5.27), we get three more conditions on the parameters:

- x_2^2 term

$$a_{22}^{2b} = \frac{g_2^-}{\bar{u}_2^b - d_2} \tag{5.53}$$

- x_2 term

$$b_2^{2b} + k_2^{2b} = 0 \tag{5.54}$$

- Constant term

$$c^{2b} + k_0^{2b} = c^{1c-2b} \tag{5.55}$$

Finally, using Boundary Condition (5.52) we get two more conditions:

- x_2 terms

$$(\mu_2 - d_2) a_{22}^{2b} - d_1 a_{12}^{2b} + (r+p) k_2^{2b} = \frac{g_2^-}{(\bar{u}_2^b - d_2)} (\mu_2 (1 - \frac{\mu_1}{d_1}) - d_2) \tag{5.56}$$

- Constant terms

$$(\mu_2 - d_2)(k_2^{2b} + b_2^{2b}) - d_1(\frac{(r+p)}{d_1}k_0^{2b} + k_1^{2b} + b_1^{2b}) = 0 \qquad (5.57)$$

We now have nine independent equations to solve for the nine unknown coefficients. The solution yields a unique value for each coefficient. It should be noted though, that the constant term in the solution contains c^{1c-2b} which is an arbitrary constant that represents the one degree of freedom in the solution of the HJB equation that is explained in Section 2.3.

Since (5.43) satisfies both the HJB equations and the boundary conditions, it represents the differential cost function in Region $2b$. The quadratic coefficients in Region $2b$ can be obtained by solving equations (5.53), (5.46), (5.45) and in that order. It is interesting to note that the solution to these equations simplifies to:

$$\begin{aligned}
a_{11}^{2b} &= \frac{g_1^+}{d_1} + \frac{g_2^-(\bar{\mu}_2 - d_2)^2}{d_1^2(\bar{\mu}_2^b - d_2)} - \frac{g_2^-(\bar{\mu}_2 - d_2)}{d_1^2} \\
a_{12}^{2b} &= \frac{-g_2^-}{d_1} + \frac{g_2^-(\bar{\mu}_2 - d_2)}{(\bar{\mu}_2^b - d_2)d_1} \\
a_{22}^{2b} &= \frac{g_2^-}{(\bar{\mu}_2^b - d_2)}
\end{aligned} \qquad (5.58)$$

This implies that the quadratic part of the differential cost for the stochastic system in Region $2b$ is *exactly the same* as the differential cost in the corresponding AD system. There is an intuitively explanation for this relationship. For points that are far away from the hedging the quadratic term dominates the solution in (5.43). A surplus trajectory that starts at some such point encounters many failures and repairs of the machine. The average behavior over many such failures and repairs is similar to that in a AD system. Hence the quadratic terms of the stochastic system must be the same as those in the AD system.

5.7.3 Solution in Region $1b$

Consider a surplus trajectory that starts from the point (x_1, x_2) in Region $1b$ as shown in Figure 5-9. When the machine is up, it devotes all its capacity to Part Type 1 causing the surplus to move in a direction parallel to $(\mu_1 - d_1, -d_2)$. During a machine failure, the trajectory moves in a direction parallel to $(-d_1, -d_2)$. In either case, the surplus trajectory eventually reaches the $1b - 1c$ boundary after which it moves into

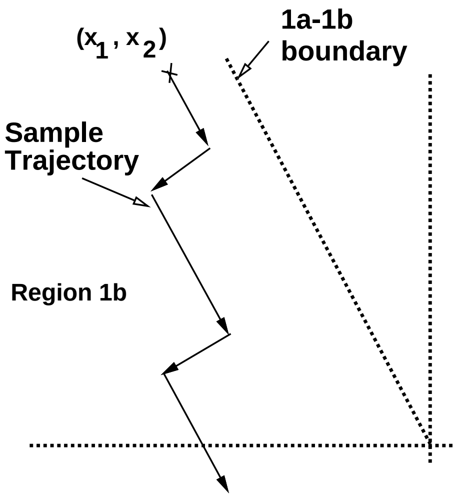

Figure 5-9: Surplus behavior in Region $1b$

Region $1c$. Part Type 1 in Region $2b$ has the same qualitative behavior as Part Type 2 in Region $1b$.

Consequently we assume a solution in Region $1b$ that is based on (5.43) and is given by

$$
\begin{aligned}
W((x_1,x_2),1) &= (k_1^{1b}x_1 + k_2^{1b}x_2 + k_0^{1b})e^{-\frac{r+p}{d_2}x_2} \\
&\quad + \frac{1}{2}a_{11}^{1b}x_1^2 + a_{12}^{1b}x_1x_2 + \frac{1}{2}a_{22}^{1b}x_2^2 + b_1^{1b}x_1 + b_2^{1b}x_2 + c^{1b}
\end{aligned} \tag{5.59}
$$

There are again nine equations in nine unknowns. Substituting (5.59) into the HJB equation we have

$$k_2^{1b}(r+p)d_2 + k_1^{1b}(rd_1 - p(\mu_1 - d_1)) = 0 \tag{5.60}$$

$$-d_2a_{22}^{1b} + \left(\frac{-d_1p + r(\mu_1 - d_1)}{r+p}\right)a_{12}^{1b} = -g_2^+ \tag{5.61}$$

$$-d_2a_{12}^{1b} + \left(\frac{-d_1p + r(\mu_1 - d_1)}{r+p}\right)a_{11}^{1b} = g_1^- \tag{5.62}$$

$$
\begin{aligned}
\bar{J}(r+p) - d_2g_2^+ + d_1g_1^- &= -d_2^2a_{22}^{1b} + d_1(\mu_1 - d_1)a_{11}^{1b} + d_2(\mu_1 - 2d_1)a_{12}^{1b} \\
&\quad -(r+p)d_2b_2^{1b} + (-d_1p + r(\mu_1 - d_1))b_1^{1b}
\end{aligned} \tag{5.63}
$$

Equations (5.60)–(5.63) are similar to (5.44)–(5.47) in Region $2b$.

The boundary conditions at the $1b - 1c$ boundary are different from those on the $1c - 2b$ boundary because the control vector is continuous at the $1c - 2b$ boundary. The continuity of the differential cost, $W(x, 1)$, at the boundary requires that

$$\lim_{x_2 \to 0,\ \boldsymbol{x} \in \text{Region } 1b} W((x_1, x_2), 1) = \lim_{x_2 \to 0,\ \boldsymbol{x} \in \text{Region } 1c} W((x_1, x_2), 1) \tag{5.64}$$

In addition, since there is no discontinuity in the control vector,

$$\lim_{x_2 \to 0,\ \boldsymbol{x} \in \text{Region } 1b} \frac{\partial W((x_1, x_2), 1)}{\partial x_1} = \lim_{x_2 \to 0,\ \boldsymbol{x} \in \text{Region } 1c} \frac{\partial W((x_1, x_2), 1)}{\partial x_1} \tag{5.65}$$

and

$$\lim_{x_2 \to 0,\ \boldsymbol{x} \in \text{Region } 1b} \frac{\partial W((x_1, x_2), 1)}{\partial x_2} = \lim_{x_2 \to 0,\ \boldsymbol{x} \in \text{Region } 1c} \frac{\partial W((x_1, x_2), 1)}{\partial x_2}. \tag{5.66}$$

Condition (5.64) yields three more equations given by

$$a_{11}^{1b} = a_{11}^{1c} \tag{5.67}$$

$$b_1^{1b} + k_1^{1b} = 0 \tag{5.68}$$

$$c^{1b} + k_0^{1b} = c^{1c} \tag{5.69}$$

Condition (5.66) yields two more equations given by

$$-\frac{r+p}{d_2} k_1^{1b} + a_{12}^{1b} = a_{12}^{1c} \tag{5.70}$$

$$-\frac{r+p}{d_2} k_0^{1b} + k_2^{1b} + b_2^{1b} = 0 \tag{5.71}$$

Equation (5.65) does not lead to any additional conditions, which shows that it is redundant.

Thus we have a system of nine independent equations for the nine unknown constants that can be solved to determine the differential cost in this region. It can be verified that $k_1^{1b} = k_2^{1b} = b_1^{1b} = 0$. Therefore the solution in Region $1b$ is of the form

$$W((x_1, x_2), 1) = k_0^{1b} e^{-\frac{r+p}{d_2} x_2} + \frac{1}{2} a_{11}^{1b} x_1^2 + a_{12}^{1b} x_1 x_2 + \frac{1}{2} a_{22}^{1b} x_2^2 + b_2^{1b} x_2 + c^{1b} \tag{5.72}$$

Also, the quadratic coefficients are given by

$$
\begin{aligned}
a_{11}^{1b} &= \frac{g_1^-}{\bar{\mu}_1 - d_1} + \frac{g_2^- d_2}{(\bar{\mu}_1 - d_1)^2} + \frac{g_2^- d_2^2}{(\bar{u}_2^b - d_2)(\bar{\mu}_1 - d_1)^2} \\
a_{22}^{1b} &= \frac{g_2^-}{\bar{u}_2^b - d_2} + \frac{g_2^-}{d_2} + \frac{g_2^+}{d_2} \\
a_{12}^{1b} &= \frac{g_2^-}{(\bar{\mu}_1 - d_1)^2} + \frac{g_2^- d_2}{(\bar{u}_2^b - d_2)(\bar{\mu}_1 - d_1)}.
\end{aligned}
$$

Again, the quadratic terms of the differential cost for the stochastic system in Region $1b$ are the same as those in Region $1b$ of the corresponding AD system.

5.8 Analysis of Regions $2a$ and $1a$

In this section, we derive expressions for the differential cost functions in Regions $2a$ and $1a$. The bulk of this section is devoted to a detailed analysis of Region $2a$. The derivation of the solution in Region $2a$ is tougher than that in Region $2b$ because a trajectory from point (x_1, x_2) in Region $2b$ always hits the $1c - 2b$ boundary after a fixed time $\frac{x_1}{d_1}$. A solution from a point in Region $2a$ could hit either the $0b - 2a$ or $2a - 2b$ boundaries. The time when these boundaries are reached is a random variable as also is the state of the machine at that time. Consequently, we expect the boundary conditions in Region $2a$ to be more complicated than those in Region $2b$. Solutions having the same form as the solution in Region $2b$ (a sum involving quadratic, linear, constant terms and their products with exponential terms) do not satisfy the Bellman equation and boundary conditions in Region $2a$. In Section 5.8.7, we suggest a method to use the results for Region $2a$ to solve for the differential cost in Region $1a$.

5.8.1 Surplus behavior in Region $2a$

Consider a sample surplus trajectory that starts at a point $\boldsymbol{x} = (x_1, x_2)$ in Region $2a$ at time $t = 0$. The control vectors in this region are given by

$$
\begin{aligned}
u^{2a}(\boldsymbol{x}, 1) &= (0, \mu_2), \\
u^{2a}(\boldsymbol{x}, 0) &= (0, 0).
\end{aligned}
$$

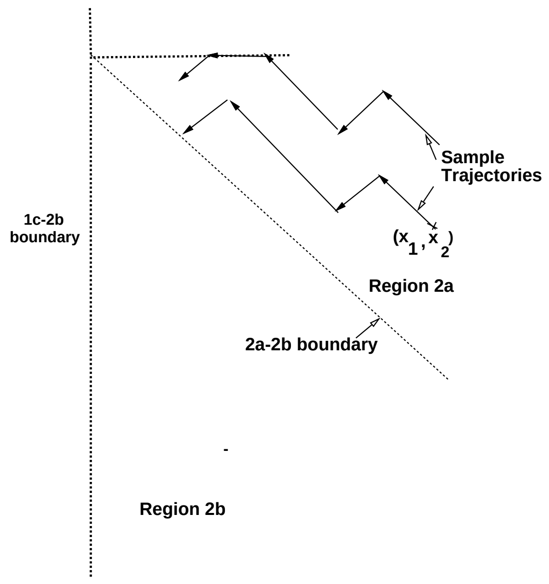

Figure 5-10: Surplus behavior in Region $2a$

For $\alpha = 1$, this control guides the trajectory along the direction $(-d_1, \mu_2 - d_2)$, towards the $0b - 2a$ boundary. A machine failure at any time causes the surplus to move away from the boundary along the direction $(-d_1, -d_2)$. If machine down times are very small compared to machine up times, then the trajectory remains in Region $2a$ and eventually hits the $0b - 2a$ boundary at some time instant. A long machine failure on the other hand causes the trajectory to cross the $2a - 2b$ subregion boundary and enter Subregion $2b$. The subsequent behavior of the trajectory in this case is as discussed in Section 5.7 and the trajectory cannot enter Region $2a$ again.

The differential cost function in Region $2a$ thus depends on two sets of boundary conditions: one along the $0b - 2a$ boundary and another along the $2a - 2b$ boundary. The surplus behavior in Region $2a$ is illustrated in Figure 5-10.

5.8.2 Dynamics of the cost-to-go function

The control vector in Region $2a$ is the same as in Region $2b$. Thus the second order form of the HJB equation in Region $2a$ is the same as that in Region $2b$ and is given by (5.19).

Equation (5.19) is a second order linear hyperbolic partial differential equation with constant coefficients. The two characteristic directions (ξ,η) of this equation are given by (Zauderer, 1983)

$$\begin{aligned}\xi(x_1, x_2) &= x_2 + \frac{(\mu_2 - d_2)}{d_1} x_1 \\ \eta(x_1, x_2) &= x_2 - \frac{d_2}{d_1} x_1.\end{aligned}$$

The $2a - 2b$ boundary corresponds to one of these characteristic directions and can be represented by the equation

$$\xi = 0$$

Boundary conditions on the $2a - 2b$ boundary There is no discontinuity in the control vector $\boldsymbol{u}$ at the $2a - 2b$ boundary. Therefore, the differential cost function is continuously differentiable across this boundary. In other words, the functions $W(\boldsymbol{x}, \alpha)$ and their partial derivatives must be continuous at the boundary.

The continuity conditions can be expressed as:

$$\lim_{x_2 \to -\frac{(\mu_2 - d_2)}{d_1} x_1 \ \boldsymbol{x} \in \text{ Region } 2a} W((x_1, x_2), 1) = \lim_{x_2 \to -\frac{(\mu_2 - d_2)}{d_1} x_1 \ \boldsymbol{x} \in \text{ Region } 2b} W((x_1, x_2), 1) \tag{5.73}$$

$$\lim_{x_2 \to -\frac{(\mu_2 - d_2)}{d_1} x_1 \ \boldsymbol{x} \in \text{ Region } 2a} W((x_1, x_2), 0) = \lim_{x_2 \to -\frac{(\mu_2 - d_2)}{d_1} x_1 \ \boldsymbol{x} \in \text{ Region } 2b} W((x_1, x_2), 0) \tag{5.74}$$

Using the first order form of the HJB equation given by (5.16), the continuity condition in (5.74) can also be expressed as

$$\begin{aligned}&\lim_{x_2 \to -\frac{(\mu_2 - d_2)}{d_1} x_1 \ \boldsymbol{x} \in \text{ Region } 2a} (\mu_2 - d_2) \frac{\partial W((x_1, x_2), 1)}{\partial x_2} - d_1 \frac{\partial W((x_1, x_2), 1)}{\partial x_1} \\ &= \lim_{x_2 \to -\frac{(\mu_2 - d_2)}{d_1} x_1 \ \boldsymbol{x} \in \text{ Region } 2b} (\mu_2 - d_2) \frac{\partial W((x_1, x_2), 1)}{\partial x_2} - d_1 \frac{\partial W((x_1, x_2), 1)}{\partial x_1}\end{aligned} \tag{5.75}$$

The right side of equation (5.75) is a known function since the differential cost function in Region $2b$ is known from Section 5.7.

Equation (5.75) states that the directional derivative of $W(\boldsymbol{x}, 1)$ is continuous along the characteristic direction $(\mu_2 - d_2, -d_1)$. From the theory of hyperbolic partial differential equations (Myint-U, 1987), a twice continuously differentiable solution to a hyperbolic partial differential equation can be obtained from the value of a function and its directional derivative along one of the characteristic directions, at the boundary. Conditions on directional derivatives along other directions are therefore redundant. Thus if (5.73) and (5.75) are satisfied, $W(\boldsymbol{x}, 1)$ is continuously differentiable on the $2a - 2b$ boundary.

Boundary conditions on the $0b - 2a$ boundary The boundary conditions that have been described thus far are *explicit conditions* that give the value of a function or its derivative along some boundary in terms of known functions. The surplus behavior along the $0b - 2a$ boundary leads to *implicit boundary conditions* along the boundary, *i.e.*, conditions in terms of the unknown function $W(\boldsymbol{x}, 1)$ in Region $2a$.

Consider a trajectory that at time t, is at some point $\boldsymbol{x} = (x_1, 0)$, on the $0b - 2a$ boundary. Let the machine be up at time t. The control vector at time t is given by

$$\boldsymbol{u}^{0b-2a}(\boldsymbol{x}, 1) = (0, d_2) \tag{5.76}$$

This control keeps the trajectory on the boundary until the next machine failure. Let $\dot{\boldsymbol{x}}^{0b-2a}$ denote the rate of change of the surplus vector along the boundary for $\alpha = 1$. Thus,

$$\dot{\boldsymbol{x}}^{0b-2a} = (-d_1, 0)$$

The dynamics of the cost-to-go function along the $0b - 2a$ boundary can be expressed as

$$\begin{aligned}
&J((x_1, 0), 1, t, T) = \\
&\quad g_1^+ x_1 \delta t + (1 - p\delta t) J((x_1 + \dot{x}_1^{0b-2a} \delta t, 0), 1, t + \delta t, T) \\
&\quad + p\delta t J((x_1 + \dot{x}_1^{0b-2a} \delta t, 0), 0, t + \delta t, T)
\end{aligned}$$

Here the left side denotes the cost-to-go at time 0. The right side is an expression for the same cost obtained by conditioning on the two possible machine state transitions in the next δt time units. The first term on the right hand side is the incremental cost in δt. The second term is the cost-to-go at time δt if there is no machine failure in $(t, t+\delta t)$ and the third term is the cost-to-go if there is a machine failure in $(t, t+\delta t)$.

Since $\boldsymbol{u}(\boldsymbol{x}, 1)$ on the boundary is continuous along the boundary at all points other than the origin, the cost function along the boundary is differentiable with respect to

x_1 at all points other than the origin (Rishel, 1975). Using Taylor series expansion to expand the terms, we find

$$\begin{aligned}
J((x_1,0),1,t,T) &= g_1^+ x_1 \delta t + (1-p\delta t) J((x_1,0),1,t,T) \\
&\quad + p\delta t J((x_1 + \dot{x}_1^{0b-2a}\delta t, 0), 0, t, T) \\
&\quad + (1-p\delta t)(\frac{\partial J((x_1,0),1,t,T)}{dx_1}\dot{x}_1^{0b-2a} + \frac{\partial J((x_1,0),1,,t,T)}{\partial t})\delta t \\
&\quad + p\delta t(\frac{\partial J((x_1,0),0,t,T)}{\partial x_1}\dot{x}_1^{0b-2a} + \frac{\partial J((x_1,0),1,t,T)}{\partial t})\delta t + o(\delta t).
\end{aligned}$$

Ignoring the second order terms in δt, dividing by δt and taking the limit of both sides in the above expression as $\delta t \to 0$, we get

$$\begin{aligned}
-\frac{\partial J((x_1,0),1,t,T)}{\partial t} &= g_1^+ x_1 + \frac{\partial J((x_1,0),1,t,T)}{\partial x_1}\dot{x}_1^{0b-2a} \\
&\quad - p(J((x_1,0),1,t,T) - J((x_1,0),0,t,T))
\end{aligned} \tag{5.77}$$

Let $W^{0b-2a}(x_1,\alpha)$ denote the differential cost of state $((x_1,0),\alpha)$ along the $0b-2a$ boundary. Thus,

$$W^{0b-2a}(x_1,\alpha) = \lim_{T\to\infty}(J^{0b-2a}(x_1,\alpha,0,T) - \bar{J}T)$$

if the limit exists. Taking limits of both sides of (5.77) as $T \to \infty$ and substituting in terms of $W(\boldsymbol{x},\alpha)$, we get

$$\bar{J} = g_1^+ x_1 + \frac{dW^{0b-2a}(x_1,1)}{dx_1}\dot{x}_1^{0b-2a} - pW^{0b-2a}(x_1,1) + pW^{0b-2a}(x_1,0) \tag{5.78}$$

Equation (5.78) represents the dynamics of the cost-to-go function for a point on the $0b-2a$ boundary.

The continuity conditions along the $0b-2a$ boundary can be expressed as

$$\lim_{x_2\to 0\ x_2\in \text{ Region } 2a} W((x_1,x_2),1) = W^{0b-2a}(x_1,1) \tag{5.79}$$

and

$$\lim_{x_2\to 0\ x_2\in \text{ Region } 2a} W((x_1,x_2),0) = W^{0b-2a}(x_1,0). \tag{5.80}$$

Using equation (5.78) to substitute for $W(\boldsymbol{x}, 0)$, (5.80) can equivalently be expressed as

$$\lim_{x_2 \to 0 \quad x_2 \in \text{ Region } 2a} (\mu_2 - d_2) \frac{\partial W((x_1, x_2), 1)}{\partial x_2} - d_1 \frac{\partial W((x_1, x_2), 1)}{\partial x_1} = -d_1 \frac{dW^{0b-2a}(x_1, 1)}{dx_1}. \tag{5.81}$$

In addition, the function W^{0b-2a} must satisfy two more boundary conditions at the origin. The first condition comes from continuity of W^{0b-2a} at the origin and can be expressed as

$$\lim_{x_1 \to 0^+} W^{0b-2a}(x) = W^{0b-2a}(0). \tag{5.82}$$

It can be noted that $W^{0b-2a}(0)$ is known from Section 5.5 (and is given by the value of c^{1c-2b}). The second condition can be derived in the same way as (5.26) in Section 5.5 and is given by

$$\lim_{x_1 \to 0^+} \frac{dW^{0b-2a}(x_1)}{dx_1} = 0. \tag{5.83}$$

Equations (5.79), (5.81) (5.82) and (5.83) constitute the boundary conditions on the $0b-2b$ boundary. The first two of these conditions are implicit conditions because they are in terms of the unknown function $W(\boldsymbol{x}, 1)$ in Region $2a$.

In the following sections, we use the boundary conditions along both the $0b-2a$ and $2a-2b$ boundaries to construct a solution for the second order hyperbolic partial differential equation (5.19) in Region $2a$.

5.8.3 Transformation to canonical form

Consider a new coordinate system whose origin is at $(0, 0)$ of the (x_1, x_2) coordinate system and whose axes are the two characteristic directions, ξ and η. The equation of the characteristics was given in Section 5.8.2. It follows that the relation between this coordinate system and the old coordinate system is given by:

$$\begin{aligned} \xi &= x_2 + \frac{(\mu_2 - d_2)}{d_1} x_1 \\ \eta &= x_2 - \frac{d_2}{d_1} x_1 \end{aligned} \tag{5.84}$$

Let $\hat{W}(\xi,\eta)$ represent the differential cost function in this new system. Let function $\mathcal{W}(\xi,\eta)$ be defined as:

$$\mathcal{W}(\xi,\eta) = \hat{W}((\xi,\eta),1)e^{\frac{r\xi - p\eta}{\mu_2}} \tag{5.85}$$

Rewriting the HJB equation (5.19), in the new coordinate system in terms of the function $\mathcal{W}(\xi,\eta)$, (Appendix B) we get

$$\frac{\partial^2 \mathcal{W}(\xi,\eta)}{\partial\xi\partial\eta} = -\frac{pr}{\mu_2^2}\mathcal{W}(\xi,\eta) + G(\xi,\eta) \tag{5.86}$$

where

$$\begin{aligned} G(\xi,\eta) &= \frac{e^{\frac{r\xi-p\eta}{\mu_2}}}{\mu_2^2}\Bigg[\bar{J}(r+p) - (\frac{g_1^+ d_1 - g_2^- d_2}{\mu_2})\xi \\ &\quad + (\frac{g_1^+ d_1 + g_2^-(\mu_2 - d_2)}{\mu_2})\eta - d_1 g_1^+ + d_2 g_2^-\Bigg] \end{aligned} \tag{5.87}$$

Equation (5.86) is the second order form of the HJB equation in terms of the function $\mathcal{W}(\xi,\eta)$. This is the non-homogeneous *telegraph equation* that can be found in a number of other engineering applications (Myint-U, 1987).

As explained in Section 5.8.2, the $2a-2b$ boundary can be represented by $\xi = 0$ in the new co-ordinate system. The $0b-2a$ boundary, which is given by $x_2 = 0$ in the (x_1,x_2) coordinate system, can be represented by the equation:

$$\eta = -\frac{d_2}{\mu_2 - d_2}\xi$$

Boundary conditions on the $2a-2b$ boundary

Let $\mathcal{W}^{2a-2b}(\eta)$ denote the value of $\mathcal{W}(\xi,\eta)$ at the point $(0,\eta)$ on the $2a-2b$ boundary.

Let

$$\mathcal{W}_\eta^{2a-2b}(\eta) = \lim_{\xi\to 0,\ (\xi,\eta)\in \text{ Region } 2b} \frac{\partial \mathcal{W}(\xi,\eta)}{\partial\eta}$$

Thus $\mathcal{W}_\eta^{2a-2b}(\eta)$ represents the partial derivative of the function $\mathcal{W}(\xi,\eta)$ with respect to η at the point $(0,\eta)$.

Boundary Conditions (5.73) and (5.75) can be expressed in the new system as:

$$\lim_{\xi\to 0,\ (\xi,\eta)\in \text{ Region } 2a} \mathcal{W}(\xi,\eta) = \mathcal{W}^{2a-2b}(\eta) \tag{5.88}$$

$$\lim_{\xi\to 0,\ (\xi,\eta)\in \text{ Region } 2a} \frac{\partial \mathcal{W}(\xi,\eta)}{\partial \eta} = \mathcal{W}_{\eta}^{2a-2b}(\eta) \tag{5.89}$$

The function $\mathcal{W}(\xi,\eta)$ in Region $2b$ is known from (5.43) and (5.85) and therefore the right side of equations (5.88) and (5.89) are known functions.

Boundary conditions on the $0b-2a$ boundary
Let $\mathcal{W}^{0b-2a}(\xi)$ and $\hat{W}^{0b-2a}(\xi)$ denote the values of $\mathcal{W}(\xi,\eta)$ and $\hat{W}(\xi,\eta)$ at the point $(\xi, \frac{-d_2}{\mu_2-d_2}\xi)$ along the $0b-2a$ boundary. The function $\mathcal{W}^{0b-2a}$ is not known and has to be determined by solving the HJB equation. Using (5.85) we have

$$\hat{W}^{0b-2a}(\xi) = \mathcal{W}^{0b-2a} e^{-\frac{(r+\frac{pd_2}{\mu_2-d_2})}{\mu_2}} \tag{5.90}$$

Boundary conditions (5.79) can be expressed in the new coordinate system as

$$\lim_{\eta\to -\frac{\xi d_2}{d_1},\ (\xi,\eta)\in \text{ Region } 2a} \mathcal{W}(\xi,\eta) = \mathcal{W}^{0b-2a}(\xi) \tag{5.91}$$

After some manipulations (see Appendix B), Equation (5.81) can be expressed as

$$\lim_{\eta\to -\frac{\xi d_2}{d_1},\ (\xi,\eta)\in \text{ Region } 2a} \frac{\partial \mathcal{W}(\xi,\eta)}{\partial \eta} = k_1 \frac{d\mathcal{W}^{0b-2a}(\xi)}{d\xi} + k_2 W^{0b-2a}(\xi) \tag{5.92}$$

where k_1, k_2 are constants whose values are determined by the system parameters.

To summarize, the function $\mathcal{W}$ has to satisfy the HJB equation, (5.86), two boundary conditions specified by (5.88) and (5.89) on the $2a-2b$ boundary and two more boundary conditions specified by (5.91) and (5.92) on the $0b-2a$ boundary.

5.8.4 The Reiman method

In this section, we use a methodology from the theory of partial differential equations called *Reiman's method* (Myint-U, 1987) for solving linear hyperbolic partial differential equations with constant coefficients like the one considered here.

Reiman's method gives the solution to the partial differential equation over some region in terms of the boundary conditions and another function called the *Reiman function* that is defined for each point in the region and which satisfies some special conditions. We now give an overview of Reiman's method for our system.

Consider any point $P = (\xi_0, \eta_0)$ in Region $2a$. Let R be the point of intersection of the line $\eta = \eta_0$ with $\xi = 0$ and let Q be the point of intersection of $\eta = -\frac{\xi d_2}{(\mu_2-d_2)}$

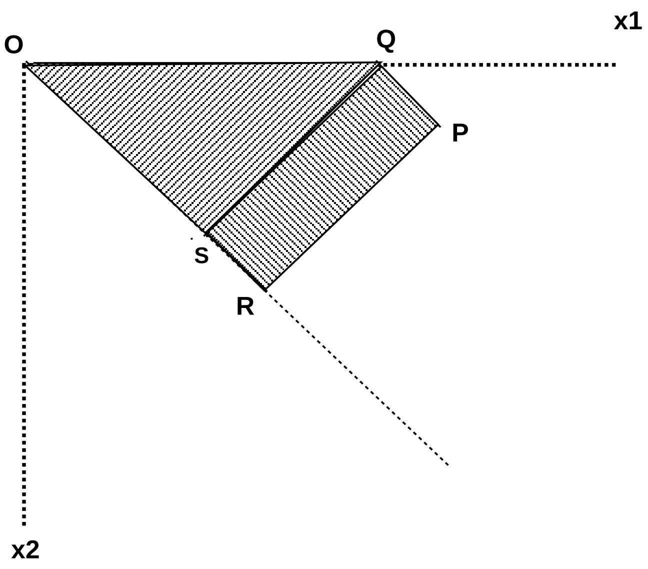

Figure 5-11: Domain of influence for Region $2a$ solution

with $\xi = \xi_0$. Thus the coordinates of R and Q are $(0, \eta_0)$ and $(\xi_0, -\frac{d_2}{(\mu_2-d_2)\xi_0})$. Let S be the point $(0, \frac{-d_2}{(\mu_2-d_2)}\xi_0)$ on the $2a - 2b$ boundary These points are illustrated in Figure 5-11. The domain of influence for Point P is now the area included between the line segments OQ, QP, PR and RO. This area is represented by $OQPR$. Let $v(\xi, \eta, \xi_0, \eta_0)$ denote a special function called the *Reiman function* corresponding to the point (ξ_0, η_0). This function is examined in detail immediately after this overview. Reiman's method gives the value of the function $\mathcal{W}(\xi, \eta)$ at point P as

$$\begin{aligned} \mathcal{W}_P &= [\mathcal{W}v]_R - \int_O^R (\mathcal{W}\frac{\partial v}{\partial \xi}d\xi \,+\, v\frac{\partial \mathcal{W}}{\partial \eta}d\eta) \\ &\quad + \int\int_{OQPR} vGd\xi d\eta - \int_Q^O (\mathcal{W}\frac{\partial v}{\partial \xi}d\xi \,+\, v\frac{\partial \mathcal{U}}{\partial \eta}d\eta) \end{aligned} \tag{5.93}$$

In the above equation, the arguments for the functions have been suppressed for notational convenience. Each term of this equation is explained in detail in Sections 5.8.5 and 5.8.6. To evaluate the right hand side of (5.93), one needs to know the value of the function $\mathcal{W}$ and its partial derivative with respect to η along the two boundaries and also the Reiman function for point P along and its partial derivatives. For the system considered here, the boundary conditions on the $0b - 2a$ boundary are implicit conditions. Reiman's method therefore provides a solution for $\mathcal{W}(\xi, \eta)$ in terms of the unknown function $\mathcal{W}^{0b-2a}(\xi)$ and its derivative along the boundary. But continuity condition (5.91) implies that for any point (ξ, η), the solution $\mathcal{W}(\xi, \eta)$

found by Reiman's method must must be equal to $\mathcal{W}^{0b-2a}(\xi)$ as $\eta \to -\frac{d_2}{\mu_2 - d_2}\xi$. This condition leads to a one-dimensional integral equation for $\mathcal{W}^{0b-2a}(\xi)$ along the $0b-2a$ boundary.

We now define the Reiman function for each point in Region $2a$.

Determination of the Reiman function

The Reiman function is a function that satisfies the homogeneous part of the HJB equation (5.19) and certain special boundary conditions (Myint-U, 1987). Thus, v is a solution to the following partial differential equation

$$\frac{\partial^2 v(\xi, \eta, \xi_0, \eta_0)}{\partial \xi \partial \eta} = -\frac{pr}{\mu_2^2} v(\xi, \eta, \xi_0, \eta_0) \tag{5.94}$$

with boundary conditions given by

$$\begin{aligned} \frac{\partial v(\xi, \eta, \xi_0, \eta_0)}{\partial \xi} &= 0 \quad \text{if } \eta = \eta_0, \\ \frac{\partial v(\xi, \eta, \xi_0, \eta_0)}{\partial \eta} &= 0 \quad \text{if } \xi = \xi_0, \\ v(\xi_0, \eta_0, \xi_0, \eta_0) &= 1. \end{aligned}$$

Let us assume solution for $v(\xi, \eta, \xi_0, \eta_0)$ of the form

$$v(\xi, \eta, \xi_0, \eta_0) = v((\xi - \xi_0)(\eta - \eta_0))$$

Let $\hat{v}$ denote the Reiman function in terms of a parameter s where

$$s = (\xi - \xi_0)(\eta - \eta_0)$$

Therefore

$$v((\xi - \xi_0)(\eta - \eta_0)) = \hat{v}(s)$$

Substituting in (5.94) we get

$$s\frac{d^2\hat{v}}{ds^2} + \frac{d\hat{v}}{ds} + \frac{pr}{\mu_2^2}\hat{v} = 0 \tag{5.95}$$

Let

$$k = \frac{pr}{\mu_2^2}$$

and

$$\lambda = \sqrt{4ks}$$

Substituting in (5.95) in terms of variable λ and using the notation

$$\hat{v}(s) = \tilde{v}(\lambda)$$

we get,

$$\frac{d^2\tilde{v}}{d\lambda^2} + \frac{1}{\lambda}\frac{\tilde{v}}{d\lambda} + \tilde{v} = 0 \tag{5.96}$$

From the theory of Bessel functions (Zauderer, 1983), the solution to (5.96) is given by

$$\tilde{v}(\lambda) = J_0(\lambda)$$

where $J_0(x)$ is the Bessel function of order 0 with parameter x. This solution satisfies the boundary conditions for the Reiman function. Thus the Reiman function is given by

$$v(\xi, \eta, \xi_0, \eta_0) = J_0\left(\sqrt{4k(\xi - \xi_0)(\eta - \eta_0)}\right) \tag{5.97}$$

Differentiating (5.97) with respect to η, we get

$$\frac{\partial v(\xi, \eta, \xi_0, \eta_0)}{\partial \eta} = \frac{\sqrt{k}(\xi - \xi_0)}{\sqrt{(\xi - \xi_0)(\eta - \eta_0)}} J_0^{'}\left(\sqrt{4k(\xi - \xi_0)(\eta - \eta_0)}\right) \tag{5.98}$$

where

$$J_0^{'}(\lambda) = \frac{dJ_0}{d\lambda}$$

Similarly, differentiating (5.97) with respect to ξ, we get

$$\frac{\partial v(\xi, \eta, \xi_0, \eta_0)}{\partial \xi} = \frac{\sqrt{k}(\eta - \eta_0)}{\sqrt{(\xi - \xi_0)(\eta - \eta_0)}} J_0^{'}\left(\sqrt{4k(\xi - \xi_0)(\eta - \eta_0)}\right) \tag{5.99}$$

In the remainder of this section, we derive expressions for the Reiman function and its partial derivatives along the two boundaries. These expressions are used in Sections 5.8.5 and 5.8.6 to evaluate the right side of equation (5.93).

Reiman function along the $0b - 2a$ boundary:

Let point $P = (\xi_0, \eta_0)$ be in the interior of Region $2a$. Let v^{0b-2a} be the Reiman function defined at P for values of ξ and η such that

$$\eta = -\frac{d_2}{(\mu_2 - d_2)}\xi.$$

From (5.97) we have

$$v^{0b-2a}(\xi, \xi_0, \eta_0) = J_0\left(\sqrt{4k(\xi - \xi_0)(-\frac{d_2}{(\mu_2 - d_2)}\xi - \eta_0)}\right) \tag{5.100}$$

For a point (ξ_0, η_0) on the $0b - 2a$ boundary,

$$\eta_0 = -\frac{d_2}{(\mu_2 - d_2)}\xi_0.$$

Substituting these values in equation (5.100), we get

$$v^{0b-2a}(\xi, \xi_0, -\frac{d_2}{(\mu_2 - d_2)}\xi_0) = J_0\left(\sqrt{\frac{-4kd_2}{(\mu_2 - d_2)}(\xi - \xi_0)^2}\right)$$

This is a function of ξ and ξ_0 only. Rather than introducing a new symbol to denote this function, we denote this function by v^{0b-2a} with only two arguments, ξ and ξ_0. Therefore

$$v^{0b-2a}(\xi, \xi_0) = J_0\left(\sqrt{\frac{-4kd_2}{(\mu_2 - d_2)}(\xi - \xi_0)^2}\right)$$

$$v^{0b-2a}(\xi, \xi_0) = J_0\left(i\sqrt{\frac{4kd_2}{(\mu_2 - d_2)}}(\xi - \xi_0)\right)$$

where $i = \sqrt{-1}$. From the theory of Bessel functions,

$$J_0(ix) = I_0(x)$$

where $I_\nu(x)$ is the modified Bessel function of order ν. Therefore,

$$v^{0b-2a}(\xi, \xi_0) = I_0\left(\sqrt{\frac{4kd_2}{(\mu_2 - d_2)}}\,(\xi - \xi_0)\right).$$

Let

$$m = \sqrt{\frac{4kd_2}{(\mu_2 - d_2)}}.$$

Thus,

$$v^{0b-2a}(\xi, \xi_0) = I_0\left(m(\xi - \xi_0)\right). \tag{5.101}$$

Let $v_\xi^{0b-2a}(\xi, \xi_0, \eta_0)$ denote the partial derivative of $v(\xi, \eta, \xi_0, \eta_0)$ with respect to ξ along the boundary. Therefore,

$$v_\xi^{0b-2a}(\xi, \xi_0, \eta_0) = \frac{\partial v(\xi, \eta, \xi_0, \eta_0)}{\partial \xi}\Big|_{\eta = \frac{-d_2}{(\mu_2 - d_2)}\xi}$$

Using (5.99) to substitute for the right side, we get

$$v_\xi^{0b-2a}(\xi, \xi_0, \eta_0) = \frac{\sqrt{k}(\eta - \eta_0)}{\sqrt{(\xi - \xi_0)(\frac{-d_2}{(\mu_2 - d_2)}\xi - \eta_0)}} J_0'\left(\sqrt{4k(\xi - \xi_0)(\frac{-d_2}{(\mu_2 - d_2)}\xi - \eta_0)}\right) \tag{5.102}$$

If in addition, the point (ξ_0, η_0) lies on the $0b - 2a$ boundary, we have

$$\eta_0 = -\frac{d_2}{(\mu_2 - d_2)}\xi_0.$$

Substituting in (5.102), the Reiman function along the boundary for a point on the boundary can be expressed as

$$v_\xi^{0b-2a}(\xi, \xi_0, -\frac{d_2}{(\mu_2 - d_2)}\xi_0) = -i\sqrt{\frac{kd_2}{\mu_2 - d_2}} J_0'\left(im(\xi - \xi_0)\right)$$

The function on the left hand side depends only on ξ and ξ_0. For simplicity, we denote it by $v^{0b-2a}(\xi, \xi_0)$. Therefore

$$v_\xi^{0b-2a}(\xi, \xi_0) = -i\sqrt{\frac{kd_2}{\mu_2 - d_2}} J_0'\left(im(\xi - \xi_0)\right)$$

From the theory of Bessel functions,

$$J_0'(\lambda) = -J_1(\lambda)$$

and

$$I_1(\lambda) = -iJ_1(i\lambda)$$

So

$$v_\xi^{0b-2a}(\xi,\xi_0) = -\sqrt{\frac{kd_2}{(\mu_2 - d_2)}} I_1\left(m(\xi - \xi_0)\right) \tag{5.103}$$

Reiman function along the $2a - 2b$ boundary:
Let v^{2a-2b} denote the Reiman function at some point in Region $2a$. Along the $2a - 2b$ boundary, we have

$$\xi = 0$$

Substituting in equation (5.97), we get

$$v^{2a-2b}(\eta,\xi_0,\eta_0) = J_0\left(\sqrt{4k(-\xi_0)(\eta - \eta_0)}\right)$$

Equivalently,

$$v^{2a-2b}(\eta,\xi_0,\eta_0) = I_0\left(\sqrt{4k\xi_0(\eta - \eta_0)}\right) \tag{5.104}$$

Similarly setting $\xi = 0$ in (5.99) yields the partial derivative of v with respect to ξ along the $2a - 2b$ boundary at some point $(0,\eta)$ on the boundary. Therefore,

$$\frac{\partial v^{2a-2b}(\eta,\xi_0,\eta_0)}{\partial \xi} = -\frac{\sqrt{k}(\eta - \eta_0)}{\sqrt{\xi_0(\eta - \eta_0)}} I_1\left(\sqrt{4k\xi_0(\eta - \eta_0)}\right) \tag{5.105}$$

5.8.5 Solution along the $0b - 2a$ boundary

In this section, we use Reiman's method to express the solution along the $0b - 2a$ boundary in terms of the Reiman function and the boundary conditions.

Consider any point $P(\xi_0, \frac{-d_2}{(\mu_2 - d_2)}\xi_0)$ on the boundary. Let R be the point of intersection of the line $\eta = \frac{-d_2}{(\mu_2 - d_2)}\xi_0$ with $\xi = 0$. Thus the coordinates of R are given by $(0, \frac{-d_2}{(\mu_2 - d_2)}\xi_0)$. Let O represent the origin. The *domain of influence* (Zauderer, 1987) for point P is the area included between the three line segments OP, OR and PR. This triangular region is denoted by OPR. This is illustrated in Figure 5-12.

Equation (5.93) for point P on the $0b - 2a$ boundary becomes

$$\begin{aligned} \mathcal{W}_P &= [\mathcal{W}v]_R - \int_O^R (\mathcal{W}\frac{\partial v}{\partial \xi} d\xi + v\frac{\partial \mathcal{W}}{\partial \eta} d\eta) \\ &\quad + \int\int_{OPR} vG d\xi d\eta - \int_P^O \mathcal{W}\frac{\partial v}{\partial \xi} d\xi + v\frac{\partial \mathcal{W}}{\partial \eta} d\eta \end{aligned} \tag{5.106}$$

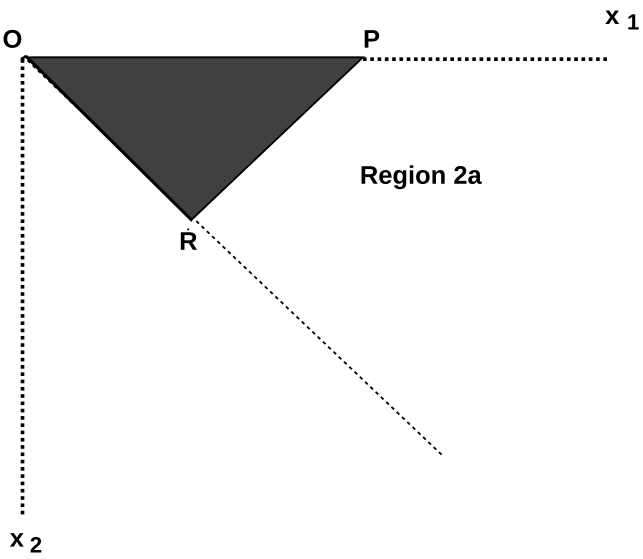

Figure 5-12: Domain of influence for boundary $0b - 2a$

In equation (5.106), the arguments of the functions have been suppressed for notational convenience. The first term on the right side of (5.106) is the product of the functions $\mathcal{W}$ and v evaluated at the point R. Using boundary condition (5.88), we get

$$[\mathcal{W}v]_R(\xi_0) = \mathcal{W}^{2a-2b}\left(\frac{-d_2}{(\mu_2 - d_2)}\xi_0\right) v(0, \frac{-d_2}{(\mu_2 - d_2)}\xi_0, \xi_0, \frac{-d_2}{(\mu_2 - d_2)}\xi_0)$$

Substituting for v from (5.97), and using $I_0(0) = 1$, the first term becomes

$$[\mathcal{W}v]_R(\xi_0) = \mathcal{W}^{2a-2b}\left(\frac{-d_2}{\mu_2 - d_2}\xi_0\right) \tag{5.107}$$

The second term on the right side of equation (5.106) is a line integral along the $2a - 2b$ boundary. Let L_1 denote this line integral. Along the $2a - 2b$ boundary, $\xi = 0$ and $d\xi = 0$. Thus $L_1(\xi_0)$ is one dimensional integral with respect to η along the $2a - 2b$ axis. Using boundary condition (5.89) we get

$$L_1(\xi_0) = -\int_0^{\frac{-d_2}{(\mu_2 - d_2)}\xi_0} \frac{\partial \mathcal{W}^{2a-2b}(\eta)}{\partial \eta}\; v^{2a-2b}\left(\xi_0, \eta, \frac{-d_2}{(\mu_2 - d_2)}\xi_0\right) d\eta \tag{5.108}$$

The first term in the integrand is a known function. The second term is given by

(5.105). Thus the integral on the right side of (5.108) can be evaluated for any value of ξ_0.

Let L_2 denote the third term in (5.106) Thus L_2 is a double integral over the area OPR and can be expressed as:

$$L_2(\xi_0) = \int_{\eta=0}^{\frac{-d_2}{(\mu_2-d_2)}\xi_0} \int_{\xi=0}^{\frac{-(\mu_2-d_2)}{d_2}\eta} G(\xi,\eta) v(\xi,\eta,\xi_0,\frac{-d_2}{(\mu_2-d_2)}\xi_0)\; d\xi\, d\eta \tag{5.109}$$

The function $G(\xi,\eta)$ is known from (5.87). The function v is given by (5.97). Thus $L_2(\xi_0)$ can be calculated for each ξ_0.

Let $F(\xi_0)$ be defined as:

$$F(\xi_0) = [\mathcal{W}v]_R(\xi_0) + L_1(\xi_0) + L_2(\xi_0) \tag{5.110}$$

From (5.107), (5.108) and(5.109), $F(\xi_0)$ is a known function of ξ_0.

Let L_3 denote the fourth term in (5.106). L_3 involves a line integral along the $0b-2a$ boundary over the line segment OP. Using boundary condition (5.91), L_3 can be expressed as:

$$L_3(\xi_0) = \int_0^{\xi_0} \mathcal{W}^{0b-2a}(\xi) v_\xi^{0b-2a}(\xi,\xi_0)\, d\xi \;+\; v^{0b-2a}(\xi,\xi_0)\frac{\partial \mathcal{W}(\xi,\eta)}{\partial \eta}\, d\eta$$

Substituting for $\frac{\partial \mathcal{W}(\xi,\eta)}{\partial \eta}$ on the boundary using (5.92), we get

$$L_3(\xi_0) = \int_0^{\xi_0} \mathcal{W}^{0b-2a}(\xi) v_\xi^{0b-2a}(\xi,\xi_0)\, d\xi + v^{0b-2a}(\xi,\xi_0)\left[k_1\mathcal{W}^{0b-2a}(\xi) + k_2\frac{d\mathcal{W}^{0b-2a}(\xi)}{d\xi}\right] d\eta \tag{5.111}$$

Along the $0b-2a$ boundary,

$$d\eta = -\frac{d_2}{(\mu_2-d_2)}d\xi$$

Therefore, (5.111) yields

$$\begin{aligned} L_3(\xi_0) \;&=\; \int_0^{\xi_0} \mathcal{W}^{0b-2a}(\xi) v_\xi^{0b-2a}(\xi,\xi_0)\, d\xi \\ &\quad - \frac{d_2}{(\mu_2-d_2)}\int_0^{\xi_0} v^{0b-2a}(\xi,\xi_0)\left[k_1\mathcal{W}^{0b-2a}(\xi) + k_2\frac{d\mathcal{W}^{0b-2a}(\xi)}{d\xi}\right] d\xi \end{aligned}$$

Substituting for v^{0b-2a} and v_η^{0b-2a} from equations (5.101) and (5.103), we get

$$\begin{aligned} L_3(\xi_0) = \quad & \int_0^{\xi_0} -\mathcal{W}^{0b-2a}(\xi)\sqrt{\frac{kd_2}{(\mu_2-d_2)}} I_1(m(\xi-\xi_0))\, d\xi \\ & - \frac{d_2}{(\mu_2-d_2)} \int_0^{\xi_0} \left[I_0(m(\xi-\xi_0)) \left(k_1 \mathcal{W}^{0b-2a}(\xi) + k_2 \frac{d\mathcal{W}^{0b-2a}(\xi)}{d\xi} \right) \right] d\xi \end{aligned}$$

Rearranging terms, we get

$$\begin{aligned} L_3(\xi_0) = \quad & \int_0^{\xi_0} -\mathcal{W}^{0b-2a}(\xi) \left[\sqrt{\frac{kd_2}{(\mu_2-d_2)}} I_1(m(\xi-\xi_0)) \; - \; \frac{k_1 d_2}{(\mu_2-d_2)} I_0(m(\xi-\xi_0)) \right] d\xi \\ & - \frac{k_2 d_2}{(\mu_2-d_2)} \int_0^{\xi_0} \left[I_0(m(\xi-\xi_0)) \frac{d\mathcal{W}^{0b-2a}(\xi)}{d\xi} \right] d\xi \end{aligned}$$

Integrating the second integral on the right hand side by parts, and using $I_0'(x) = I_1(x)$, we get

$$\begin{aligned} L_3(\xi_0) = \quad & \int_0^{\xi_0} -\mathcal{W}^{0b-2a}(\xi)[\sqrt{\frac{kd_2}{(\mu_2-d_2)}} I_1(m(\xi-\xi_0)) \; - \; \frac{k_1 d_2}{(\mu_2-d_2)} I_0(m(\xi-\xi_0))] d\xi \\ & - \int_0^{\xi_0} \frac{k_2 d_2}{(\mu_2-d_2)} I_1(m(\xi-\xi_0)) \mathcal{W}^{0b-2a}(\xi)\, d\xi \\ & - \frac{k_2 d_2}{(\mu_2-d_2)} [\mathcal{W}^{0b-2a}(\xi_0) I_0(0) - \mathcal{W}^{0b-2a}(0) I_0(-m\xi_0)] \end{aligned}$$

Substituting $I_0(0) = 1$, $\mathcal{W}^{0b-2a}(0) = 0$ and rearranging terms

$$\begin{aligned} & L_3(\xi_0) = \\ & \quad \int_0^{\xi_0} \mathcal{W}^{0b-2a}(\xi) \left[(\frac{-k_2 d_2}{(\mu_2-d_2)} - \sqrt{\frac{kd_2}{(\mu_2-d_2)}} I_1(m(\xi-\xi_0)) \; - \; \frac{k_1 d_2}{(\mu_2-d_2)} I_0(m(\xi-\xi_0)) \right] d\xi \\ & \quad - \frac{k_2 d_2}{(\mu_2-d_2)} \left[\mathcal{W}^{0b-2a}(\xi_0) \right] \end{aligned} \tag{5.112}$$

The right hand side of equation (5.112) involves the unknown function $\mathcal{W}^{0b-2a}$ which is a consequence of the implicit boundary condition on the $0b-2a$ boundary. Let $R(\xi, \xi_0)$ be a function defined as:

$$R(\xi,\xi_0) = (\frac{-k_2 d_2}{(\mu_2-d_2)} - \sqrt{\frac{kd_2}{(\mu_2-d_2)}} I_1(m(\xi-\xi_0)) \; - \; \frac{k_1 d_2}{(\mu_2-d_2)} I_0(m(\xi-\xi_0))$$

Thus, (5.112) can be written in a compact form as:

$$L_3(\xi_0) = \int_0^{\xi_0} \mathcal{W}^{0b-2a}(\xi) R(\xi, \xi_0) d\xi \; - \; \frac{k_2 d_2}{(\mu_2 - d_2)} \left[\mathcal{W}^{0b-2a}(\xi_0) \right] \tag{5.113}$$

Using (5.110) and (5.113), equation (5.106) can now be expressed as

$$\mathcal{W}_P = F(\xi_0) + L_3(\xi_0) \tag{5.114}$$

Boundary condition (5.91) implies that

$$\mathcal{W}_P = \mathcal{W}^{0b-2a}(\xi_0)$$

Thus (5.106), (5.112) and (5.114) yield

$$\mathcal{W}^{0b-2a}(\xi_0)(1 + \frac{k_2 d_2}{(\mu_2 - d_2)}) \; = \; F(\xi_0) + \int_0^{\xi_0} \mathcal{W}^{0b-2a}(\xi) R(\xi, \xi_0) d\xi \tag{5.115}$$

Multiplying both sides by $e^{(-\frac{r + \frac{p d_2}{\mu_2 - d_2}}{\mu_2})\xi_0}$ and using (5.90), (5.115) becomes

$$\begin{aligned} \hat{W}^{0b-2a}(\xi)(1 + \frac{k_2 d_2}{(\mu_2 - d_2)}) &= \\ e^{(-\frac{r + \frac{p d_2}{\mu_2 - d_2}}{\mu_2})\xi_0} F(\xi_0) &+ \int_0^{\xi_0} e^{(-\frac{r + \frac{p d_2}{\mu_2 - d_2}}{\mu_2})(\xi_0 - \xi)} W^{0b-2a}(\xi) R(\xi, \xi_0) d\xi \end{aligned} \tag{5.116}$$

This solution must satisfy the remaining two boundary conditions (5.82) and (5.83) at the origin. In the new coordinate system, condition (5.82) can be expressed as:

$$\lim_{\xi \to 0^+} \hat{W}^{0b-2a}(\xi) = \hat{W}^{0b-2a}(0) \tag{5.117}$$

Condition (5.83) can be expressed as (Appendix B)

$$\lim_{\xi \to 0^+} \frac{d\hat{W}^{0b-2a}(\xi)}{d\xi} = 0 \tag{5.118}$$

The function $\hat{W}(\xi, \eta)$ and its derivative along the $0b - 2a$ boundary $\frac{d\hat{W}^{0b-2a}(\xi)}{d\xi}$ can be obtained by numerically solving the one dimensional integral equation (5.116) subject to the initial conditions (5.117) and (5.118). Finally, one can use equation (B.8) to obtain the differential cost function in the x_1, x_2 coordinate system which is given by:

$$W^{0b-2a}(x_1) = \hat{W}^{0b-2a}(\frac{d_1}{\mu_2 - d_2}\xi) \tag{5.119}$$

Similarly using (B.9), we get

$$\frac{dW^{0b-2a}(x_1)}{dx_1} = \frac{d\hat{W}^{0b-2a}(\xi)}{d\xi}\frac{(\mu_2 - d_2)}{d_1} \tag{5.120}$$

5.8.6 Solution in Region $2a$

In this section, we derive an expression for $\mathcal{W}(\xi, \eta)$ in Region $2a$ in terms of the value of the function and its derivatives along the $0b - 2a$ boundary. The boundary condition along the $0b - 2a$ boundary is in terms of the function $\mathcal{W}^{0b-2a}$ along the boundary and its derivative which are known from Section 5.8.5.

The value of the function $\mathcal{W}$ at Point P is given by (5.93). We now examine each term on the right side of (5.93). The first term on the right side is similar to the first term in (5.106). Using similar arguments as in Section 5.8.5, we can express this term as

$$[\mathcal{W}v]_R(\xi_0, \eta_0) = \mathcal{W}^{2a-2b}(\eta_0)\, v(0, \eta_0, \xi_0, \eta_0)$$

Substituting for v from (5.97), and using $I_0(0) = 1$, the first term becomes

$$[\mathcal{W}v]_R(\xi_0, \eta_0) = \mathcal{W}^{2a-2b}(\eta_0) \tag{5.121}$$

Let L_4 denote the second term in (5.93). This term is similar to the second term in (5.106) and can therefore be expressed as:

$$L_4(\xi_0, \eta_0) = -\int_0^{\eta_0} \frac{\partial \mathcal{W}^{2a-2b}(\eta)}{\partial \eta}\, v^{2a-2b}(\eta, \xi_0, \eta_0)\, d\eta. \tag{5.122}$$

The first factor in the above integrand is a known function. The second factor is given by (5.105). Thus the above integral can therefore be evaluated for any value of (ξ_0, η_0).

The third term on the right side of (5.122) is a double integral over the domain of influence similar to the third term on the right side of (5.106). Let L_5 denote this term. From Figure 5-11, the area of integration $OQPR$ is the union of two areas: the triangular area OQS and the area enclosed by the parallelogram $SQPR$. L_5 can therefore be expressed as:

$$
\begin{aligned}
L_5(\xi_0,\eta_0) &= \int_{\eta=0}^{\frac{-d_2}{(\mu_2-d_2)}\xi_0} \int_{\xi=0}^{\frac{-(\mu_2-d_2)}{d_2}\eta} G(\xi,\eta)v(\xi,\eta,\xi_0,\eta_0)d\xi d\eta \\
&+ \int_{\eta=\frac{-d_2}{(\mu_2-d_2)}\xi_0}^{\eta_0} \int_{\xi=0}^{\xi_0} G(\xi,\eta)v(\xi,\eta,\xi_0,\eta_0)d\xi d\eta. \qquad (5.123)
\end{aligned}
$$

The function $G(\xi,\eta)$ is known from (5.87). The function $v(\xi,\eta,\xi_0,\eta_0)$ is given by (5.97). It should be noted that the arguments for the v function here are different from those in (5.109).

Let $\mathcal{F}(\xi_0,\eta_0)$ be defined as

$$
\mathcal{F}(\xi_0,\eta_0) = [Uv]_R(\xi_0,\eta_0) + L_4(\xi_0,\eta_0) + L_5(\xi_0,\eta_0) \qquad (5.124)
$$

The right hand side of the above equation can be computed from (5.121), (5.122) and (5.123).

Let L_6 denote the fourth term in equation (5.93). This term is similar to the fourth term in (5.106) except now the function $\mathcal{W}^{0b-2a}(\xi)$ and its derivative $\dot{U}^{2a-2b}(\xi)$ have been calculated as described in Section 5.8.5 and are therefore known functions. Thus, we have

$$
\begin{aligned}
L_6(\xi_0,\eta_0) &= \int_0^{\xi_0} \mathcal{W}^{0b-2a}(\xi)v_\xi^{0b-2a}(\xi,\xi_0,\eta_0)\,d\xi \\
&+ v^{0b-2a}(\xi,\xi_0,\eta_0)\left[k_1\mathcal{W}^{0b-2a}(\xi) + k_2\frac{d\mathcal{W}^{0b-2a}(\xi)}{d\xi}\right]d\eta \quad (5.125)
\end{aligned}
$$

Along the $0b-2a$ boundary,

$$
d\eta = -\frac{d_2}{(\mu_2-d_2)}d\xi.
$$

Therefore, (5.125) becomes

$$
\begin{aligned}
L_6(\xi_0,\eta_0) &= \int_0^{\xi_0} \mathcal{W}^{0b-2a}(\xi)v_\xi^{0b-2a}(\xi,\xi_0,\eta_0)\,d\xi \\
&- \frac{d_2}{(\mu_2-d_2)}\int_0^{\xi_0} v^{0b-2a}(\xi,\xi_0,\eta_0)\left[k_1\mathcal{W}^{0b-2a}(\xi) + k_2\frac{d\mathcal{W}^{0b-2a}(\xi)}{d\xi}\right]d\xi \\
&\qquad (5.126)
\end{aligned}
$$

The functions v^{0b-2a} and v_ξ^{0b-2a} are known from equations (5.101) and (5.102). The functions $\mathcal{W}^{0b-2a}(\xi)$ and $\dot{\mathcal{W}}^{0b-2a}(\xi)$ are known from Section 5.8.5. Thus $L_6(\xi_0, \eta_0)$ can be calculated for each point (ξ_0, η_0).
Using (5.124) and (5.126), equation (5.93) can now be expressed as:

$$\mathcal{W}(\xi_0, \eta_0) = \mathcal{F}(\xi_0, \eta_0) + L_6(\xi_0, \eta_0) \tag{5.127}$$

The right sides of (5.124) and (5.126) are in terms of known functions. Therefore (5.127) gives the value of the function $\mathcal{W}(\xi_0, \eta_0)$ at Point (ξ_0, η_0) in Region $2a$.

Finally one can use (5.85) to get

$$\hat{W}(\xi, \eta) = \mathcal{W}(\xi, \eta) e^{-\frac{(r\xi - p\eta)}{\mu_2}} \tag{5.128}$$

To summarize, we have used the expressions for the Reiman function and its derivatives along the $0b - 2a$ and $2a - 2b$ boundaries along with the results from Section (5.8.5) to evaluate the right side of expression (5.93). The solution from (5.128) is in the (ξ, η) coordinate system. Using (5.84) the solution in the (x_1, x_2) coordinate system is given by

$$W(x_1, x_2, 1) = \hat{W}\left(\frac{d_1}{\mu_2}(\xi - \eta), (\frac{d_2}{\mu_2}\xi + (1 - \frac{d_2}{\mu_2})\eta)\right), \tag{5.129}$$

5.8.7 Solution in Region $1a$

In this section we suggest a method to use the results for Region $2a$ to solve for the differential cost in Region $1a$. The derivation of the solution in Region $1a$ is tougher than that in Region $1b$ because a trajectory from point (x_1, x_2) in Region $1b$ always hits the $1c - 1b$ boundary after a fixed time $\frac{x_2}{d_2}$. A solution from a point in Region $1a$ could hit either the $0a - 1a$ or $1a - 1b$ boundaries. The time when these boundaries are reached is a random variable as also is the state of the machine at that time. Consequently, we expect the boundary conditions in Region $1a$ to be more complicated than those in Region $1b$. Solutions having the same form as the solution in Region $2b$ (a sum involving quadratic, linear, constant terms and their products with exponential terms) do not satisfy the Bellman equation and boundary conditions in Region $1a$.

Consider a trajectory that originates at some point $\boldsymbol{x}$ in Region $1a$ at time 0 with $\alpha(0) = 1$. This trajectory could either reach the $0a - 1b$ boundary or $1a - 1b$ boundary

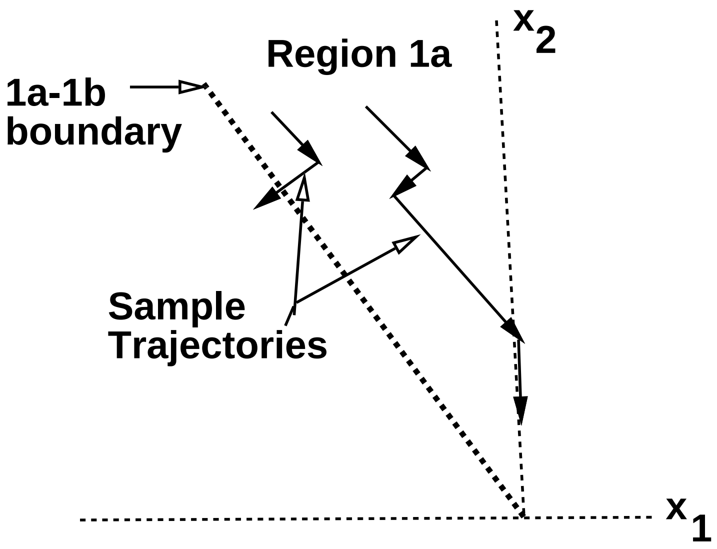

Figure 5-13: Surplus behavior in Region $1a$

depending on the sample realization of the machine failures and repairs as shown in Figure 5-13. In the first case, the OODS policy maintains it on the $0a-1b$ boundary until the hedging point is reached or until a machine failure forces the trajectory back into Region $1a$ whichever happens earlier. If the trajectory reaches the $1a-1b$ boundary it moves into Region $1b$ and subsequently does not return to Region $1a$. It can be seen that this behavior is *symmetrical* to that in Region $2a$ in the sense that the behavior of Part Type 1 in Region $2a$ is the similar to that of Part Type 2 in Region $1a$ and vice versa. Consequently, the derivation of the expression for $\mathcal{W}$ follows the same steps as those given in Section 5.8.6. Let us define a new coordinate system in Region $1b$ as

$$\begin{aligned} \xi &= x_1 + \frac{(\mu_1 - d_1)}{d_2} x_2 \\ \eta &= x_1 - \frac{d_1}{d_2} x_2 \end{aligned} \tag{5.130}$$

This expression can be obtained from (5.84) by interchanging the parameters of the two part types. Let $\hat{W}(\xi, \eta)$ represent the differential cost function in Region $1a$ this new system. Let function $\mathcal{W}(\xi, \eta)$ be defined as:

$$\mathcal{W}(\xi, \eta) = \hat{W}((\xi, \eta), 1) e^{\frac{r\xi - p\eta}{\mu_1}} \tag{5.131}$$

Following the same steps as in (5.86), the HJB equation in Region $1a$ becomes

$$\frac{\partial^2 \mathcal{W}(\xi,\eta)}{\partial\xi\partial\eta} = -\frac{pr}{\mu_1^2}\mathcal{W}(\xi,\eta) + G(\xi,\eta) \tag{5.132}$$

where

$$\begin{aligned} G(\xi,\eta) &= \frac{e^{\frac{r\xi-p\eta}{\mu_1}}}{\mu_1^2}\Bigg[\bar{J}(r+p) - (\frac{g_2^+d_2 - g_1^-d_1}{\mu_1})\xi \\ &\quad + (\frac{g_2^+d_2 + g_1^-(\mu_1-d_1)}{\mu_1})\eta - d_2g_2^+ + d_1g_1^-\Bigg] \end{aligned} \tag{5.133}$$

Let $\mathcal{W}^{1a-1b}$ and $\mathcal{W}^{0a-1a}$ denote denote the function $\mathcal{W}$ along the $1a-1b$ boundary and the $0a-1a$ boundaries respectively. The boundary conditions along the $1a-1b$ boundary can be derived in the same way as (5.88) and (5.89) and can be expressed as

$$\lim_{\xi\to 0,\ (\xi,\eta)\in \text{ Region } 1a} \mathcal{W}(\xi,\eta) = \mathcal{W}^{1a-1b}(\eta) \tag{5.134}$$

and

$$\lim_{\xi\to 0,\ (\xi,\eta)\in \text{ Region } 1a} \frac{\partial\mathcal{W}(\xi,\eta)}{\partial\eta} = \mathcal{W}_\eta^{1a-1b}(\eta) \tag{5.135}$$

The boundary conditions along the $0a-1a$ boundary are similar to (5.91) and (5.92) and are given by

$$\lim_{\eta\to -\frac{\xi d_1}{d_2},\ (\xi,\eta)\in \text{ Region } 1a} \mathcal{W}(\xi,\eta) = \mathcal{W}^{0a-1a}(\xi) \tag{5.136}$$

and

$$\lim_{\eta\to -\frac{\xi d_1}{d_2},\ (\xi,\eta)\in \text{ Region } 1a} \frac{\partial\mathcal{W}(\xi,\eta)}{\partial\eta} = k_3\frac{d\mathcal{W}^{0a-1a}(\xi)}{d\xi} + k_4^{0a-1a}(\xi) \tag{5.137}$$

where k_3 and k_4 are constants whose values are determined by the system parameters (Appendix B).

The solution method to evaluate $\mathcal{W}$ follows the same steps as in Sections (5.8.5) and (5.8.6) and is therefore not repeated here. Once $\mathcal{W}$ is known, the function $\hat{W}$ can be obtained using equation (5.131). Thus

$$\hat{W}((\xi,\eta),1) = \mathcal{W}(\xi,\eta)e^{-\frac{r\xi-p\eta}{\mu_1}} \tag{5.138}$$

Finally, using (5.130) to transform this solution to the (x_1,x_2) coodinates, the differential cost in Region $1a$ is given by

$$W(x_1, x_2, 1) = \hat{W}\left(\frac{d_2}{\mu_1}(\xi - \eta), (\frac{d_1}{\mu_1}\xi + (1 - \frac{d_1}{\mu_1})\eta)\right). \tag{5.139}$$

In the next section, we use the solution along the $0a - 1a$ and $0b - 2a$ boundaries to obtain the solutions in Regions $0a$ and $0b$.

5.9 Analysis of Regions $0b$ and $0a$

5.9.1 Surplus behavior in Region $0b$

Consider a sample surplus trajectory that, at time t, is at some point $\boldsymbol{x} = (x_1, x_2)$ in Region $0b$. The control vectors in this region is given by

$$\begin{aligned} u^{2a}(\boldsymbol{x}, 1) &= (0, 0), \\ u^{2a}(\boldsymbol{x}, 0) &= (0, 0). \end{aligned}$$

For $\alpha = 1$ or $\alpha = 0$, this control guides the trajectory along the direction $(-d_1, -d_2)$, towards the $0b - 2a$ boundary. The trajectory could reach the boundary with the machine being up or down. If the machine state is up, the trajectory remains on the $0b - 2a$ boundary until the next machine failure. If the machine state is down, then the trajectory enters Region $2a$ and its subsequent behavior is as described in Section 5.8.

The differential cost function in Region $0b$ thus depends only on the boundary condition along the $0b - 2a$ boundary. The surplus behavior in Region $0b$ is illustrated in Figure 5-10 for two trajectories. One of these trajectories reaches the $0b - 2a$ boundary with the machine being up and subsequently stays on the boundary whereas the other reaches the $0b - 2a$ boundary with the machine being down and subsequently moves into Region $2a$.

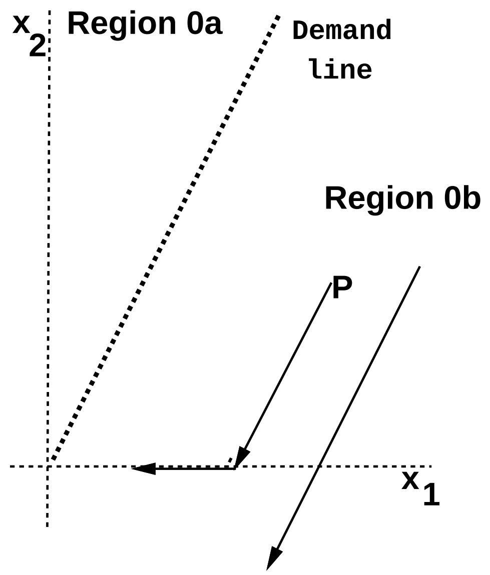

Figure 5-14: Surplus behavior in Region $0b$

5.9.2 Solution in Region $0b$

Dynamics of surplus

Consider a trajectory at some time t such that $\boldsymbol{x}(t) \in$ Region $0b$, $\alpha(t) = 1$. Let $t + t_1$ be the first time, after t, such that

$$x_2(t + t_1) = 0$$

During the interval $(t, t + t_1)$ the rate of change of surplus is given by $(-d_1, -d_2)$. Thus we have

$$0 = x_2(t + t_1) = x_2 - d_2 t_1$$

So

$$t_1 = \frac{x_2}{d_2}.$$

Also, the dynamics of x_1 implies that

$$x_1(t + t_1) = x_1 - d_1 t_1.$$

Substituting for t_1 yields

$$x_1(t + t_1) = x_1 - \frac{d_1}{d_2} x_2.$$

Dynamics of the machine state

The machine state at time t_1 can be either up or down. Let q_1 denote the probability that the machine is up at time $t+t_1$ given it is up at time t. Thus

$$q_1 = \text{prob}\,\{\alpha(t+t_1) = 1|\alpha(t) = 1\}$$

The probability that the machine is down at time $t+t_1$ given that the machine is up at time t is therefore given by $1-q_1$.

It can be recalled from Chapter 2 that during the interval $(t, t+t_1)$, machine failures and repairs evolve according to a Markov process. Thus q_1 is the probability that the Markov process is in state 1 at time $t+t_1$ given that it is in state 1 at time t. From the transient analysis of the Markov process (Kleinrock, 1975a), this probability is given by

$$q_1 = \frac{r}{r+p} + \frac{p}{r+p}e^{-\frac{(r+p)x_2}{d_2}} \tag{5.140}$$

Cost dynamics

Let $W(\boldsymbol{x}, \alpha)$ be the differential cost function for any state $(\boldsymbol{x}, \alpha)$ where $\boldsymbol{x} \in$ Region $0b$ and let $W^{0b-2a}(x, \alpha)$ denote the differential cost function of a state (x, α) on the $0b-2a$ boundary.

The definition of the cost-to-go function in Chapter 2 implies that the cost-to-go from state $(\boldsymbol{x}, 1)$ at time t can be expressed as a sum of two costs: expected cost incurred over the interval $(t, t+t_1)$ and the expected cost-to-go at time $t+t_1$ The machine state at time $t+t_1$ is a random variable which can be 1 with a probability q_1 or be 0 with a probability $1-q_1$. Thus we have

$$\begin{aligned} J((x_1,x_2),1,t,T) \;\; = \;\; & \int_t^{t+t_1} g(x(s))\,ds + q_1 J((x_1(t+t_1), x_2(t+t_1)), 1, t+t_1, T) \\ & +(1-q_1)J((x_1(t+t_1), x_2(t+t_1)), 0, t+t_1, T) \end{aligned}$$

Substituting for $x_1(t+t_1)$ and $x_2(t+t_1)$, we get

$$\begin{aligned} J((x_1,x_2),1,t,T) \;\; = \;\; & \int_t^{t+t_1} g(x(s))\,ds + q_1 J((x_1 - d_1 t_1, 0), 1, t+t_1, T) \\ & +(1-q_1)J((x_1 - d_1 t_1, 0), 0, t+t_1, T) \end{aligned}$$

Taking limits of both sides as $T \to \infty$ and using (2.6), we can express the above equation as

$$\begin{aligned} W((x_1,x_2),1)+\bar{J}(T-t) &= \int_t^{t+t_1} g(x(s))\,ds + q_1 W^{0b-2a}(x_1-d_1t_1,1)+q\bar{J}(T-(t+t_1)) \\ &\quad +(1-q_1)W^{0b-2a}(x_1-d_1t_1,0)+(1-q)\bar{J}(T-(t+t_1)) \end{aligned}$$

$\bar{J}$ is the average cost under the OODS policy and is known from (A.11). Canceling common terms from both sides and rearranging, we get

$$\begin{aligned} W((x_1,x_2),1) &= \int_t^{t+t_1} g(x(s))\,ds + q_1 W^{0b-2a}(x_1-d_1t_1,1) \\ &\quad +(1-q_1)W^{0b-2a}(x_1-d_1t_1,0)-\bar{J}t_1 \end{aligned} \tag{5.141}$$

The additivity of the instantaneous costs implies that the integral in the first term on the right side can be expressed as:

$$\int_t^{t+t_1} g(x(t))\,dt = \int_t^{t+t_1} g_1^+ x_1(s))\,ds + \int_t^{t+t_1} g_2^+ x_2(s)\,ds \tag{5.142}$$

The dynamics of $\boldsymbol{x}$ in $(t,t+t_1)$ is given by:

$$\begin{aligned} x_1(s) &= x_1-d_1s \\ x_2(s) &= x_2-d_2s \end{aligned}$$

Substituting in (5.142) and evaluating the integral, we get

$$\int_t^{t+t_1} g(x(t))\,dt = \frac{g_1^+x_1x_2}{d_2}-\frac{g_1^+d_1x_2^2}{2d_2^2}+\frac{g_2^+x_2^2}{2d_2} \tag{5.143}$$

Finally, substituting (5.143) and (5.140) in (5.141) and simplifying leads to the following expression for the differential cost in Region $0b$.

$$\begin{aligned} W(\boldsymbol{x},1) &= \frac{g_1^+x_1x_2}{d_2}-\frac{g_1^+d_1x_2^2}{2d_2^2}+\frac{g_2^+x_2^2}{2d_2}-\bar{J}\frac{x_2}{d_2} \\ &\quad +\left(\frac{r}{r+p}+\frac{p}{r+p}e^{-\frac{(r+p)x_2}{d_2}}\right)W^{0b-2a}(x_1-\frac{d_1x_2}{d_2},1) \\ &\quad +\left(\frac{p}{r+p}-\frac{p}{r+p}e^{-\frac{(r+p)x_2}{d_2}}\right)W^{0b-2a}(x_1-\frac{d_1x_2}{d_2},0) \end{aligned} \tag{5.144}$$

W^{0b-2a} is a function of a scalar variable and is known from (5.119) in Section 5.8. Let $\dot{W}^{0b-2a}$ denote its derivative which is known from (5.120).

Using (5.78) to substitute for $W(x,0)$ in equation (5.144) gives

$$\begin{aligned} W(\boldsymbol{x},1) &= \frac{g_1^+ x_1 x_2}{d_2} - \frac{g_1^+ d_1 x_2^2}{2d_2^2} + \frac{g_2^+ x_2^2}{2d_2} - \bar{J}\frac{x_2}{d_2} + W^{0b-2a}(x_1 - \frac{d_1 x_2}{d_2}, 1) \\ &+ \left(\frac{p}{r+p} - \frac{p}{r+p}e^{-\frac{(r+p)x_2}{d_2}}\right)\left[\bar{J} - g_1^+ x_1 - \dot{W}^{0b-2a}(x_1 - \frac{d_1 x_2}{d_2})(u_1^{0b-2a} - d_1)\right] \end{aligned}$$

All functions on the right side of (5.144) are known. Therefore (5.145) is an expression for the differential cost function for state $(\boldsymbol{x},1)$ in Region $0b$.

5.9.3 Solution in Region $0a$

The surplus behavior in Region $0a$ is symmetrical to that in Region $0b$. A trajectory that originates in Region $0a$ always reaches the $0a-1a$ boundary after a fixed time. The differential cost analysis in Region $0b$ is similar to that in Region $0b$ Similar to (5.144) in Region $0b$, the differential cost in Region $0a$ is given by:

$$\begin{aligned} W(\boldsymbol{x},1) &= \frac{g_2^+ x_2 x_1}{d_1} - \frac{g_2^+ d_2 x_1^2}{2d_1^2} + \frac{g_1^+ x_1^2}{2d_1} - \bar{J}\frac{x_1}{d_1} + W^{0a-1a}(x_2 - \frac{d_2 x_1}{d_1}, 1) \\ &+ \left(\frac{p}{r+p} - \frac{p}{r+p}e^{-\frac{(r+p)x_1}{d_1}}\right) W^{0a-1a}(x_2 - \frac{d_2 x_1}{d_1}, 0) \end{aligned} \quad (5.145)$$

The differential costs along the $0b-2a$ and $0a-1a$ boundaries satisfy continuity conditions at the origin. That is

$$\lim_{x_1 \to 0} W^{0b-2a}(x_1, \alpha) = W(0, \alpha) = \lim_{x_2 \to 0} W^{0a-1a}(x_2, \alpha).$$

Using (5.144) and (5.145), it is therefore easy to verify that

$$\lim_{\frac{x_2}{d_2} \to \frac{x_1}{d_1}, \boldsymbol{x} in \text{ Region } 0b} W(x_1, x_2, 1) = \lim_{\frac{x_2}{d_2} \to \frac{x_1}{d_1}, \boldsymbol{x} in \text{ Region } 0a} W(x_1, x_2, 1)$$

In other words, the differential cost function $W(x,1)$ automatically satisfies the continuity condition along the $0b-0a$ boundary.

Let $\dot{W}^{0a-1a}$ denote the derivative of the function W^{0a-1a} and let u_2^{0a-1a} denote the production rate for Part Type 2 along the $0a-1a$ boundary. Using arguments similar to those for Region $0b$, equation (5.144) can also be expressed as

$$\begin{aligned} W(\boldsymbol{x},1) &= \frac{g_2^+ x_2 x_1}{d_1} - \frac{g_2^+ d_2 x_1^2}{2d_1^2} + \frac{g_1^+ x_1^2}{2d_1} - \bar{J}\frac{x_1}{d_1} + W^{0a-1a}(x_2 - \frac{d_2 x_1}{d_1}, 1) \\ &+ \left(\frac{p}{r+p} - \frac{p}{r+p} e^{-\frac{(r+p)x_2}{d_2}}\right) \left[\bar{J} - g_2^+ x_2 - \dot{W}^{0a-1a}(x_2 - \frac{d_2 x_1}{d_1})(u_2^{0b-2a} - d_2)\right] \end{aligned}$$

Since all functions on the right side are known from Section 5.8, equation (5.146) can be used to evaluate the differential cost in Region $0a$.

5.10 Numerical results

In this section, we describe an example and illustrate the method discussed in the previous sections to compute the differential cost in the various regions.

Consider a system that has the following parameter values.

$$\begin{aligned} g_1^+ &= 5.0 & g_2^+ &= 5.5 \\ g_1^- &= 2.0 & g_2^- &= 0.8 \\ \mu_1 &= 3.0 & \mu_2 &= 6.0 \\ d_1 &= 1.0 & d_2 &= 2.0 \\ r &= 0.9 & p &= 0.1 \end{aligned}$$

The fraction of time that the machine needs to be busy to sustain these demand rates is given by

$$\frac{d_1}{\mu_1} + \frac{d_2}{\mu_2} = 0.666$$

The fraction of time that the machine is available on an average $(\frac{r}{r+p})$ is equal to 0.9. The system can therefore meet demand in the long run.

The definition of the regions of constant production rates for the OODS policy does not depend on specific parameter values and are defined in Table 5.1. The subregion boundaries $0a-0b$, $2a-2b$ and $1a-1b$ defined in Table 5.2 do depend on the choice of the capacity set. For this example, the subregion boundaries are given in Table 5.8.

From equation (5.27) in Section 5.5, the differential cost along the $1c-2b$ boundary

Subregion	Equation
$0a - 0b$	$x_1 - 0.5x_2 = 0$
$1a - 1b$	$x_1 + x_2 = 0$
$2a - 2b$	$x_1 + 0.25x_2 = 0$

Table 5.8: Example subregion boundaries

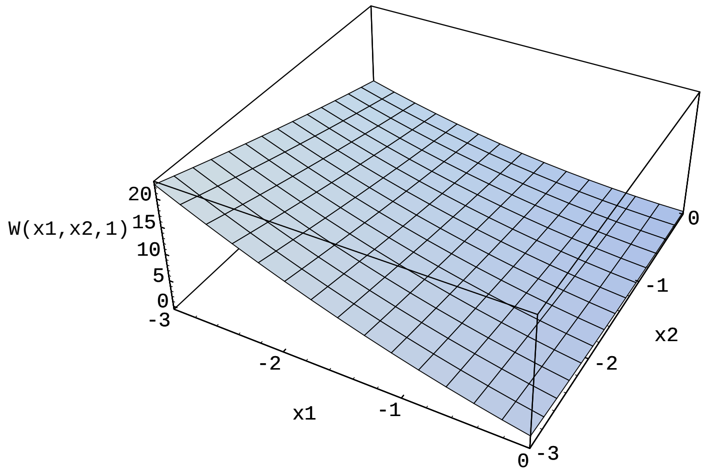

Figure 5-15: Solution in Region $1c$

is given by

$$W^{1c-2b}(x_2, 1) = 0.29x_2^2 + c^{1c-2b} \tag{5.146}$$

We choose $c^{1c-2b} = 0$. Using (5.38), the differential cost for Region $1c$ can be calculated as

$$W(\boldsymbol{x}, 1) = 1.26x_1^2 + 1.85x_1x_2 + 0.29x_2^2$$

This solution is illustrated in Figure 5-15. In Region $2b$, the solution is of the form (5.43) where the coefficients are obtained by the method explained in Section5.7. For this example, the solution becomes

$$W(\boldsymbol{x}, 1) = (0.48x_1 - 0.34x_2 - 2.06)e^{-x_1} + 4.44x_1^2 + 1.14x_1x_2 + 0.29x_2^2 + 1.58x_1 + 0.34x_2 + 2.06 \tag{5.147}$$

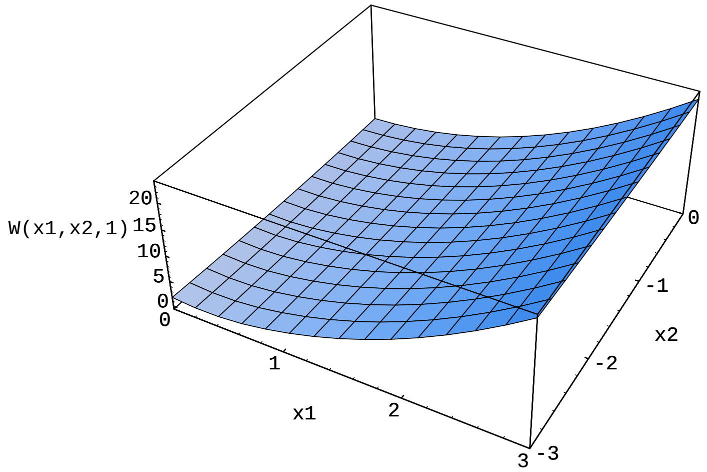

Figure 5-16: Solution in Region 2

Similarly in Region $1b$ the solution is given by

$$W(\boldsymbol{x}, 1) = -9.1e^{-0.5x_2} + 1.26x_1^2 + 1.85x_1x_2 + 1.86x_2^2 + 4.55x_2 + 9.10$$

The solutions in the other subregions are calculated numerically. Figure 5-16 illustrates the solution in Region 2 As seen in Section 5.8, the differentiable cost function is twice continuously differentiable in the interior of subregions $2a$ and $2b$ in Region 2 and is continuously differentiable at the $2a - 2b$ boundary. The behavior of the second derivative in the interior of Region $2a$ is investigated later in this section. Figure 5-17 shows the solution in Regions $1a$ and $1b$. Figure 5-18 illustrates the solution in Regions $0a$ and $0b$. The solution over the entire (x_1, x_2) space is as shown in Figure 5-19.

Dependence on failure rate Figure 5-20 shows the variation of the differential cost along the $2a - 2b$ boundary for different values of the failure rates when all the other parameters are held constant. The horizontal axis (d) represents the x_1 coordinates of points that lie on the $2a - 2b$ boundary. The vertical axis gives the differential cost function at such points. The dot-dashed line corresponds to the case where $p = 0.1$. The dashed line corresponds to $p = 0.001$ and the solid line corresponds to $p = 0$. As p decreases to 0, the solution approaches that of the RD system described in Section 5.2.1. This is intuitive because as $p \to 0$, failures become

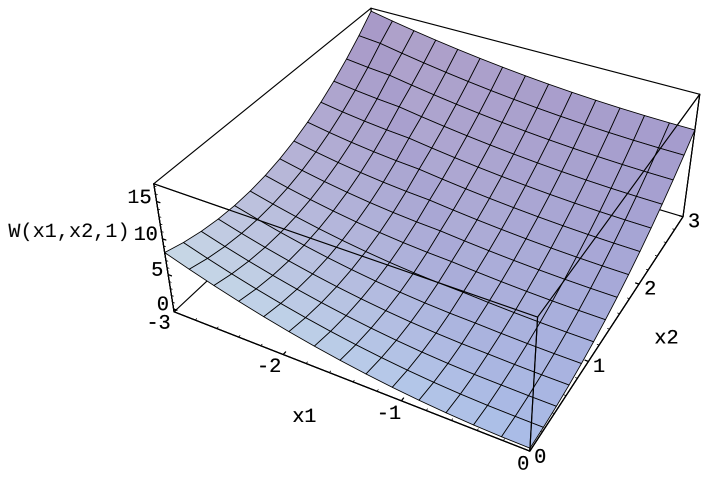

Figure 5-17: Solution in Region 1

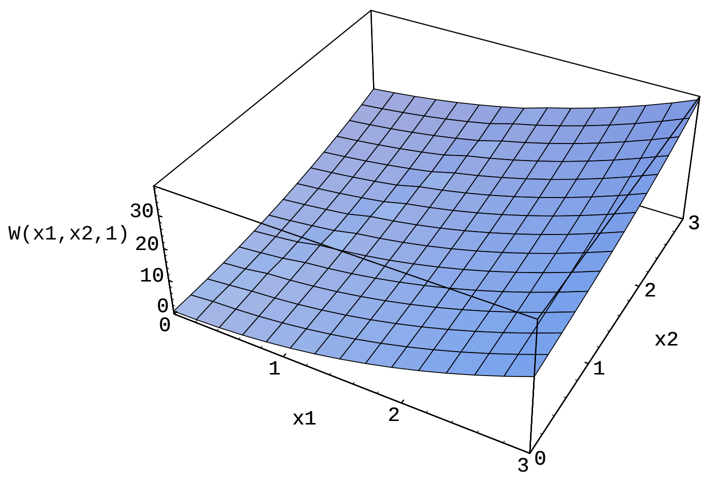

Figure 5-18: Solution in Region 0

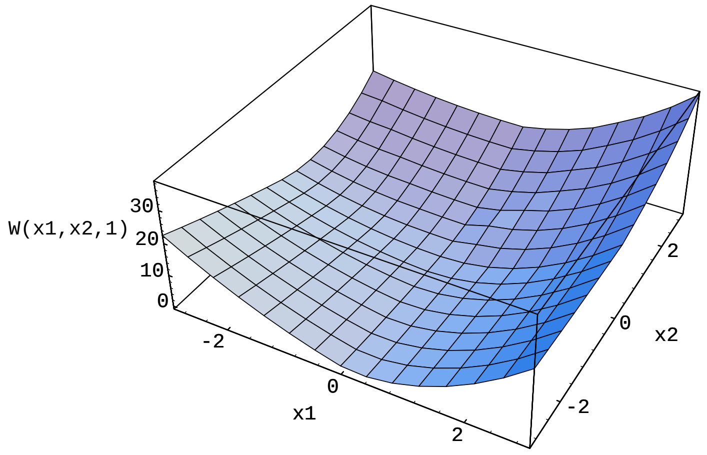

Figure 5-19: Solution over the entire surplus space

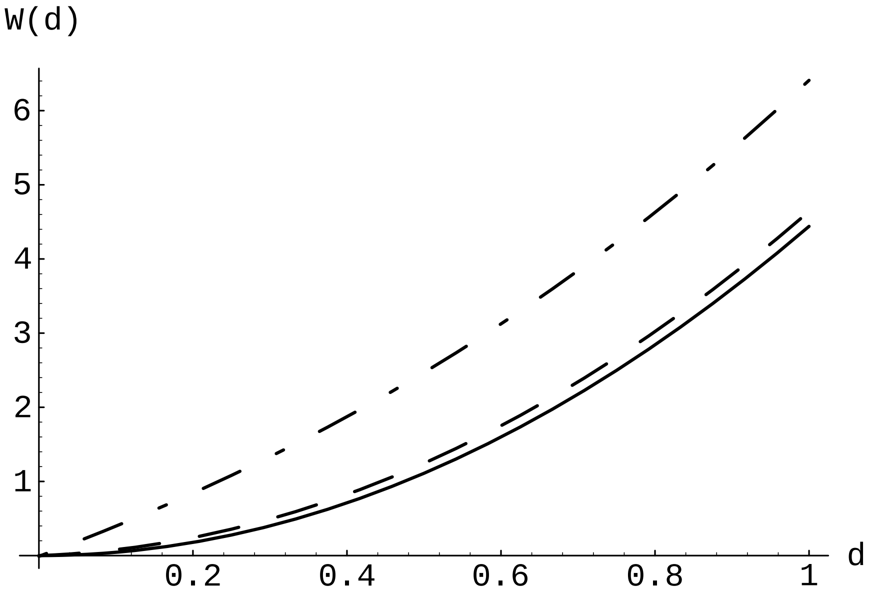

Figure 5-20: Variation with failure rate

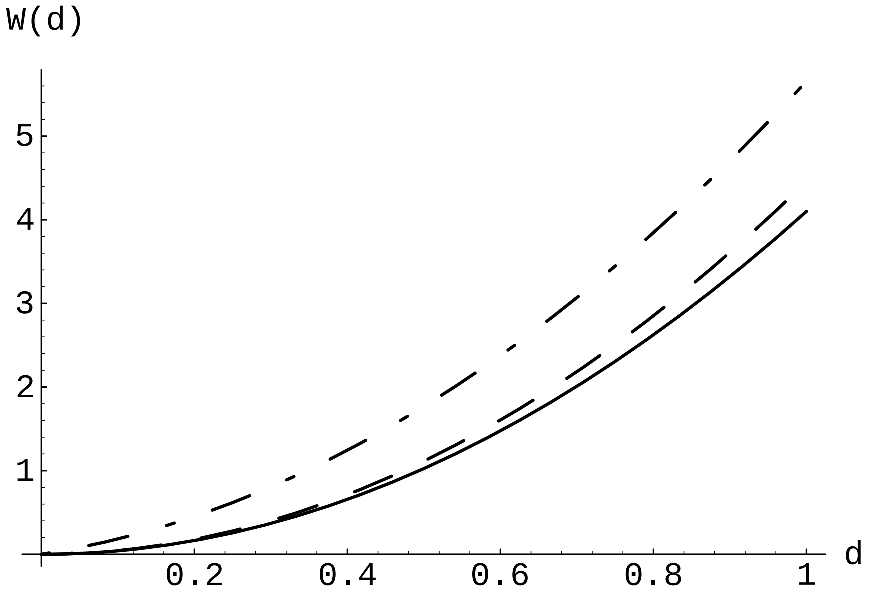

Figure 5-21: Comparison with RD and AD systems near hedging point

very infrequent and the behavior of the surplus approaches that of the RD system.

For any value of $p > 0$, the differential cost of the stochastic system is greater than that of the RD system. This can be explained by interpreting the differential cost of the OODS policy at some surplus state as the cost incurred by the OODS policy to guide the system from that state to a desired state (the hedging point). If the differential cost for the stochastic system were strictly lower, then that would imply the existence of some trajectory in the RD system that is different from the OODS trajectory and has a lower cost. This would contradict the fact that the OODS policy is optimal for the RD system. Hence, the differential cost for the RD system is a lower bound on the differential cost of the stochastic system.

Comparison with RD and AD systems We now examine the relation between the differential costs of the stochastic system and the AD and RD systems of Section 5.2. This comparison is illustrated in Figures 5-21 and 5-22. In both these figures, the horizontal axis (d) represents the x_1 coordinates of points that lie on the $2a - 2b$ boundary and the vertical axis gives the differential cost function at such points. A dashed line is used for the stochastic system, a dot-dash line for the AD system and a solid line for the RD system. Figure 5-21 compares the three differential costs close to the origin (for small values of d). Figure 5-22 shows the comparison over a larger range of distances from the origin (large values of d).

The value of the differential cost function of the stochastic system is between those

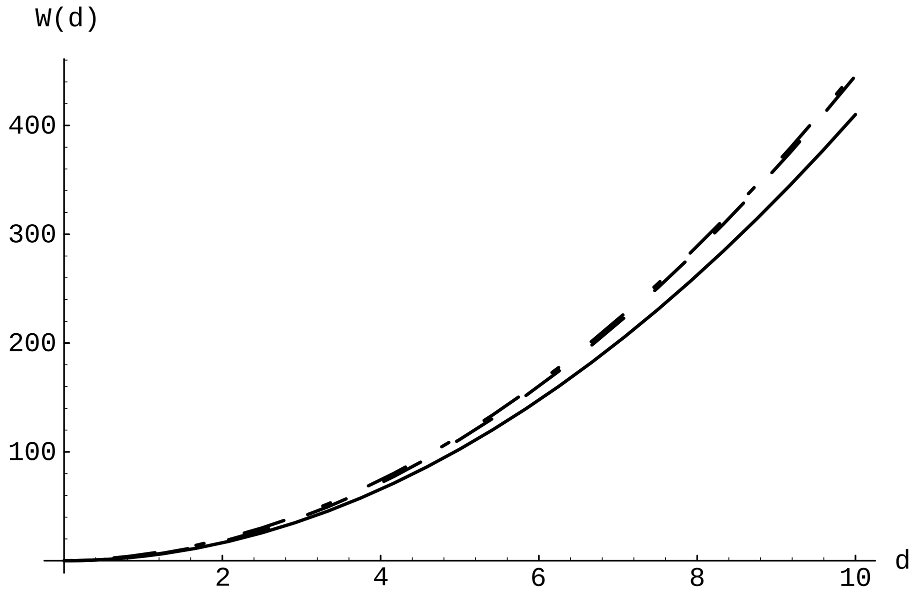

Figure 5-22: Comparison with RD and AD systems away from hedging point

of the RD and the AD systems. For point close to the origin, the differential cost of the stochastic system is close to that of the RD system. An intuitive reason for this behavior is that the probability of a machine failure or repair before the trajectory reaches the hedging point is small for points close to the origin. For points far from the origin, the differential cost of the stochastic system is close to that of the AD system. This can be explained as follows: There are many machine failures and repairs before a trajectory originating from a distant point with reaches the origin. The average surplus behavior over many failure-repair cycles approaches that of the AD system.

Variation of second derivative in Region 2 We now explore the relationship between the stochastic system and the AD and RD systems in the interior of Region 2. For the AD system, the $2a - 2b$ boundary is given by

$$\frac{x_2}{\frac{r}{r+p}\mu_2 - d_2} = -\frac{x_1}{d_1} \tag{5.148}$$

From Section 5.2.2, we know that the second derivative of the differential cost function in the AD system is discontinuous across this boundary but is continuous in Regions $2a$ and $2b$ on either side of it.

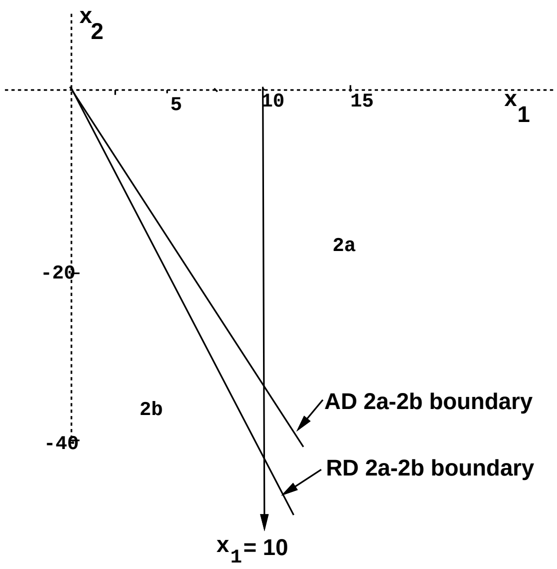

Figure 5-23: $2a - 2b$ boundaries in AD and RD systems

For the RD system the $2a - 2b$ boundary is given by

$$\frac{x_2}{\mu_2 - d_2} = -\frac{x_1}{d_1} \tag{5.149}$$

From Section 5.2.1 we know that the second derivative of the differential cost for the RD system is discontinuous across this boundary.

The $2a - 2b$ boundary for the stochastic system is same as that for the RD system and is given by (5.149). The $2a - 2b$ boundaries for the AD and RD (which is the same as the stochastic) systems are shown in Figure 5-23.

Figure 5-24 shows the variation of of $\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_2^2}$ in the interior of Region 2 for the stochastic and RD systems. The value of this second derivative is plotted on the vertical axis. The horizontal axis represents the distance along the negative x_2 direction from the $0b-2a$ boundary for points whose x_1 coordinate is fixed at $x_1 = 10$. This direction is illustrated in Figure 5-23. The solid line represents the stochastic system and the dashed line represents the RD system. For both these systems, the $2a-2b$ boundary is along $x_2 = -4.0x_1$. For $x_1 = 10$, the value of x_2 for a point on the boundary is -40. From the figure, it can be observed that $\frac{\partial^2 W(\boldsymbol{x},1)}{\partial x_2^2}$ for both systems is continuous in Region $2a$ and $2b$ and is discontinuous across the $2a - 2b$ boundary. The value of this second derivative in both systems is equal for points whose x_2 values are less than 40, that is, for points in Region $2b$ of both systems.

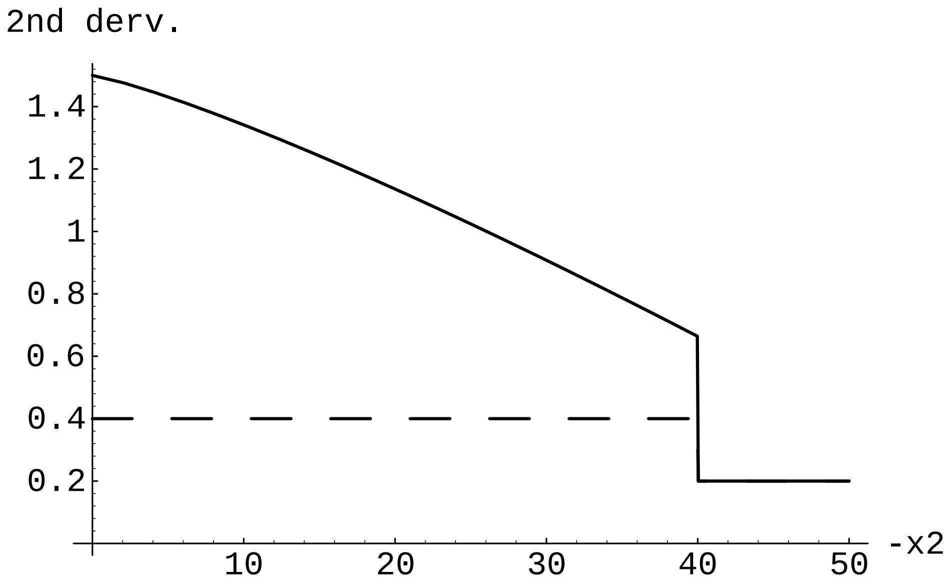

Figure 5-24: Variation of second derivative in Region 2

Based on the relation between the stochastic and the AD systems along the $2a-2b$ boundary illustrated in Figure 5-22, one would expect the two systems to exhibit similar behavior at points (in the respective Regions $2a$) that are far away from the origin. This would suggest a similar behavior of differential cost in both systems for such points. Since there is a discontinuity in the second derivative of W of the AD system at its $2a - 2b$ boundary, the above relation would imply a near discontinuity in $\frac{\partial^2 W}{\partial x^2}$ of the stochastic system along the line

$$\frac{x_2}{\frac{r}{r+p}\mu_2 - d_2} = -\frac{x_1}{d_1}$$

which is in the interior of its Region $2a$. Also, by the same logic, the discontinuity in $\frac{\partial^2 W}{\partial x^2}$ at the $2a - 2b$ boundary of the stochastic system should also vanish for points far away from the origin.

We are not able to confirm this hypothesis in our experiments. The second derivative in Figure 5-24 does not show any discontinuity in the interior of its Region $2a$. This discrepancy between expected and observed results could either be due to one or both of the following reasons: a) For points far away from the origin, the numerical computation of the differential cost and its derivatives may not be accurate; or b) The logic used in explaining the relationship between the stochastic system and the AD system in Region $2b$ may not hold in Region $2a$. A trajectory in the AD system that starts in its Region $2b$ always hits the $2a - 2b$ boundary. The same is true with a trajectory in the stochastic system that starts in its Region $2b$. But the qualitative behavior of two trajectories, one in the AD system that starts in its Region $2a$, and the other in the stochastic system that starts in its Region $2a$ are different. The former reaches its $0b - 2a$ boundary before reaching the hedging point whereas the latter may or may not reach its $0b - 2a$ boundary. A prolonged machine failure, for instance, would cause the trajectory to enter Region $2b$ and subsequently the trajectory would never return to Region $2a$. This qualitative difference in surplus behavior is a differentiating factor, in the relationship between the stochastic and AD systems, between Regions $2a$ and $2b$. Further research is necessary to analyze these reasons and study the continuity properties of the second derivative in Region $2a$.

In this section, we have presented some numerical results based on the methods developed in the previous section. In the next section, we derive optimality conditions for the OODS policy using expressions for the differential cost function.

5.11 Optimality conditions

In this section, we study conditions for the OODS policy to be optimal. Consider the first order form of the HJB equations given by (2.7) and (2.8). As seen in Chapter 2, the HJB conditions lead to the linear program (2.9). Also from Chapter 2, we know that the HJB conditions are necessary conditions if the optimality criterion is as given in (2.1). Here, we consider subregions for which the analytical expression for $W(\boldsymbol{x}, 1)$ is known and use the linear program (2.9) to derive necessary conditions on the parameters for which the OODS policy is optimum. Throughout this section, we assume that the system can satisfy demand in the long-run. That is, the demand rates of the two part types satisfy

$$\frac{d_1}{\mu_1} + \frac{d_2}{\mu_2} \leq \frac{r}{r+p}. \tag{5.150}$$

The effect of this condition on stability of hedging point policies is examined in detail in Section 6.2. It can be verified that if (5.150) is satisfied, then the coefficients of all quadratic terms in $W(\boldsymbol{x}, 1)$ (from (5.40), 5.59) and (5.73)) in Subregions $1c$, $2b$, $1b$, $0a$ and $0b$ are all non-negative.

Subregion 1c This subregion is defined by

$$x_1 \le 0, \; x_2 \le 0$$

The control vector given by the OODS policy in Region $1c$ is $(\mu_1, 0)$ for $\alpha = 1$. This is an optimal solution to the linear program (2.9) in Region $1c$ if and only if the differential cost function in Region $1c$ is such that

$$\frac{\partial W(\boldsymbol{x}, 1)}{\partial x_1} \le 0, \tag{5.151}$$

and

$$\mu_1 \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_1} \le \mu_2 \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_2}. \tag{5.152}$$

Substituting for $W(\boldsymbol{x}, 1)$ from (5.38) in (5.151) leads to

$$a_{11}^{1c} x_1 + a_{12}^{1c} x_2 \le 0$$

This condition is always satisfied since a_{11}^{1c} and a_{12}^{1c} are both non-negative and in Region $1c$, x_1 and x_2 are both non-positive.

Substituting for $W(\boldsymbol{x}, 1)$ from (5.38) in (5.152) yields:

$$\mu_1(a_{11}^{1c} x_1 + a_1 2^{1c} x_2) \le \mu_2(a_{12}^{1c} x_1 + a_{22}^{1c} x_2)$$

Multiplying both sides by $\frac{r}{r+p}$,

$$\bar{\mu}_1(a_{11}^{1c} x_1 + a_1 2^{1c} x_2) \le \bar{\mu}_2(a_{12}^{1c} x_1 + a_{22}^{1c} x_2)$$

which becomes

$$(\mu_1 a_{11}^{1c} - \mu_2 a_{12}^{1c} x_1) x_1 \le (\mu_2 a_{22}^{1c} - \mu_1 a_{12}^{1c}) x_2$$

Plugging in expressions for a_{11}^{1c}, a_{12}^{1c} and a_{22}^{1c} from (5.40), it can be verified that the coefficient of x_2 reduces to 0 and the coefficient of x_1 reduces to $g_1^- \bar{\mu}_1 - g_2^- \bar{\mu}_2$ Thus (5.152) simplifies to

$$(g_1^- \bar{\mu}_1 - g_2^- \bar{\mu}_2) x_1 \le 0$$

which is always satisfied if

$$g_1^- \bar{\mu}_1 \le g_2^- \bar{\mu}_2$$

or

$$g_1^- \mu_1 \leq g_2^- \mu_2 \tag{5.153}$$

This is the '$c\mu$' rule found in many scheduling problems (Wein, 1992) and is a necessary and sufficient condition for optimality of the OODS policy in the deterministic version of the two-part-type system considered here (Connolly, 1992). Applying this result to the AD system implies that (5.153) is a necessary and sufficient condition for optimality of the policy in Table 5.4.

For the stochastic system considered here, (5.153) is a necessary condition. In Chapter 6, we further prove that as long as this condition is satisfied in the stochastic system, the optimal production vector for $x_1 < 0$ is given by $(\mu_1, 0)$.

Region $2b$ The production rate vector given by the OODS policy in Region $2b$, for $\alpha = 1$, is given by $(0, \mu_2)$. This is an optimum solution to the linear program if and only if the differential cost in Region $2b$ satisfies

$$\frac{\partial W(\boldsymbol{x}, 1)}{\partial x_2} \leq 0 \tag{5.154}$$

and

$$\mu_2 \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_2} \leq \mu_1 \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_1}. \tag{5.155}$$

Substituting from (5.43), condition (5.154) becomes

$$k_2^{2b} e^{-\frac{r+p}{d_1} x_1} + b_2^{2b} + a_{12}^{2b} x_1 + a_{22}^{2b} x_2 \leq 0 \tag{5.156}$$

where coefficients are given by (5.59). Appendix C describes a method to express (5.156) as a single condition, (C.15) of the parameters.

Condition (5.154) can be expressed as

$$\begin{aligned} &\mu_2 \left(k_2^{2b} e^{-\frac{r+p}{d_1} x_1} + b_2^{2b} + a_{12}^{2b} x_1 + a_{22}^{2b} x_2 \right) \\ &\leq \mu_1 \left(-\frac{r+p}{d_1} (k_1^{2b} x_1 + k_2^{2b} x_2 + k_0^{2b}) e^{-\frac{r+p}{d_1} x_1} + k_1^{2b} e^{-\frac{r+p}{d_1} x_1} \right. \\ &\left. + a_{11}^{2b} x_1 + a_{12}^{2b} x_2 + b_1^{2b} \right) \end{aligned} \tag{5.157}$$

which can be rearranged to give

$$\frac{r+p}{d_1}(\mu_1 k_1^{2b} x_1 + \mu_1 k_2^{2b} x_2 + \mu_1 k_0^{2b}) e^{-\frac{r+p}{d_1} x_1} + (\mu_2 k_2^{2b} - \mu_1 k_1^{2b}) e^{-\frac{r+p}{d_1} x_1}$$
$$+(\mu_2 a_{12}^{2b} - \mu_1 a_{11}^{2b}) x_1 + (\mu_2 a_{22}^{2b} - \mu_1 a_{12}^{2b}) x_2 - \mu_1 b_1^{2b} + \mu_2 b_2^{2b} \quad \leq 0. \tag{5.158}$$

The values of a_{11}^{2b}, a_{12}^{2b}, a_{22}^{2b}, k_1^{2b}, k_2^{2b}, k_0^{2b}, b_1^{2b}, b_2^{2b} are all known in terms of the system parameters from Section 5.7. Section C.1 describes a method to express (5.158) as a set of conditions on the parameters of the system.

Region $1b$ The production rate under the OODS policy in Region $1b$ is given by $(\mu_1, 0)$ for $\alpha = 1$. This is an optimal solution to the linear program (2.9) if and only if the differential cost function is such that

$$\frac{\partial W(\boldsymbol{x}, 1)}{\partial x_1} \leq 0 \tag{5.159}$$

and

$$\mu_1 \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_1} \leq \mu_2 \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_2}. \tag{5.160}$$

Substituting from (5.59), condition (5.159) becomes

$$a_{11}^{1b} x_1 + a_{12}^{1b} x_2 \leq 0 \tag{5.161}$$

In Region $1b$,

$$x_2 \leq -\frac{-d_2}{\mu_1 - d_1} x_1$$

Mutiplying both sides by a_{12}^{1b} which is non-negative gives

$$a_{12} x_2 \leq -\frac{-d_2}{\bar{\mu}_1 - d_1} a_{12} x_1$$

Therefore (5.161) is satisfied if

$$\left(a_{11}^{1b} - \frac{-d_2}{\bar{\mu}_1 - d_1} \right) x_1 \leq 0. \tag{5.162}$$

Using (5.59) to substitute for a_{11}^{1b} and a_{12}^{1b} , it can be verified that

$$a_{11}^{1b} - \frac{-d_2}{\bar{\mu}_1 - d_1} = \frac{g_1^-}{\bar{\mu}_1 - d_1} \geq 0.$$

In Region $1b$, $x_1 \leq 0$. Therefore,

$$\left(a_{11}^{1b} - \frac{-d_2}{\bar{\mu}_1 - d_1} \right) x_1 \leq 0$$

Thus from (5.161) and (5.162) we find that (5.159) is always satisfied.

Consider the condition given in (5.160). Using the expression for the differential cost in equation (5.72), this condition becomes

$$-\mu_2 \frac{(r+p)}{d_2} k_0^{1b} e^{-\frac{(r+p)}{d_2} x_2} + \mu_2 a_{12}^{1b} x_1 + \mu_2 a_{22}^{1b} x_2 + \mu_2 b_2^{1b} - \mu_1 a_{11}^{1b} x_1 - \mu_1 a_{12}^{1b} x_2 \leq 0 \quad (5.163)$$

where a_{11}^{1b}, a_{12}^{1b} and a_{11}^{1b} are given by (5.73). Section C.2 describes a method to express condition (5.163) as a condition on the parameters of the system.

In Region $0a$ and $0b$, the OODS policy has a production rate $(0, 0)$. This is an optimum solution to (2.9) if and only if the differential cost function satisfies the following conditions.

$$\begin{aligned} \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_1} &\geq 0, \\ \frac{\partial W(\boldsymbol{x}, 1)}{\partial x_2} &\geq 0. \end{aligned}$$

Since the value of $W(\boldsymbol{x}, 1)$ in Regions $1a$, $2a$, $0a$ and $0b$ are all obtained numerically, conditions in these regions do not lead to simple expressions involving the system parameters.

In this section, we have considered necessary conditions for the OODS policy to be optimal for the objective function in (2.1). In Chapter 6, we derive conditions that any optimal hedging point policy to the two-part-type system must satisfy.

5.12 Summary

In this chapter, we have described the OODS policy with its regions and subregions and analyzed the surplus behavior in each region. The HJB equations in each subregion and along region boundaries was considered separately and the boundary conditions were identified. The differential cost along the $0b-2a$ boundary was determined using results from Chapter 4. This solution was then propagated in a particular sequence over the entire domain using the appropriate boundary conditions. A solution that is analytical in some regions and numerical in others was obtained. Finally, the analytical expressions were used to derive conditions on the system parameters necessary for this policy to be optimal.

In the next chapter, we derive conditions on the optimal policy for any set of parameter values and use those conditions to partially characterize the optimal policy.

Chapter 6

Structural properties of an optimal hedging point policy

6.1 Introduction

In this chapter, we assume that an optimal control policy to the production control problem described in Chapter 5 is a hedging point policy. We derive results that help characterize the hedging point policy in the non-transient region of the surplus space. In Section 6.2, we derive conditions under which any hedging point policy leads to a stable behavior of the surplus. In Section 6.3, we compare costs of trajectories generated by two non-optimal policies for the single-part-type system. This comparison is done over intervals in which the trajectories satisfy certain conditions. This cost comparison is used to show that one of the policies incurs a lower expected cost than the other during the interval. In Section 6.4, we use this result to compare surplus behaviors in hedging point policies with different $1-2$ boundaries in the two-part-type system. A partial characterization of the $1-2$ boundary for an optimal hedging point policy is obtained. Section 6.5 describes the complete characterization of an optimal policy for a two-part type system with special parameter values. Finally, Section 6.6 describes a conjecture on the complete characterization of the optimal policy for general parameter values.

6.2 Stability of hedging point policies

The structure of the optimal policy for a two-part-type is described in Section 5.1. We refer to this policy as a hedging point policy for the two-part-type system. A characterization of the policy requires the specification of the hedging point $\boldsymbol{z}$ and the three region boundaries.

Without loss of generality we assume for the remainder of this section that the system always starts with the surplus at the hedging point. By definition of the hedging point policy, the surplus remains at the hedging point until the next machine failure. When the machine is failed, the surplus moves in the $-\boldsymbol{d}$ direction until the machine next gets repaired. Depending on the shape of the $1-2$ boundary (see Figure 5-4), the surplus is in either Region 1 or 2 when the machine next gets repaired. If there is no further machine failure, the surplus is guided to the $1-2$ boundary and stays on it until the surplus reaches the hedging point. A machine failure at some instant before the surplus reaches the hedging point would cause the surplus to again decrease at a rate $\boldsymbol{d}$ until the next machine repair. In either case, it can be seen if the surplus starts at the hedging point, the surplus behavior is completely determined by the hedging point and the $1-2$ boundary. Since the machine is fully utilized in Regions 1 and 2, and the control on the boundary is a convex conbination of the controls in these regions, therefore the machine is fully utilized on the $1-2$ boundary until the surplus reaches the hedging point.

Following the definition in Chapter 2, we call a hedging point policy is stable if the behavior of the surplus is such that

$$\lim_{t\to\infty} \frac{E\{x_i(t)^2\}}{t} = 0 \quad i = 1, 2.$$

In this section, we derive conditions under which a hedging point policy with a finite hedging point is stable. Since we assume that the initial surplus level is at the hedging point, the hedging point and the $1-2$ boundary determine the surplus behavior. The conditions for stability are derived in two steps. First, we define a hedging point policy with some hedging point and a specific $1-2$ boundary (Policy a) and derive conditions under which this policy is stable. Then we use the fact that for all hedging point policies that follow the description in Section 5.1, the machine is utilized at full capacity at all times except when the surplus is at the hedging point. This property is used to show that a weighted sum of the surpluses of the two part types is the same under any such policy at all times. This enables us to derive the stability conditions

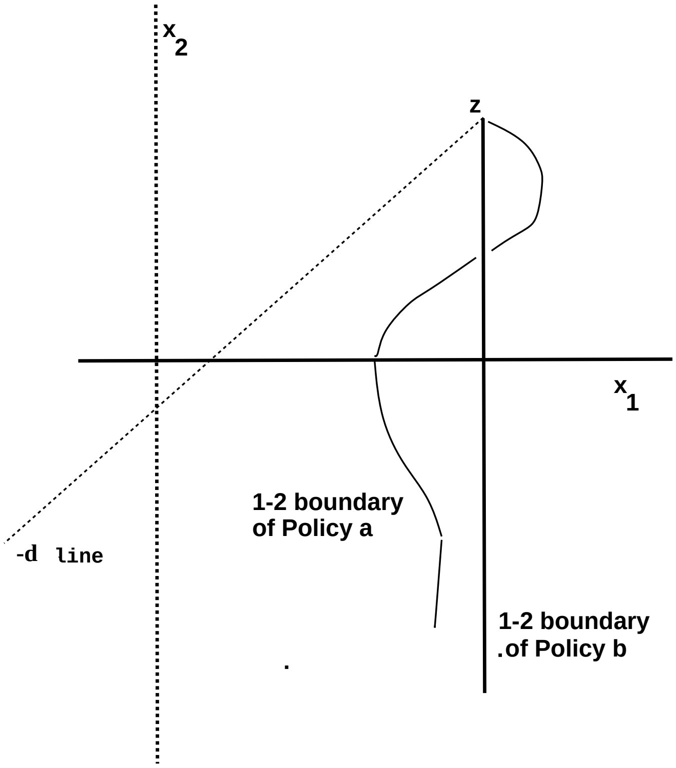

Figure 6-1: Stability in a two-part-type system

for a hedging point policy with an arbitrary $1-2$ policy based on those for Policy a.

Let Policy a be a hedging point policy with a finite hedging point $\boldsymbol{z} = (z_1, z_2)$ and for which the $1-2$ boundary is defined by the vertical half line, $x_1 = z_1, x_2 \leq z_2$ as shown in Figure 6-1. We first derive stability conditions for Policy a. Let C^a be the random variable which denote the times between successive entries of the process $(\boldsymbol{x}, \alpha)$ into state $(\boldsymbol{z}, 1)$ and C_1^a denote the amount of time it takes for the surplus to reach a point on the boundary from the instant it leaves the boundary.

We first consider the stability of Part Type 1 under Policty a. Since the $1-2$ boundary is a straight line, the production rate of Part Type 1 under Policy a is given by

$$u_1 = \begin{cases} \mu_1 & x_1 < z_1 \\ d_1 & x_1 = z_1 \\ 0 & x_1 > z_1 \end{cases}$$

This is a hedging point policy for Part Type 1. The control for Part Type 1 is independent of the value of the Part Type 2 surplus, x_2. The behavior of Part Type 1 is therefore not influenced by Part Type 2 and Part Type 1 behaves like the single-part-type system in Chapter 4 operated under a hedging point policy with hedging point z_1, maximum production rate μ_1, demand rate d_1, exponential machine failure times with rate $\frac{1}{p}$ and exponential machine repair times with mean $\frac{1}{r}$. A sample trajectory of Part Type 1 is as shown in Figure 4-1 in Chapter 4. Using Theorem 2 and Corollary 3 for Part Type 1 leads to the following lemma.

Lemma 4 *Policy a leads to a stable behavior for Part Type 1 if*

$$\frac{d_1}{\mu_1} < \frac{r}{r+p}$$

In that case,

$$E\{C_1^a\} = \frac{\mu_1}{r(\mu_1 - d_1) - pd_1}$$

Now consider the stability of Part Type 2. Under Policy a, the machine does not produce Part Type 2 as long as $x_1 < z_1$. When $x_1 = z_1$, Part Type 1 is maintained at z_1 and the remaining machine capacity is allocated to Part Type 2. Thus when the machine is up and the surplus $\boldsymbol{x}^a$ is on the $1-2$ boundary, Part Type 2 is produced at a rate u_2 where

$$u_2 = \mu_2(1 - \frac{d_1}{\mu_1})$$

If the machine fails when $\boldsymbol{x}^a$ is on the $1-2$ boundary, Part Type 2 'sees' a machine failure until the trajectory reaches the boundary again. Part Type 2 therefore behaves like a single-part-type system in Chapter 4 with an exponential failure rate p, general repair times of length C_1^a, demand rate d_2 and a maximum production rate u_2.

The following lemma gives conditions on the system parameters for which Policy a is stable.

Lemma 5 *Policy a is stable if*

$$\sum_{i=1}^{2} \frac{d_i}{\mu_i} < \frac{r}{r+p}$$

Proof: Lemma 4 gives the condition under which Policy a is stable for Part Type 1.

Using Theorem 2, Policy a leads to stable behavior of Part Type 2 if

$$\frac{d_2}{u_2} < \frac{1}{1 + pE\{C_1^a\}}$$

Substituting for $E\{C_1^a\}$ from Lemma 4, we get,

$$\frac{d_2}{u_2} < \frac{r(\mu_1 - d_1) - pd_1}{(p+r)(\mu_1 - d_1)}$$

Substituting for u_2, we get

$$\frac{d_1}{\mu_1} + \frac{d_2}{\mu_2} < \frac{r}{r+p} \tag{6.1}$$

The condition for stability of Part Type 2 is stronger than that for Part Type 1. Thus Policy a is stable for both part types if (6.1) is satisfied.

Condition (6.1) can also be written in terms of the processing times of the two part types as

$$\tau_1\mu_1 + \tau_2\mu_2 < \frac{r}{r+p}$$

If this condition is satisfied, Theorem (2) applied to Part Type 2 gives

$$E\{(C^a)^2\} < \infty. \tag{6.2}$$

Theorem 7 *A hedging point policy with a finite hedging point is stable for a manufacturing system if the demand satisfies:*

$$\sum_{i=1}^{2} \frac{d_i}{\mu_i} < \frac{r}{r+p} \tag{6.3}$$

Proof: Let Policy b denote a hedging point policy with a finite hedging point $\boldsymbol{z}$ and an arbitrary $1-2$ boundary. Let C^b be the random variable which denote the times between successive entries of the process $(\boldsymbol{x}, \alpha)$ into state $(\boldsymbol{z}, 1)$. Consider two trajectories that start at $\boldsymbol{z}$ and evolve according to Policies a and b respectively. The two trajectories remain at the hedging point until a machine failure causes both trajectories to move away from the hedging point. At any time between the instant when the machine fails and the instant when the surplus of each trajectory next reaches $\boldsymbol{z}$, each trajectory is in its own Region 1 or Region 2 or on its $1-2$ boundary. In each of these situations, the machine works at full capacity under either policy. Thus, when the trajectories are not at their respective hedging points, we have:

$$\sum_{i=1}^{2} \tau_i u_i^a(t) = \sum_{i=1}^{2} \tau_i u_i^b(t) \quad (= 1)$$

Subtracting the constant term $\sum_{i=1,2} \tau_i d_i(t)$ from both sides, we get

$$\sum_{i=1,2} \tau_i \dot{x}_i^a(t) = \sum_{i=1,2} \tau_i \dot{x}_i^b(t)$$

Integrating both sides and using $\boldsymbol{x}^a(0) = \boldsymbol{x}^b(0) = \boldsymbol{z}$, we get

$$\sum_{i=1,2} \tau_i x_i^a(t) = \sum_{i=1,2} \tau_i x_i^b(t) \tag{6.4}$$

Equation (6.33) implies that both trajectories reach the hedging point again at exactly the same instant. After that instant both trajectories are again maintained at $\boldsymbol{z}$ until the next machine failure. Thus (6.4) holds at all times. It follows that

$$\lim_{t\to\infty} E\{x_i^a(t)\}^2 \to \infty \Longleftrightarrow \lim_{t\to\infty} E\{x_i^b(t)\}^2 \to \infty$$

Using the definition of stability, Policy b is stable if Policy a is stable. The conditions for stability of Policy a in Lemma 5 are also the conditions for stability of Policy b and the theorem is proved.

A demand that satisfies condition (6.3) is called a *feasible* demand. For a stable system, the optimal control for $\alpha = 1$ guides the surplus towards the hedging point. Equation (6.4) implies that

$$C^a = C^b$$

This along with (6.2) imply that the hedging point is positive recurrent for a stable system. Therefore the limit for the differential cost $W(\boldsymbol{x}, \alpha)$ defined in (2.6) exists for a feasible demand. In the remaining sections we assume a feasible demand.

6.3 Comparison of trajectories in a single-part-type system

In Section 6.4, we compare trajectories generated by two specific policies in a two-part-type system. At any time, these trajectories are either identical or satisfy some other relationship with one another. It is shown that the difference in cost between the policies over intervals when the trajectories are not identical can be obtained from an analysis of only Part Type 1 behavior under the two policies.

In this section, we compare the expected costs incurred by trajectories generated by two different policies in a single-part-type system. The costs are compared over specific time intervals. During these intervals, the two trajectories are related to each other in the same way as two Part Type 1 trajectories generated by the policies in Section 6.4, in a two-part-type system. From Chapter 3, we know that a hedging point policy is optimal for a single-part-type system. Let the optimal hedging point be denoted by Z. Our goal in this section is to define two policies neither of which is restrcited to be a hedging point policy and compare their costs.

The following notation is used for comparing trajectories.

- X^a and X^b denote trajectories generated by Policies a and b. $x^a(t)$ and $x^b(t)$ denotes the surplus of the respective trajectories at time t.
- The expected cost of policy a in the interval $[s, f]$ is represented as:

$$J^a(s,f) = E\{\int_s^f g(x^a(t))dt | x^a(s) = x, \alpha(s) = \alpha\}$$

  (In this section, we always compare trajectories with the same initial conditions. Thus for notational simplicity, the dependence of this cost on the initial conditions $x^a(s), \alpha(s)$ is neglected.)

- The difference in the expected costs of policies a and b in the interval $[s, f]$ is represented by

$$\Delta J^{ab}(s,f) = J^a(s,f) - J^b(s,f)$$

The comparison is done in several steps. In Lemma 6, we prove a basic result for two trajectories over a specified interval. One of the two policies is restricted to be a hedging point policy over the interval. This restriction also holds in Lemma 7, but the cost comparison is done for a different set of initial and final conditions. In Lemma 8, we relax this restriction and generalize the result of Lemma 7 to cases where two trajectories satisfy a certain condition, called **Condition c1**. In Section 6.4, it is shown that this condition is satisfied by the trajectories for Part Type 1 under specific policies in a two-part-type system. Finally, Theorem 8 extends the result of Lemma 8 to include all possible initial and final conditions satisfied by the Part Type 1 trajectories in Section 6.4.

We first prove Lemma 6 which compares the costs of two trajectories, one of which, X^a is generated by the optimal hedging point policy and the other, X^b, by some other feasible policy. This comparison is done over a time interval such that

Trajectory X^a is nearer to the hedging point than X^b at the start of the interval and both Trajectories are at the hedging point at the end of the interval.

Lemma 6 *Consider an interval* $[0, S]$ *and two trajectories,* X^a *and* X^b *such that:* X^a *is generated by the hedging point policy,* X^b *is generated by some other feasible policy,*

1. $\alpha(0) = \alpha(S) = 1$,
2. $x^b(0) \leq x^a(0) \leq Z$ *or* $x^b(0) \geq x^a(0) \geq Z$,
   and
3. $x^a(S) = x^b(S) = Z$.

Then

$$\Delta J^{ab}(0, S) \leq 0 \tag{6.5}$$

Proof:
X^a is the trajectory generated by the optimal hedging point policy. Therefore from Chapter 3, there exist functions $W(x, 0)$ and $W(x, 1)$ and a value J^* such that

$$J^* = g(x) + \frac{dW(x,1)}{dx}(u^a - d) - pW(x, 1) + pW(x, 0)$$

and

$$\frac{dW(x,1)}{dx}(u^a - d) = \min_i \frac{dW(x,1)}{dx}(u^i - d)$$

for any feasible policy i and for all x.
From Condition 1 in Lemma 6, the machine is up at the beginning and end of the time interval. Therefore Dynkin's formula (A.12) over $[0, S]$ becomes

$$J^a(0, S) = J^*S + W(x^a(0), 1) - W(x^a(S), 1) \tag{6.6}$$

$$J^b(0, S) \geq J^*S + W(x^b(0), 1) - W(x^b(S), 1) \tag{6.7}$$

where J^a and J^b are as defined above From Condition 3 of Lemma 6, we have

$$W(x^b(S), 1) = W(x^a(S), 1) = W(Z, 1)$$

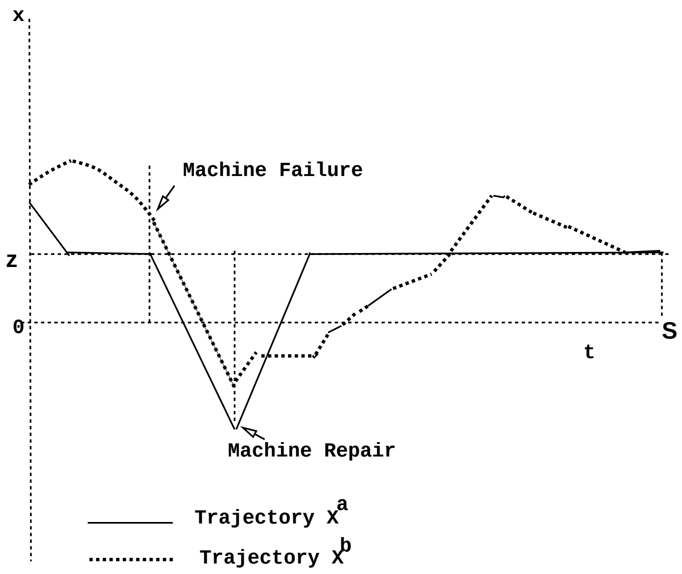

Figure 6-2: Illustration of Lemma 6

Subtracting (6.6) from (6.7),

$$J^b(0,S) - J^a(0,S) \geq W(x^b(0),1) - W(x^a(0),1) \tag{6.8}$$

Using Condition 2 in Lemma 6 and the expression for $W(x,1)$ given by (3.28)-(3.32) with time dependent failures, it can be verified that

$$W(x^b(0),1) - W(x^a(0),1) \geq 0$$

which gives

$$J^b(0,S) - J^a(0,S) \geq 0$$

This lemma is illustrated in Figure 6-2 for the case when $x^b(0) \geq x^a(0) \geq Z$.

Thus we have shown that the expected cost of X^a is a lower bound on that incurred by X^b over $[0,S]$.

For a single-part-type system operated under the hedging policy, the region $x > Z$ is transient. In other words, a trajectory that originates in this region eventually reaches Z at some instant. After that instant the trajectory remains in the region $x \leq Z$. It is therefore necessary to point out the need for the condition $x^b(0) \geq x^a(0) \geq Z$ in the lemma. The initial conditions on Trajectories X^a and X^b considered in the lemma arise from conditions on trajectories of Part Type 1 in the two-part-type system in Section 6.4. It is seen in Section 6.4 that in the context of the two-part-type system, $x^a > z$ is not a transient region and it is necessary to consider the case when x^a starts above the hedging point at time 0.

We now use the results from Lemma 1 to compare costs for two trajectories with a different set of initial and final conditions than those in Lemma 6. Again these initial and final conditions arise from conditions satisfied by Part Type 1 trajectories in the two-part-type system in Section 6.4.

Lemma 7 *Consider trajectories X^a and X^b and an interval $[0, S]$ such that: X^a is generated by the hedging point policy, X^b is generated by some other feasible policy,*

1. $\alpha(0) = \alpha(S) = 1$,
2. $x^b(0) \geq x^a(0) \geq Z$
3. $x^b(S) \geq x^a(S) = Z$

Then,

$$\Delta J^{ab}(0, S) \leq 0 \tag{6.9}$$

Proof: There are two cases to be considered.

- **Case** 1: Trajectory X^b does not go below Z in $[0, S]$
- **Case** 2: Trajectory X^b goes below Z at least once in $[0, S]$

Case 1: $x^b(t) \geq Z \quad 0 \leq t \leq S$. (See Figure 6.3 for illustration)
Let t_m be some time instant when the Trajectory X^b takes its minimum value in $[0, S]$ and let D_m be the difference between that minimum value and Z. That is,

$$t_m = \min_t \ argmin\{x^b(t)\} \quad 0 \leq t \leq S$$

and

$$D_m = x^b(t_m) - Z$$

Define a new trajectory X^c such that

$$x^c(t) = x^b(t) - D_m \text{ for } 0 \leq t \leq S$$

X^c is obtained by shifting X^b down by D_m.

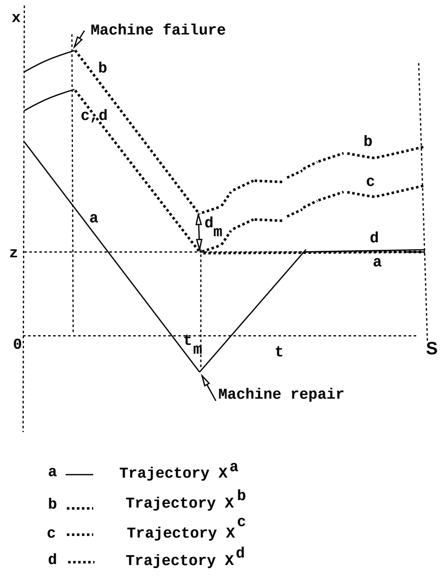

Figure 6-3: Illustration of Lemma 7: Case 1

Define a trajectory X^d such that

$$x^d(t) = \begin{cases} x^c(t) & 0 \leq t \leq t^m \\ Z & t_m \leq t \leq S \end{cases}$$

X^d is the same as X^c until time t_m and is equal to Z for the remainder of the interval. A motivation for defining X^d this way is so that Trajectories X^a and X^d have the same initial and final conditions in the interval $[0, S]$ as the two trajectories considered in Lemma 6.

The definition of the trajectories gives

$$x^b(t) \geq x^c(t) \geq x^d(t) \geq Z \quad 0 \leq t \leq S$$

Therefore

$$g(x^b(t)) \geq g(x^d(t)) \quad 0 \leq t \leq S$$

which gives,

$$J^b(0, S) - J^d(0, S) \geq 0 \tag{6.10}$$

Trajectories X^d and X^a satisfy the conditions in Lemma 6 in the interval $[0, S]$.

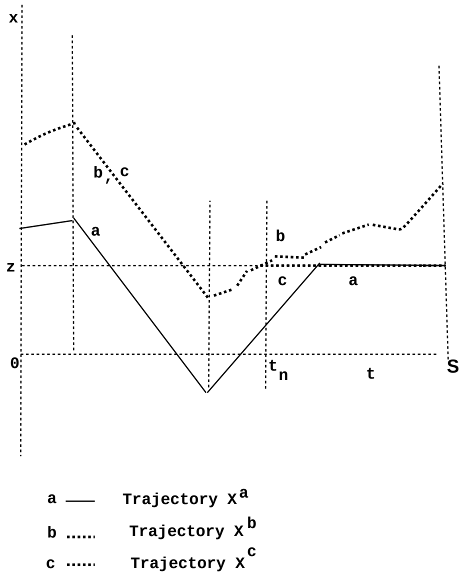

Figure 6-4: Illustration of Lemma 7: Case 2

Therefore from Lemma 6,

$$J^d(0,S) - J^a(0,S) \geq 0 \tag{6.11}$$

From (6.10) and (6.11),

$$J^b(0,S) - J^a(0,S) \geq 0$$

Case 2: $x^b(t) < Z$ for some t, $0 < t < S$. (See Figure 6.3)
Condition 3 in Lemma 7 implies that there exists at least one time instant in $(0, S)$ when X^b crosses Z from below. Let t_n be last such instant in $(0, S)$. In other words t_n is such that,

$$x^b(t) > Z \quad t_n < t \leq S$$

Define a trajectory X^c such that

$$x^c(t) = \begin{cases} x^b(t) & 0 \leq t \leq t_n \\ Z & t_n \leq t \leq S \end{cases}$$

Again, by construction

$$x^b(t) \geq x^c(t) \geq Z \quad 0 \leq t \leq S$$

Therefore

$$g(x^b(t)) \geq g(x^c(t)) \quad 0 \leq t \leq S$$

which gives,

$$J^b(0,S) - J^c(0,S) \geq 0 \tag{6.12}$$

Trajectories X^c and X^a satisfy the conditions in Lemma 6 in the interval $[0,T]$. Therefore from Lemma 6,

$$J^c(0,S) - J^a(0,S) \geq 0 \tag{6.13}$$

From (6.12) and (6.13),

$$J^b(0,S) - J^a(0,S) \geq 0$$

Thus the lemma is proved.

In Lemmas 6 and 7, one of the two trajectories considered is generated by a hedging point policy over the considered time interval. We relax this restriction in Lemma 8 and specify that the two trajectories satisfy a condition defined below:

Definition 1 *Two trajectories X^a and X^b satisfy Condition* **C1** *if they are such that,*

1. X^a is generated by the optimal hedging point policy for all t such that $x^a(t) < Z$.

2. $x^a(t) > x^b(t)$ for all t such that $x^a(t) > Z$.

It can be seen that if X^a is generated by a hedging point policy, then Condition **C1** is automatically satisfied.

Lemma 8 *Consider two trajectories X^a and X^b and an interval $[0,S]$ such that: X^a and X^b satisfy Condition* **C1** *over $[0,S]$,*

1. $\alpha(0) = \alpha(S) = 1$

2. $x^b(0) \geq x^a(0) \geq Z$

3. $x^b(S) \geq x^a(S) = Z$

Then,

$$\Delta J^{ab}(0,S) \leq 0 \tag{6.14}$$

Proof:

The initial and final conditions on X^a imply that there exits an instant τ in $[0,S]$ such that

$$\begin{aligned} x^a(t) \quad &\geq Z \qquad t \in [0,\tau) \\ &\leq Z \qquad t \in [\tau,S] \end{aligned}$$

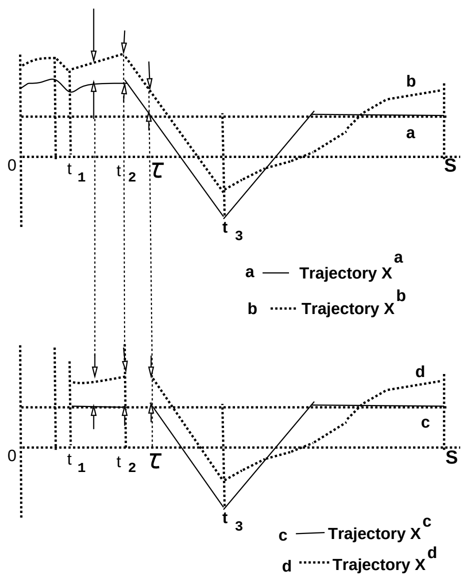

Figure 6-5: Illustration of Lemma 8

Two trajectories satisfying the conditions of the Lemma 8 are shown in Figure 6-5. If $\tau = S$, this implies that $X^a \geq Z$ over $[0, S]$. Using Condition **C1**, this implies that $x^b(t) \geq x^a(t) \geq Z$ for all $t \in [0, S]$. Therefore

$$\Delta J^{ab}(0, S) \leq 0$$

We now consider the case where $\tau < S$. Let us define the following terms:

- n: The number of machine "up" periods in $[0, \tau]$.
- t_1: The beginning of the nth up period.
- t_2: The end of the nth up period.
- t_3: The end of the nth down period.

Let

$$\delta_1 = t_2 - t_1.$$

$$\delta_2 = t_3 - \tau.$$

δ_1 is the machine up time and is therefore exponentially distributed with rate p. δ_2 is the residual life of the machine repair time and, because of the memoryless property of the exponential distribution is exponentially distributed with rate r.

By definition,

$$\Delta J^{ab}(0,S) = \Delta J^{ab}(0,t_1) + \Delta J^{ab}(t_1,t_2) + \Delta J^{ab}(t_2,\tau) + \Delta J^{ab}(\tau,S) \tag{6.15}$$

From **Condition C1** and the definition of t_1,

$$x^a(t) \le x^b(t) \quad \text{for all } 0 \le t \le t_1$$

which gives

$$g(x^a(t)) \le g(x^b(t)) \quad \text{for all } 0 \le t \le t_1$$

Therefore,

$$\Delta J^{ab}(0,t_1) \le 0 \tag{6.16}$$

Similarly,

$$\Delta J^{ab}(t_2,\tau) \le 0 \tag{6.17}$$

Let $\mathcal{I}$ be the interval:

$$\mathcal{I} = [t_1,t_2] \bigcup [\tau,S]$$

Let us construct two Trajectories X^c and X^d over the interval $\mathcal{I}$ such that

$$x^c(t) = \begin{cases} Z & t_1 \le t \le t_2 \\ x^a(t) & \tau \le t \le S \end{cases}$$

$$x^d(t) = \begin{cases} Z + x^b(t) - x^a(t) & t_1 \le t \le t_2 \\ x^b(t) & \tau \le t \le S \end{cases}$$

The trajectories are illustrated in Figure 6-5. X^c and X^d are constructed such that they satisfy the conditions of Lemma 7.

The construction of trajectories X^c and X^d in $[t_1,t_2]$ gives

$$x^b(t) - x^a(t) = x^d(t) - x^c(t) \quad t_1 \le t \le t_2$$

and

$$x^b(t), x^a(t), x^d(t), x^c(t) \geq 0 \quad t_1 \leq t \leq t_2$$

Therefore,

$$\Delta J^{ab}(t_1, t_2) = \Delta J^{cd}(t_1, t_2)$$

Also the construction of X^c and X^d in $[\tau, S]$ gives

$$\Delta J^{ab}(\tau, S) = \Delta J^{cd}(\tau, S)$$

Therefore, we have

$$\Delta J^{ab}(t_1, t_2) + \Delta J^{ab}(\tau, S) = \Delta J^{cd}(t_1, t_2) + \Delta J^{cd}(\tau, S) \tag{6.18}$$

Since x^a and x^b decrease at the same rate when the machine is down, we have,

$$x^b(t_2) - x^a(t_2) = x^b(\tau) - x^a(\tau)$$

Therefore, by construction of X^c and X^d we get

$$x^d(t_2) - x^c(t_2) = x^d(\tau) - x^c(\tau) \tag{6.19}$$

Using (6.19) and Conditions 1–3 of Lemma 8, we observe that X^c and X^d satisfy the conditions of Lemma 8 in the interval $\mathcal{I}$.
Therefore from Lemma 8,

$$\Delta J^{cd}(\mathcal{I}) \leq 0 \tag{6.20}$$

From (6.15), (6.16), 6.17), (6.18) and (6.20) we get

$$\Delta J^{ab}(0, S) \leq 0$$

Thus the lemma is proved.

The conditions in Lemma 7 are obtained from those in Lemma 8 in the special case when X^a is generated by a hedging point policy. Lemma 8 is thus a generalization of Lemma 7. Now, we prove a theorem which generalizes the result in Lemma 8 to include the set of initial conditions specified in Lemma 6. The resulting conditions on the two trajectories in this theorem are exactly the conditions satisfied by Part

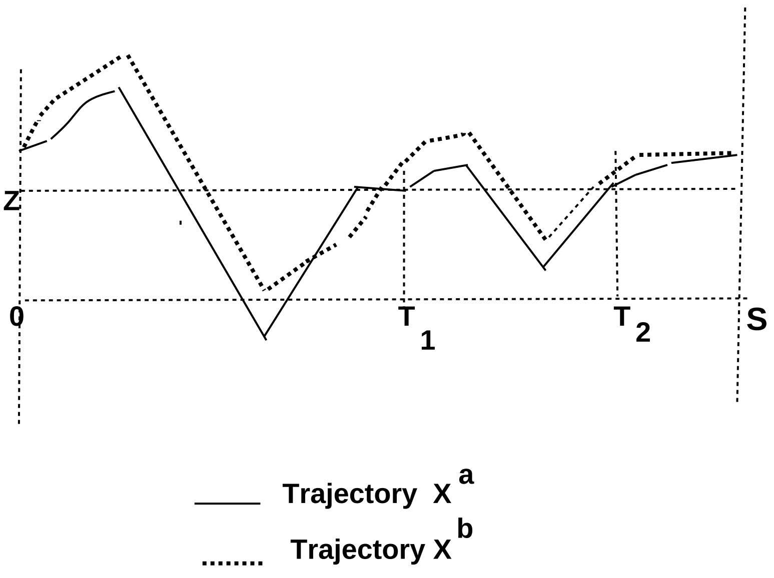

Figure 6-6: Illustration of Theorem 8, Case 1

Type 1 under the specific policies in Section 6.4 for a two-part-type system.

Theorem 8 *Let X^a and X^b be two trajectories that satisfy Condition* **C1** *.*
Consider an interval $[0, S]$ such that

1. $\alpha(0) = \alpha(S) = 1$
2. $x^a(0) = x^b(0)$
3. $x^b(S) = x^a(S) \geq Z$

Then,

$$\Delta J^{ab}(0, S) \leq 0 \tag{6.21}$$

Proof:
We have to consider two cases:
Case 1: $x^b(0) = x^a(0) \geq Z$ (See Figure 6-6)
Now, if $x^a(t) \geq Z$ for all t in $[0, S]$ then from Condition **C1**

$$x^b(t) \geq x^a(t) \geq Z \quad 0 \leq t \leq S$$

Therefore,

$$g(x^a(t)) \leq g(x^b(t)) \quad 0 \leq t \leq S$$

$$\Delta J^{ab}(0,S) \leq 0$$

The more interesting case is when Trajectory X^a goes below Z at least once in [0,S]. In this case, let n be the number of times Trajectory X^a crosses Z from below in $[0,S]$. From Condition 3 in Theorem 1, there exists at least one such crossing. Define $T_1, T_2 .... T_n$ as the time epochs when Trajectory X^a crosses Z from below.
Trajectories X^a, X^b satisfy conditions of Lemma 8 in each of the intervals:
$[0,T_1], [T_1,T_2], \ldots [T_{n-1},T_n]$.
Using Lemma 8,

$$\Delta J^{ab}(0,T_1) \leq 0$$

$$\Delta J^{ab}(T_1,T_2) \leq 0$$

$$\vdots$$

$$\Delta J^{ab}(T_{n-1},T_n) \leq 0$$

Therefore by definition,

$$\Delta J^{ab}(0,T_n) = \Delta J^{ab}(0,T_1) + \Delta J^{ab}(T_1,T_2) + \ldots + \Delta J^{ab}(T_{n-1},T_n) \leq 0$$

From the definition of T_n and from Condition **C1** , we have

$$x^b(t) \geq x^a(t) \geq Z \quad T_n \leq t \leq S$$

Therefore,

$$g(x^a(t)) \leq g(x^b(t)) \quad T_n \leq t \leq S$$

$$\Delta J^{ab}(T_n,S) \leq 0$$

Therefore

$$\Delta J^{ab}(0,S) \leq 0$$

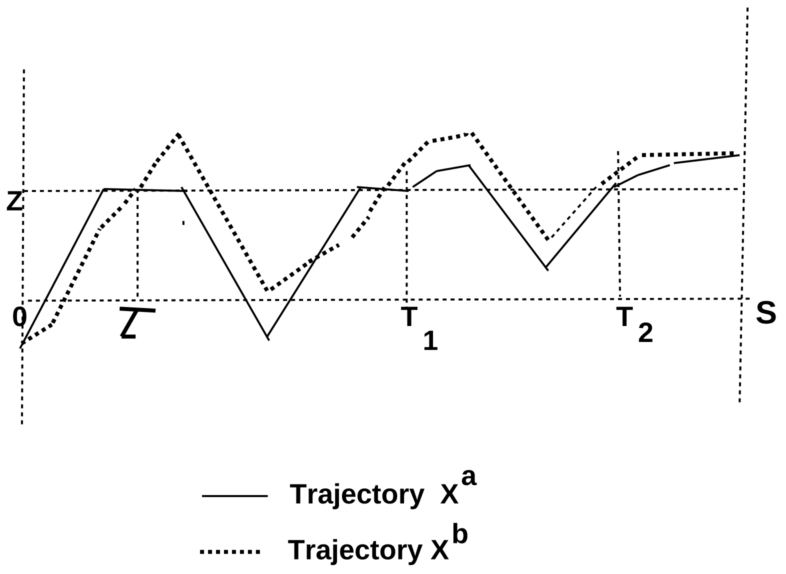

Figure 6-7: Illusration of Theorem 8, Case 2

Case 2: $x^b(0) = x^a(0) \leq Z$. (See Figure 6-7).
Let τ be the first instant in $(0, S]$ such that

$$x^a(\tau) = x^b(\tau) = Z$$

and

$$\alpha(\tau) = 1$$

The existence of τ is guaranteed by Condition **C1** and Condition 3 of Theorem 1. In the interval $[0, \tau]$, Trajectories X^a and X^b satisfy conditions of Lemma 6. Therefore,

$$\Delta J^{ab}(0, \tau) \leq 0 \tag{6.22}$$

In $[\tau, S]$, we can use the arguments from **Case 1**.
Therefore

$$\Delta J^{ab}(\tau, S) \leq 0 \tag{6.23}$$

From (6.22) and (6.23)

$$\Delta J^{ab}(0, S) \leq 0$$

Thus the theorem is proved.

We have shown that if two trajectories that originate at the sane point at time 0 are observed over an an interval $[0, S]$ over which they satisfy Condition **C1** where S is some time when the two trajectories are above Z then the cost incurred by X^a over this interval is a lower bound on that incurred by X^b.

6.4 Structural properties of the optimal hedging point policy in a two-part-type system

In this section we investigate the structure of an optimal hedging point policy in the two-part-type system described in Section 5.1. Without loss of generality, let the two part types be numbered such that

$$\frac{g_1^-}{\tau_1} \geq \frac{g_2^-}{\tau_2} \tag{6.24}$$

In Chapter 5 ,it is shown that the structure of the optimal policy for a is a hedging point policy. Here we further characterize the hedging point policy by deriving conditions that an optimal hedging point policy must satisfy. First, we prove in Theorem 9 that that an optimal hedging point policy does not idle the machine when at least one of the two part types is non-negative. Then we prove in Theorem 10 that the optimal policy always devotes all the available capacity to Part Type 1 whenever its surplus is negative. Finally, we use these results and Theorem 8 to show that the $1-2$ boundary for an optimal hedging point policy is a straight line for $x_2 < 0$.

Throughout this section, we consider the case where (6.24) is a strict inequality. That is,

$$\frac{g_1^-}{\tau_1} > \frac{g_2^-}{\tau_2} \tag{6.25}$$

If (6.24) hold as an equality, the system is is said to have *identical backlog cost*

$$\frac{g_1^-}{\tau_1} = \frac{g_2^-}{\tau_2} \tag{6.26}$$

A discussion of systems with identical backlog costs is deferred until Section 6.5.

We now prove that an optimal hedging policy never idles the machine when at least one of the part types is backlogged.

Theorem 9 *The optimal hedging point policy is such that*

$$\sum_{i=1}^{2} \tau_i u_i(\boldsymbol{x}, 1) = 1 \tag{6.27}$$

for all points $\boldsymbol{x} = (x_1, x_2)$ in its non-transient surplus space, for which at least one of the two components, x_1 or x_2, is strictly negative.

Proof: Assume that an optimal hedging point policy (Policy b) does not satisfy (6.27). In other words, there exists some region R^b in the non-transient surplus space that contains points with least one component strictly negative (say $x_1 < 0$) such that:

$$\tau_1 u_1^b(\boldsymbol{x}, 1) + \tau_2 u_2^b(\boldsymbol{x}, 1) < 1 \quad \boldsymbol{x} \in R_b$$

Let us define another policy, Policy a, as follows:

$$\boldsymbol{u}^a(t) = \left\{ \begin{array}{ll} (\alpha\mu_1(1 - \frac{u_2^a(t)}{\mu_2}), u_2^b(t)) & \text{if } x_1^a(t) < 0 \\ (\min(\alpha d_1, \alpha\mu_1(1 - \frac{u_2^a(t)}{\mu_2})\,)\,, u_2^b(t)) & \text{if } x_1^b(t) < x_1^a(t) = 0 \\ \boldsymbol{u}^b(t) & \quad \text{otherwise} \end{array} \right\}$$

The control vectors given by Policy a are dependent on time. Therefore Policy a is non-stationary.

Under Policy a, the machine is always utilized at full capacity at any time t such that $x_1^a(t) < 0$. If $x_1^a(t) = 0$, Policy a seeks to maintain X_1^a at 0 until X_1^b also reaches 0. Policy a is a feasible policy since it satisfies (2.3) and (2.4) at all times.
Consider two trajectories X^a and X^b that start at the point $\boldsymbol{x}(0)$ with machine state $\alpha(0)$ at time 0 and evolve under Policies a and b respectively.

Let $\boldsymbol{x}(0) \in R_b$ and $\alpha(0) = 1$. By definition of Region R_b and our assumption about Policy b, there is some finite time for which the machine is utilized at full capacity under Policy a, and at less than its full capacity under Policy b. The x_2 components of the two trajectories are same at all times because Policy a devotes the same capacity to Part Type 2 at all instants as Policy b. Therefore, the definition of Policy a implies that the x_1 components must satisfy

$$x_1^b(t) \le x_1^a(t)$$

for all t and for all sample realizations of failures and repairs of the machine.

Let t_1 be the first time instant such that $x_1^b(t_1) = 0$.
Until t_1, both Policy a and b devote the same capacity to Part Type 2. But unlike Policy b, Policy a devotes all the remaining capacity to Part Type 1. Therefore, for $t < t_1$ we have

$$x_2^a(t) = x_2^b(t)$$

and

$$x_1^b(t) < x_1^a(t) \leq 0$$

which gives

$$g(x^a(t)) < g(x^b(t))$$

For $t \geq t_1$

$$x_2^a(t) = x_2^b(t)$$

$$x_1^b(t) \leq x_1^a(t) \leq 0 \quad \text{if} \quad x_1^b(t) \leq 0$$

$$x_1^b(t) = x_1^a(t) > 0 \quad \text{if} \quad x_1^b(t) > 0$$

Therefore,

$$g(x^a(t)) \leq g(x^b(t))$$

Thus for any time T,

$$E\{\int_0^T g(x^a(t))dt\} < E\{\int_0^T g(x^b(t))dt\}$$

Since region R_b belongs to the non-transient space of Policy b, this implies that,

$$\lim_{T\to\infty} \frac{1}{T} E\{\int_0^T g(x^a(t))dt\} < \lim_{T\to\infty} \frac{1}{T} E\{\int_0^T g(x^b(t))dt\}. \tag{6.28}$$

Thus, we have shown that if Policy b does not satisfy (6.27), then we can construct a non-stationary policy that has a strictly lower average expected cost than Policy b. This contradicts our assumption that Policy b is an optimal policy. The theorem is therefore proved.

From Section 6.2, we know that the hedging point is a recurrent point and belongs to the non-transient region of the surplus space for a feasible demand. Also from Section 5.1, the optimal control vector is $\boldsymbol{u} = \boldsymbol{d}$ which means that the machine is utilized at less than its full capacity. Using Theorem 9 we get the following result

Corollary 5 *The hedging point is always non-negative*

We now prove that any optimal hedging point policy must devote all the available capacity at the machine to Part Type 1 whenever it has a negative surplus, irrespective of the level of the Part Type 2 surplus, x_2.

Theorem 10 *Any optimal hedging point policy satisfies*

$$u(\boldsymbol{x}, \alpha) = (\alpha\, \mu_1, 0) \tag{6.29}$$

for all points $\boldsymbol{x} = (x_1, x_2)$ in the non-transient surplus space, for which $x_1 < 0$.

Proof:
Assume an optimal hedging point policy (Policy b) does not satisfy (6.29). In other words, there exists a region R_b that contains points with $x_1 < 0$ in the non-transient surplus space of Policy b such that

$$u_1^b(\boldsymbol{x}, 1) < \mu_1 \quad \boldsymbol{x} \in R_b \tag{6.30}$$

Consider a non-stationary policy (Policy a) defined by:

$$\boldsymbol{u}^a(t) = \left\{ \begin{array}{ll} (\alpha\mu_1, 0) & \text{if } x_1^a(t) < 0 \\ (\alpha d_1, \alpha\mu_2(1 - \frac{d_1}{\mu_1})) & \text{if } x_1^a(t) = 0, x_1^b(t) < 0 \\ \boldsymbol{u}^b(t) & \text{otherwise} \end{array} \right\}$$

Consider two trajectories X^a and X^b that start from point $\boldsymbol{x}$ with machine state $\alpha(0)$ at time 0 and evolve under Policies a and b respectively. Without loss of generality, let $\boldsymbol{x} \in R_b$ and $\alpha(0) = 1$.

By definition of Policies a and b and Region R_b, there exists some non-zero interval of time during which Part Type 1 is produced at its maximum rate μ_1 under Policy a and less than its maximum rate under Policy b.

Since Policy b is a hedging point policy for a feasible demand therefore from Section 6.2, we know that the hedging point is a recurrent point. Thus Trajectory X^b must eventually reach the hedging point at some future time. From Corollary 5, it follows that there exists some time instant (t^b) such that $x_1^b(t^b) = 0$. For $t < t^b$, X^b is away from its hedging point and therefore the machine is utilized at full capacity under Policy b. Thus,

$$\sum_{i=1}^{2} \tau_i u_i^b(t) = 1, \quad 0 \le t \le t^b \tag{6.31}$$

By construction, under Policy a the machine devotes all its capacity to Part Type 1 until x_1^a reaches zero. Subsequently, the surplus of Part Type is maintained at 0 and the machine devotes all of the remaining capacity to Part Type 2. This continues until time t^b. Thus the machine is utilized at full capacity under Policy a at any time in the interval $(0, t^b)$. Therefore

$$\sum_{i=1}^{2} \tau_i u_i^a(t) = 1 \quad 0 \leq t \leq t^b \tag{6.32}$$

Using equations (6.31) and (6.32) and the initial condition, $\boldsymbol{x}^a(0) = \boldsymbol{x}^b(0)$, we get,

$$\sum_{i=1}^{2} \tau_i x_i^a(t) = \sum_{i=1}^{2} \tau_i x_i^b(t) \quad 0 \leq t \leq t^b \tag{6.33}$$

From (6.30) and the definition of Policy a, we also have:

$$x_1^b(t) < x_1^a(t) \leq 0, \quad 0 < t < t^b$$

which, along with (6.33), implies that

$$x_2^a(t) < x_2^b(t) 0 < t < t^b$$

At time t^b, by construction, we have

$$x_1^a(t^b) = x_1^b(t^b) = 0.$$

Since (6.33) is satisfied at time t^b, it follows that

$$x_2^a(t^b) = x_2^b(t^b)$$

Thus Trajectories X^a and X^b meet at time t^b. A machine failure at any instant before x_1^b reaches 0 causes the surplus of both trajectories to decrease at the demand rate until the machine gets repaired. All the relations derived above between the trajectories still hold for any time in the interval $(0, t^b]$. After t^b, the two trajectories are identical until a machine failure causes the Part Type 1 surplus to become negative and the two trajectories pass through another point at which (6.29) is not satisfied for Policy b. From that instant, the behaviors of X^a and X^b are exactly the same as those at time 0.

The above analysis is based on sample-path arguments and is therefore true for any sample realization of failures and repairs of the machine.

Let $\Delta g(t)$ denote the difference in instantaneous costs at time t between Policy a and Policy b.

$$\Delta g(t) = g(\boldsymbol{x}^a(t)) - g(\boldsymbol{x}^b(t))$$

We consider the difference in cost over the interval $(0, t^b)$. Over this time interval,

$$x_1^b(t) < x_1^a(t) \leq 0 \tag{6.34}$$

and

$$x_2^a(t) < x_2^b(t) \tag{6.35}$$

Equations (6.34) and (6.35) imply that for any $t \in (0, t^b)$, there exist three cases depending on the relative levels of x^a and x^b:

- Case 1: $x_2^a(t) \leq x_2^b(t) \leq 0$

  $$\Delta g(t) = -g_1^-(x_1^a(t) - x_1^b(t)) - g_2^-(x_2^a(t) - x_2^b(t))$$

  Using (6.33), we get

  $$\Delta g(t) = (x_1^a(t) - x_1^b(t))(-g_1^- + g_2^- \frac{\tau_1}{\tau_2}).$$

  Using (6.34) and (6.25) we get,

  $$\Delta g(t) < 0.$$

- Case 2: $x_2^a(t) \leq 0$, $x_2^b(t) \geq 0$

  $$\Delta g(t) = -g_1^-(x_1^a(t) - x_1^b(t)) - g_2^- x_2^a(t) - g_2^+ x_2^b(t)$$

The above expression can also be written as

$$\Delta g(t) = -g_1^-(x_1^a(t) - x_1^b(t)) - g_2^-(x_2^a(t) - x_2^b(t)) - (g_2^- + g_2^+)x_2^b(t)$$

Using (6.33), we get

$$\Delta g(t) = (x_1^a(t) - x_1^b(t))(-g_1^- - g_2^+\frac{\tau_1}{\tau_2}) - (g_2^- + g_2^+)x_2^b(t)$$

which yields

$$\Delta g(t) < 0.$$

- Case 3: $x_2^b(t) \geq x_2^a(t) \geq 0$

$$\Delta g(t) = -g_1^-(x_1^a(t) - x_1^b(t)) + g_2^+(x_2^a(t) - x_2^b(t))$$

Equations (6.34) and (6.35) imply that each of the two terms above is negative and thus

$$\Delta g(t) < 0.$$

So

$$g(x_1^a(t)) < g(x_1^b(t)) \quad 0 \leq t < t^b. \tag{6.36}$$

Since Region R_b belongs to the non-transient surplus space of Policy b, equation (6.36) implies that,

$$\lim_{T\to\infty} \frac{1}{T} E\{\int_0^T g(x^a(t))dt\} < \lim_{T\to\infty} \frac{1}{T} E\{\int_0^T g(x^b(t))dt\}. \tag{6.37}$$

Thus, we have shown that if Policy b does not satisfy (6.29), then we can construct a non-stationary policy that has a strictly lower average expected cost than Policy b. This contradicts the fact that Policy b is an optimal policy. Therefore Theorem 10 is proved.

This theorem states that whenever Part Type 1 has a negative surplus, the optimal policy allocates all the available capacity to Part Type 1. This is similar to assigning priorities to the part types by the $c\mu$ scheduling rule in queueing theory (Wein, 1991). This rule is also the optimal rule for the deterministic system in Connolly (1992).

In Section 5.1, Region 1 for a hedging point policy is defined as the subset of the surplus space where the machine devotes all its capacity to Part Type 1. The $1-2$

boundary represents a subset of the surplus space where the machine devotes at least some capacity to Part Type 2. Using Theorem 10 we get the following corollary.

Corollary 6 *The* $1-2$ *boundary for the optimal policy does not intersect the region* $x_1 < 0$.

We now prove that the $1-2$ boundary corresponding to any optimal hedging point policy is a straight line in the negative x_2 region. To be consistent with the previous theorems, we state this theorem as a result on the production rate vector of an optimal policy rather than in terms of the $1-2$ boundary.

Theorem 11 *Any optimal hedging point policy to (2.1) is of the form:*

$$\begin{aligned} u(\boldsymbol{x},1) = &\quad (\mu_1, 0) && x_1 < z_1^m, x_2 < 0 \\ u(\boldsymbol{x},1) = &\quad (d_1, \mu_2(1-\tfrac{d_1}{\mu_1})) && x_1 = z_1^m, x_2 < 0 \\ u(\boldsymbol{x},1) = &\quad (0, \mu_2) && x_1 > z_1^m, x_2 < 0 \end{aligned}$$

where z_1^m *is the optimal hedging point of the single-part-type system in Chapter 3. This single-part-type system has exponentially distributed time dependent machine failures with rate* p*, and exponential machine repairs with rate* r*. The maximum production rate of the single part types is* $\mu = \mu_1$*, its demand rate is* $d = d_1$ *and the instantaneous costs are*

$$\begin{aligned} g^+ &= g_1^+ + g_2^- \frac{\tau_1}{\tau_2} \\ g^- &= -g_1^- + g_2^- \frac{\tau_1}{\tau_2} \end{aligned} \tag{6.38}$$

This theorem states that the $1-2$ boundary corresponding to any optimal policy is a vertical half-line passing through $(z_1^m, 0)$, and extending in the negative x_2 direction. This means that in the negative x_2 region, the capacity of the machine is fully devoted to Part Type 1 when the value of its surplus is less than z_m^1 regardless of the value of Part Type 2 surplus. When the value of Part Type 1 surplus is greater than z_1^m, the machine is fully devoted to Part Type 2. When the surplus of Part Type 1 is at z_1^m, part of the capacity at the machine is used to maintain it at this level while the remaining capacity is allocated to Part Type 2. In other words, in the negative x_2 region, the behavior of Part Type 1 is the same as that of a single-part-type system under a hedging point policy with a hedging point z_1^m.

Proof:
Let Policy b be an optimal hedging point policy with hedging point $\boldsymbol{z} = (z_1, z_2)$, and a $1-2$ boundary that does not satisfy the conditions on the production rates specified in Theorem 11.

Corollary 5 implies that $z_2 \geq 0$ and thus the $1-2$ boundary for Policy b has at least one point with $x_2 = 0$. Let $S1$ denote the set of x_1 coordinates of points on the boundary whose x_2 coordinates are zero. Let $\hat{z}_1$ be defined as:

$$\hat{z}_1 = \inf\{S1\}$$

For simplicity of exposition, we first prove this theorem for for the case where $S1$ contains a single point. A method for extending the arguments in this proof to a case where $S1$ contains many points is given at the end of this proof.

There are two cases to be considered depending on the shape of the $1-2$ boundary of Policy b.

1. **Case 1:** $\hat{z}_1 > z_1^m$.

2. **Case 2:** $\hat{z}_1 \leq z_1^m$.

The proof strategy is now described. Each case is considered separately. In both cases, we first construct another policy, Policy a, that has a $1-2$ boundary which is a half-line passing through $(z_1^m, 0)$ and that satisfies the conditions of the theorem. Then we compare two trajectories that have the same initial conditions and evolve under Policies a and b respectively. Using the fact that both policies are hedging point policies, it is shown that the difference in cost between the trajectories can expressed in terms of the difference in a modified cost function of only the Part Type 1 surplus. We then use results from Section 6.3 to show that the expected cost incurred by Part Type 1 under Policy a is a lower bound on that incurred under Policy b.

We now explain these steps for Case 1.

Case 1: $\hat{z}_1 > z_1^m$.
Let Policy a be a hedging point policy with the same hedging point $\boldsymbol{z}$ as Policy b, and a $1-2$ boundary which is

- a vertical line through $(z_1^m, 0)$ for $x_2 < 0$,
- the x_1 axis for $x_2 = 0$ and $z_1^m \leq x_1 \leq \hat{z}_1$

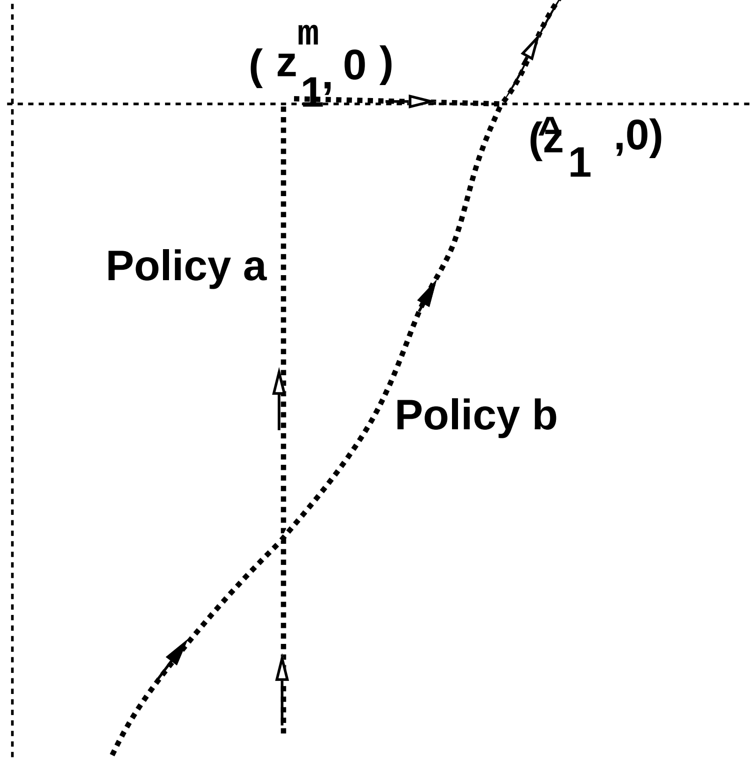

Figure 6-8: Illustration of Theorem 11, Case 1

- is the same as that of Policy a elsewhere.

The $1-2$ boundaries for Policies a and b for this case are illustrated in Figure 6-8.

Let R_b denote the portion of the non-transient surplus space under Policy b with $x_2 < 0$. Let X^a and X^b denote two trajectories that start at some point belonging to R_b at time 0 and evolve under Policies a and b respectively. Since both policies are hedging point policies, the machine is utilized at 100% of its capacity under both policies, for all $\boldsymbol{x}$ in the respective non-transient regions of the surplus space except at the hedging point. At the hedging point, $\boldsymbol{z}$, the machine is operated at the demand rate $\boldsymbol{d}$ under both policies. Using arguments similar to those in the derivation of (6.33), we get:

$$\sum_{i=1}^{2} \tau_i x_i^a(t) = \sum_{i=1}^{2} \tau_i x_i^b(t) \quad \text{for all} \quad t \tag{6.39}$$

Let s denote some time instant such that

$$\boldsymbol{x}^a(s) = \boldsymbol{x}^b(s) = \boldsymbol{x} \quad \boldsymbol{x} \in R_b$$

and such that

$$\begin{aligned} \alpha(s) &= 1 \\ u^a(\boldsymbol{x}, 1) &\neq u^b(\boldsymbol{x}, 1) \end{aligned}$$

In other words, there exists some non-zero time interval beginning at s over which the trajectories generated by the two policies are different. The point $(\hat{z}_1, 0)$ lies on the $1-2$ boundary for both policies and thus both trajectories must eventually pass through this point before reaching the hedging point. Moreoever (6.39) implies that X^a and X^b reach this point at exactly the same time instant. Let S denote this time instant. Thus S is a random variable whose value is determined by the sample realization of failures and repairs of the machine.

By definition of Policies a and b, it follows that Trajectories X^a and X^b are identical for all times t such that $t > S$ and $x_2^b(t) \geq 0$. Therefore,

$$\boldsymbol{x}^a(t) = \boldsymbol{x}^b(t) \quad t > S, \quad x_2^b(t) > 0 \tag{6.40}$$

Equation (6.40) implies that the costs incurred by the two trajectories after time S are identical until a machine failure causes both trajectories back into the region where $x_2^b < 0$. In that case, the surplus behavior at the instant when the machine next gets repaired is similar to the surplus behavior at time s.

We now consider the difference in the expected costs of Policies a and b over the interval $[s, S]$. Using the instantaneous cost function in Section 2.2, the surplus costs for $x_2 < 0$ are given by

$$g(\boldsymbol{x}) = \begin{cases} g_1^+ x_1 - g_2^- x_2 & x_1 > 0 \\ -g_1^- x_1 - g_2^- x_2 & x_1 \leq 0 \end{cases}$$

This can also be expressed as

$$g(\boldsymbol{x}) = \begin{cases} (g_1^+ + g_2^- \frac{\tau_1}{\tau_2}) x_1 - \frac{g_2^-}{\tau_2}(\tau_1 x_1 + \tau_2 x_2) & x_1 > 0, x_2 \leq 0 \\ -(g_1^- - g_2^- \frac{\tau_1}{\tau_2}) x_1 - \frac{g_2^-}{\tau_2}(\tau_1 x_1 + \tau_2 x_2) & x_1 < 0, x_2 \leq 0 \end{cases}$$

Equation (6.39) implies that the last term is the same for all $t \in [s, S]$ for both policies and can therefore be ignored in a comparison of the difference in costs. Thus the difference in costs between the two trajectories in $[s, S]$ is the same as that for system where the instantaneous cost function in the non-positive x_2 region is given by:

$$g^m(\boldsymbol{x}) = \begin{cases} g^+ x_1 & x_1 > 0, x_2 \leq 0 \\ -g^- x_1 & x_1 \leq 0, x_2 \leq 0 \end{cases}$$

where g^+, g^- are as defined in (6.38). This cost function depends only on the value of the Part Type 1 surplus, x_1.

Consider the behavior of x_1 under Policies a and b respectively. Over the interval $[s, S]$, the production rate for Part Type 1 under Policy a satisfies:

$$u_1(\boldsymbol{x}, \alpha) = \begin{cases} \alpha\mu_1 & x_1^a < z_1^m \\ \alpha d_1 & x_1^a = z_1^m \end{cases}$$

Thus, under Policy a, the behavior of Part Type 1 for any time t such that $x_1^a \leq z_1^m$, is the same as that generated by the optimal hedging point policy for a single-part-type system described in Bielecki and Kumar (1988) with an instantaneous cost function g and a hedging point z_1^m.

For any time t for which $x_1^a(t) > z_1^m$, the two trajectories satisfy

$$x_1^a(t) \leq x_1^b(t)$$

Thus trajectories X^a and X^b satisfy condition **C1** and conditions 1-3 of Theorem 8 in Section 6.3 over the interval $[s, T]$. Therefore, from Theorem 8, we get

$$J^a(s, S) \leq J^b(s, S).$$

If the initial condition is such that $\boldsymbol{x} \in R_b, x_2(s) < 0$ and $\alpha(s) = 0$, then the control under both policies is the same and the two trajectories overlap until the machine next gets repaired. At that instant the system is in the same scenario as the one considered before. Thus for any value of $\alpha(0)$,

$$J^a(\boldsymbol{x}, \alpha, s, S) \leq J^b(\boldsymbol{x}, \alpha, s, S) \quad \boldsymbol{x} \in R_b \tag{6.41}$$

Since R_b is a non-transient region under Policy b, (6.41) implies that over an interval of time $[s, T]$, where T is some large number,

$$J^a(\boldsymbol{x}, \alpha, s, T) \leq J^b(\boldsymbol{x}, \alpha, s, T) \quad \boldsymbol{x} \in R_b$$

and

$$\lim_{T\to\infty} \frac{1}{T} J^a(\boldsymbol{x}, \alpha, s, T) \leq \lim_{T\to\infty} \frac{1}{T} J^b(\boldsymbol{x}, \alpha, s, T)$$

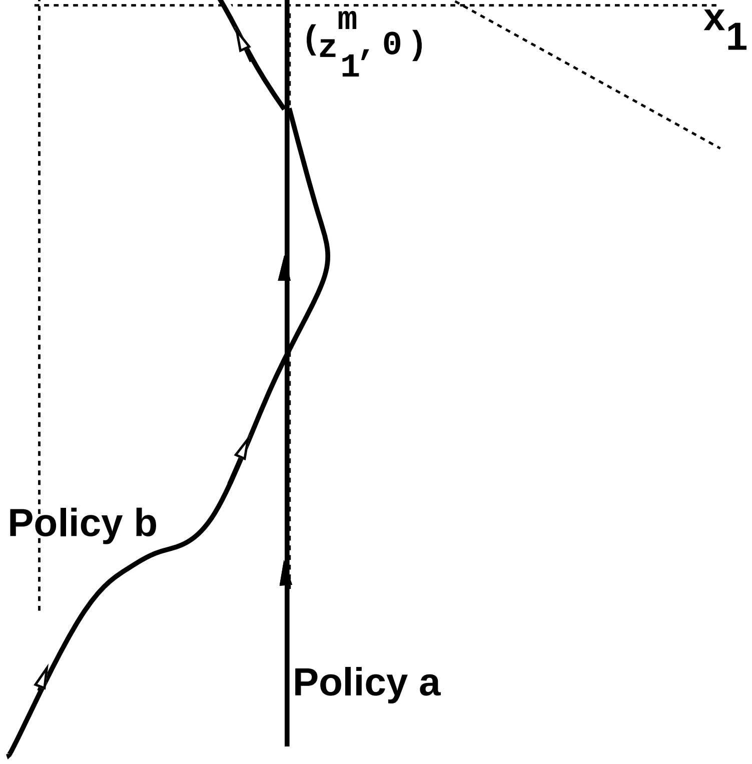

Figure 6-9: Illustration of Theorem 11, Case 2, Subcase 1

We now consider **Case** 2.

Case 2**:** $\hat{z}_1 \leq z_1^m$
This situation is illustrated in Figure 6-9. It is necessary to consider two subcases within this case to accomodate different shapes of the $1-2$ boundary of Policy b.

1. Subcase 1:The $1-2$ boundary of Policy b does not intersect the line $x_1 = z_1^m$ above the x_1 axis.

2. Subcase 2:The $1-2$ boundary of Policy b intersects the line $x_1 = z_1^m$ above the x_1 axis.

The proof strategy in each subcase is the same as in Case 1. In each subcase, we construct another policy Policy a whose $1-2$ boundary satisfies the condition of the theorem. We then compare the expected costs of trajectories generated by Policies a and b.

Subcase 1: The $1-2$ boundary of Policy b does not intersect the line $x_1 = z_1^m$ above the x_1 axis.
This is the case, if $\boldsymbol{z}$ is such that $z_1 < z_1^m$ as shown in Figure 6-9. Policy a be a hedging point policy with its hedging point $\boldsymbol{z}^a = (z_1^a, z_2^a)$ given by

$$
\begin{aligned}
z_1^a &= z_1^m \\
z_2^a &= \frac{\tau_1 z_1 + \tau_2 z_2 - \tau_1 z_1^m}{\tau_2}
\end{aligned}
$$

and with the $1-2$ boundary given by

$$x_1 = z_1^m \ , \ \ x_2 < z_2^m.$$

This policy is also illustated in Figure 6-9.
The hedging points of Policies a and b satisfy:

$$\sum_{i=1}^{2} \tau_i z_i^a = \sum_{i=1}^{2} \tau_i z_i \tag{6.42}$$

Consider an interval of time $[0,T]$ and two trajectories X^a and X^b such that: X^a originates at $\boldsymbol{z}$ at $t=0$ and evolves under Policy b; and X^b originates at $\boldsymbol{z}^a$ at $t=0$ and evolves under Policy a. Using (6.42) and the same arguments as in the derivation of (6.39), we get

$$\sum_{i=1}^{2} \tau_i x_i^a(t) = \sum_{i=1}^{2} \tau_i x_i^b(t), \quad 0 \le t \le T \tag{6.43}$$

A machine failure causes both trajectories to move away from the respective hedging points. Each trajectory is then guided back by the control policy to its $1-2$ boundary and eventually to its hedging point. There may be many machine failures and repairs before each trajectory reaches its hedging point. Since the $1-2$ boundary for Policy b does not intersect the line $x_1 = z_1^m$ above the x_1 axis, Trajectories X^a and X^b are such that

$$x_1^a(t) \ge x_1^b(t) \quad \text{for all } t \text{ such that} \quad x_2^b(t) > 0 \tag{6.44}$$

From (6.43) and (6.44), it follows that

$$x_2^b(t) \ge x_2^a(t) \quad \text{for all } t \text{ such that} \quad x_2^b(t) > 0 \tag{6.45}$$

Thus for any T, the interval $[0,T]$ can be expressed as a union of three mutually exclusive sets $\mathcal{I}_1$,$\mathcal{I}_2$, and $\mathcal{I}_1$ that are defined below:

$$\begin{aligned}
\mathcal{I}_1 &= \{t : 0 \le t \le T;\, x_2^b(t) \ge x_2^a(t) \ge 0\}, \\
\mathcal{I}_2 &= \{t : 0 \le t \le T;\, x_2^b(t) \le 0;\, x_2^a(t) \le 0\}, \\
\mathcal{I}_3 &= \{t : 0 \le t \le T;\, x_2^b(t) > 0 \ge x_2^a(t)\}.
\end{aligned}$$

We now show that the difference in costs between the two trajectories on $[0,T]$ can be expressed in terms of the difference in a modified cost function of only the Part Type 1 surplus.

The difference in instantaneous costs between the two trajectories at time t is given by

$$\Delta g(t) = g(\boldsymbol{x}^a(t)) - g(\boldsymbol{x}^b(t)). \tag{6.46}$$

For $t \in \mathcal{I}_1$

$$\Delta g(t) = g_1(x_1^a(t)) - g_1(x_1^b(t)) + g_2^+(x_2^a(t) - x_2^b(t))$$

Using (6.45), the second term is non-positive over the interval $\mathcal{I}_1$. Thus

$$\Delta g(t) \leq g_1(x_1^a(t)) - g_1(x_1^b(t))$$

Again using (6.45), the above inequality implies that, for $t \in \mathcal{I}_1$,

$$\Delta g(t) \leq g_1(x_1^a(t)) - g_1(x_1^b(t)) - g_2^-(x_2^a(t) - x_2^b(t)). \tag{6.47}$$

For $t \in \mathcal{I}_2$

$$\Delta g(t) = g_1(x_1^a(t)) - g_1(x_1^b(t)) - g_2^-(x_2^a(t) - x_b^2(t)). \tag{6.48}$$

For $t \in \mathcal{I}_3$

$$\begin{aligned} \Delta g(t) &= g_1(x_1^a(t)) - g_1(x_1^b(t)) - g_2^- x_2^a(t) + g_2^+ x_2^b(t)) \\ &= g_1(x_1^a(t) - x_1^b(t)) - g_2^-(x_2^a(t) - x_2^b(t)) + (g_2^+ + g_2^-)x_2^b(t) \end{aligned} \tag{6.49}$$

For $t \in \mathcal{I}_3$, $x_2^b(t) > 0$, Therefore, (6.49) implies that:

$$\Delta g(t) \leq g_1^+(x_1^a(t) - x_1^b(t)) - g_2^-(x_2^a(t) - x_2^b(t)) \tag{6.50}$$

From (6.47), (6.48) and (6.50), we get

$$\Delta g(t) \leq g_1(x_1^a(t)) - g_2^- x_2^a(t) - (g_1(x_1^b(t)) - g_2^-(x_2^b(t))) \quad 0 \leq t \leq T \tag{6.51}$$

The right side is the difference in the function

$$g_1(x_1(t)) - g_2^- x_2(t)$$

between the two trajectories. This function can also be expressed as

$$g_1(x_1(t)) + g_2^- \frac{\tau_1}{\tau_2} x_1(t) - \frac{g_2^-}{\tau_2}(\tau_1 x_1(t) + \tau_2 x_2(t))$$

Let $g^m(x_1)$ be an instantaneous cost function defined in Case 1. Therefore

$$g^m(x_1) = g_1(x_1) + g_2^- \frac{\tau_1}{\tau_2} x_1$$

Expressing (6.51) in terms of $g^m(x_1)$ and using (6.43) we get,

$$\Delta g(t) \le g^m(x_1^a(t)) - g^m(x_1^b(t)) \tag{6.52}$$

Integrating (6.52) from 0 to T, the left side is the difference of expected costs incurred by the two policies in $[0,T]$, *i.e*, $\Delta J^{ab}[0,T]$. Thus, we get

$$\Delta J^{ab} \le \int_0^T \left[g^m(x_1^a(t)) - g^m(x_1^b(t)) \right] dt$$

Dividing by T and taking limits of both sides as $T \to \infty$, we get

$$\lim_{T\to\infty} \frac{\Delta J^{ab}}{T} \le \lim_{T\to\infty} \frac{1}{T} \int_0^T [g^m(x_1^a(t)) - g^m(x_1^b(t))] dt \tag{6.53}$$

The integrand on the right hand side depends only on x_1, the Part Type 1 surplus.

Consider the behavior of x_1 under Policies a and b respectively. In the non-transient region under Policy a the production rate for Part Type 1 under Policy a is given by:

$$u_1 = \begin{cases} \mu_1 & x_1^a < z_1^a \\ d_1 & x_1^a = z_1^a \end{cases}$$

Thus, under Policy a, the non-transient behavior of Part Type 1 is the same as that of a single-part-type system described in Bielecki and Kumar (1988) with time dependent failures, exponential failures and repairs with an instantaneous cost function g^m and an optimal hedging point z_1^m.

By construction $z_1^a = z_1^m$, and thus Part Type 1 under Policy a follows the optimal policy in $[0,T]$ for the instantaneous cost function g^m. Over the same interval, the trajectory of Part Type 1 under Policy b evolves according to a policy that is not a hedging point policy.

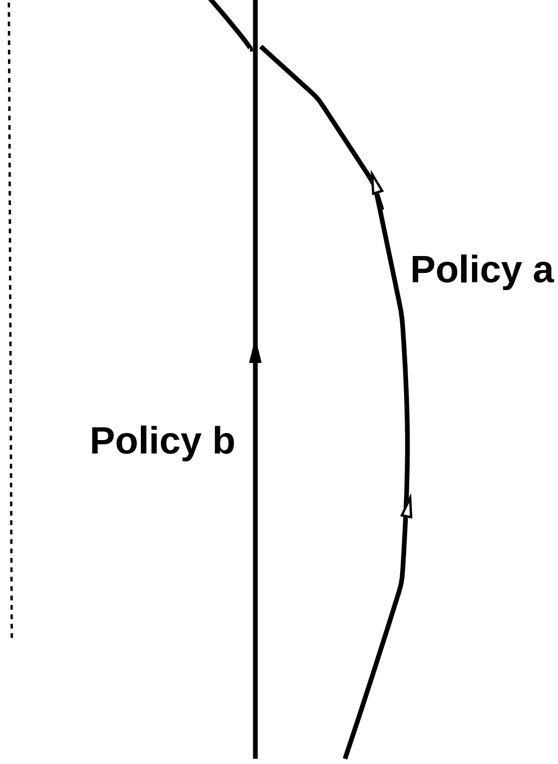

Figure 6-10: Illustration of Theorem 2, Case 2,Subcase 2

Using the results in Bielecki and Kumar (1988), we therefore get

$$\lim_{T\to\infty} \frac{1}{T} E\{\int_0^T [g^m(x_1^a(t)) - g^m(x_1^b(t))]dt\} \leq 0 \tag{6.54}$$

Equations (6.53) and (6.54) yield

$$\lim_{T\to\infty} \frac{\Delta J^{ab}}{T} \leq 0$$

Thus Policy a is a hedging point policy that satisfies Theorem 11 and has an average expected cost that is no worse than that of Policy b.

Finally we consider subcase 2 of case 2.

Subcase 2: The $1-2$ boundary of Policy b intersects the line $x_1 = z_1^m$ above the x_2 axis.

This is the case for example, when the hedging point $\boldsymbol{z}$ is such that $z_1 \geq z_1^m$ as shown in Figure 6-10. Let $S2$ denote the set of x_2 coordinates of points where the $1-2$ boundary for Policy b intersects the line $x_1 = z_1^m$. Let $\hat{x}_2$ be defined as:

$$\hat{x}_2 = \inf\{S2\}$$

Let Policy a be a hedging point policy with the same hedging point as Policy b and whose $1-2$ boundary is the vertical line $x_1 = z_1^m$ for $x_2 \leq \hat{x}_2$; and the same as that of Policy a elsewhere as shown in Figure 6-10.

Let R_b denote the portion of the non-transient surplus space under Policy b with $x_1 < 0, x_2 < 0$. Consider two trajectories, X^a and X^b, that start at the same point $\boldsymbol{x}$ at time 0 and evolve under Policies a and b respectively. Without loss of generality, let $\boldsymbol{x} \in R_b$ and $\alpha(0) = 1$. Using arguments similar to those in Case 1 it can be shown that Trajectories X^a and X^b pass through the point $(z_1^m, \hat{x}_2)$ at the same instant at some future random time S. After time S the two trajectories overlap until a machine failure causes the surplus trajectory to enter a region where the two control policies are different. This leads to the same scenario described above.

Using the same arguments as in the proof of Subcase 1 of Case 2, it follows that the difference in the instantaneous cost function between the two trajectories in the interval $[0, S]$ satisfies:

$$\Delta g(t) \leq g(x_1^a(t)) - g(x_1^b(t)) \tag{6.55}$$

Also, the surplus trajectories of Part Type 1 under Policies a and b in $[0, S]$ satisfy the conditions of Theorem 8 and therefore from Theorem 8 we get

$$\int_0^S g(x_1^a(t)) - g(x_1^b(t))dt \leq 0 \tag{6.56}$$

Integrating (6.55) and using (6.56) we get

$$\Delta J^{ab}[0, S] \leq 0$$

Since R_b belongs to the non-transient surplus space under Policy b, this implies that

$$\lim_{T \to \infty} \frac{\Delta J^{ab}[0, T]}{T} \leq 0$$

Thus Policy a is a hedging point policy that satisfies Theorem 11 and has an average expected cost that is no worse than that of Policy b.

The proof given above pertains to the case where $1-2$ boundary has a single point for which $x_2 = 0$. The extension to the case when there are many such points is relatively straightforward. For this case, one can construct Policy a as illustrated in Figure 6-11. The constuction is such that trajectories X^a and X^b are the same for the region $x_2 > 0$ while in the region $x_2 \leq 0$, the arguments of Case 1 hold. The arguments of the proof can now be applied to show that Policy a has an expected

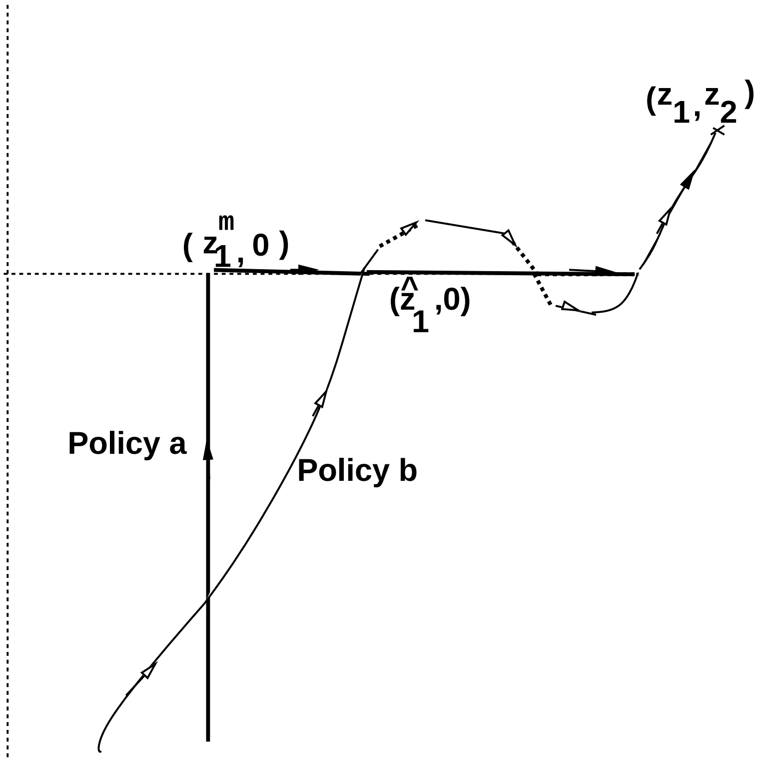

Figure 6-11: Definition of Policy a

cost that is less than or equal to that of Policy b.

The results derived so far in this chapter do not make any assumptions on specific values of the system parameters. In the following section, we describe the application of these results for systems where parameters satisfy certain conditions.

6.5 Systems with special parameter values

6.5.1 Identical backlog costs

In this section, we consider the special case for which the two part types have *identical backlog costs*, i.e., the cost coefficients satisfy (6.26).

It is easy to verify that the proofs of Theorems 9 and 11 are valid for this case as well. In particular, the analog of Theorem 11 for a system with identical backlog costs can be stated in two steps as given below:

Theorem 12 *1. The* $1-2$ *boundary corresponding to any optimal hedging point policy for a system with identical backlog costs contains the point* $(0,0)$.

2. There exists an optimal policy for a system with identical backlog costs that is *of the form:*

$$\begin{array}{lll} u(\boldsymbol{x},1) = & (\mu_1, 0) & x_1 < 0, x_2 < 0 \\ u(\boldsymbol{x},1) = & (d_1, \mu_2(1 - \frac{d_1}{\mu_1})) & \textit{for} \;\; x_1 = 0, x_2 < 0 \\ u(\boldsymbol{x},1) = & (0, \mu_2) & x_1 > 0, x_2 < 0 \end{array}$$

Condition 1 in Theorem 12 follows from the fact that the single-part-type system of Chapter 3 with time dependent exponential failures with rate p, exponential repairs with rate r, maximum production rate μ_1, demand rate d_1 and instantaneous cost coefficients given by:

$$\begin{aligned} g^+ &= g_1^+ + g_2^- \frac{\tau_1}{\tau_2} \\ g^- &= 0 \end{aligned}$$

has an optimal hedging point of zero.

Non-Uniqueness of optimal policy: Let Policy a denote a policy that satisfies Condition 2 in Theorem 12. For systems with equal backlog costs, there are many policies that have the same cost as Policy a. For example, consider another policy (Policy b) that has a $1-2$ boundary that contains $(0,0)$ and is such that it has an arbitrary shape for $x_1 < 0, x_2 < 0$; but is the same as that of Policy a elsewhere. Consider two trajectories X^a and X^b that start at the same point and evolve according to these policies. Then

$$\sum_{i=1}^{2} \tau_i x_i^a(t) = \sum_{i=1}^{2} \tau_i x_i^b(t) \quad \text{for all} \;\; t. \tag{6.57}$$

Moreover these trajectories differ only in the region $x_1 < 0, x_2 < 0$. But for this region, the difference in instantaneous cost is given by

$$\Delta g(t) = -g_1^-(x_1^a(t) - x_1^b(t)) - g_2^-(x_1^a(t) - x_1^b(t))$$

which can be expressed as

$$\Delta g(t) = -\frac{g_1^-}{\tau_1}(\tau_1 x_1^a(t) - \tau_1 x_1^b(t)) - \frac{g_2^-}{\tau_2}(\tau_2 x_2^a(t) - \tau_2 x_2^b(t)).$$

Using (6.57) and (6.26)

$$\Delta g(t) = 0.$$

Thus Policy a and b have identical costs at all times.

6.5.2 Identical backlog and surplus costs

In the remaining of this section, we consider the case where the two part have identical backlog costs and in addition the cost coefficients satisfy:

$$\frac{g_1^+}{\tau_1} = \frac{g_2^+}{\tau_2} \tag{6.58}$$

The two part types are said to have *identical costs.* We now obtain a complete characterization of an optimal policy for this system. A system with identical costs of the part types need not be symmetrical with respect to the two part types because there is no restriction on the demand rates d_1 and d_2.

Let c^+ and c^- be defined as:

$$c^+ = \frac{g_1^+}{\tau_1} = \frac{g_2^+}{\tau_2} \tag{6.59}$$

and

$$c^- = \frac{g_1^-}{\tau_1} = \frac{g_2^-}{\tau_2} \tag{6.60}$$

The following theorem characterizes the $1-2$ boundary for an optimal policy in such a system:

Theorem 13 *For a two-part-type system with identical costs, there exists an optimal hedging point policy with hedging point* $\boldsymbol{z} = (z_1, z_2)$ *such that*

$$\frac{z_1}{d_1} = \frac{z_2}{d_2} \tag{6.61}$$

and with a $1-2$ *boundary defined by:*

$$\frac{x_1}{d_1} = \frac{x_2}{d_2} \quad x_1 < z_1,\, x_2 < z_2 \tag{6.62}$$

Proof:

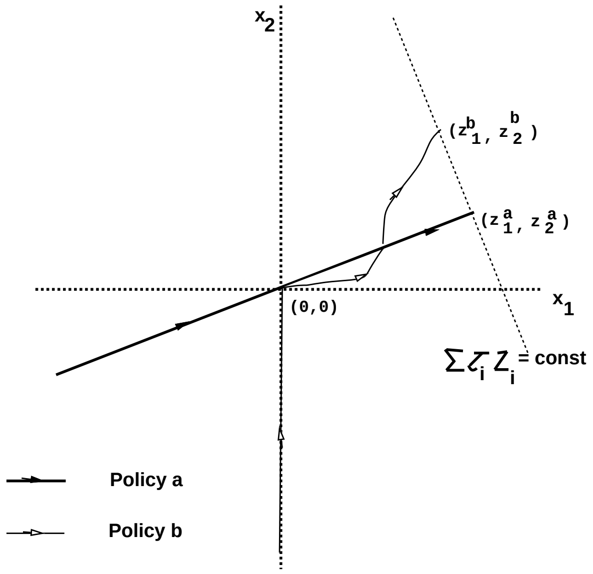

Figure 6-12: Optimal policy with identical surplus and backlog costs

Since the cost coefficients for a system with identical costs satisfy (6.26), Theorem 12 implies that any optimal hedging point policy contains the point $(0, 0)$.

Let Policy b be a hedging point policy that does not satisfy the conditions of Theorem 13, and is defined by the hedging point $\boldsymbol{z}^b = (z_1^b, z_2^b)$ and a $1 - 2$ boundary that contains $(0, 0)$.

The identical backlog costs imply that the shape of the $1 - 2$ boundary in the region $\boldsymbol{x} \leq 0$ does not make any difference to the cost. Let Policy a be another hedging point policy that has a hedging point $\boldsymbol{z}^a = (z_1^a, z_2^a)$, where

$$
\begin{aligned}
z_1^a &= \frac{d_1}{\tau_1 d_1 + \tau_2 d_2}(\tau_1 z_1^b + \tau_2 z_2^b) \\
z_2^a &= \frac{d_2}{\tau_1 d_1 + \tau_2 d_2}(\tau_1 z_1^b + \tau_2 z_2^b)
\end{aligned}
$$

and a $1 - 2$ boundary that satisfies (6.62). Policies a and b are illustrated in Figure 6-12. Policy a satisfes both conditions of Theorem 13.

Also, the definition of $\boldsymbol{z}^a$ implies that

$$\tau_1 z_1^a + \tau_2 z_2^a = \tau_1 z_1^b + \tau_2 z_2^b$$

Let X^a and X^b denote trajectories that originate at $t = 0$ at some point in the non-transient region of Policy b and evolve under Policies a and b respectively. Using arguments similar to those in derivation of (6.39), we get for any t,

$$\tau_1 x_1^a(t) + \tau_2 x_2^a(t) = \tau_1 x_1^b(t) + \tau_2 x_2^b(t) \tag{6.63}$$

Also by construction, since the $1-2$ boundary of Policy a is the demand line, the trajectory X^a never leaves this boundary. A machine failure causes the surplus to decrease along the boundary. Once the machine is repaired, the trajectory remains on the boundary while moving towards the hedging point. Thus at any time t Trajectory X^a is such that either

$$x_1^a(t) \geq 0 \quad \text{and} \quad x_2^a(t) \geq 0$$

or

$$x_1^a(t) \leq 0 \quad \text{and} \quad x_2^a(t) \leq 0$$

We compare the instantaneous cost of the two trajectroies for each case separately and show that in each case, the instantaneous cost of Policy a at any time is a lower bound on that of Policy b.

Case 1: $x_1^a(t) \geq 0 \quad \text{and} \quad x_2^a(t) \geq 0$

In this case

$$\tau_1 x_1^a(t) + \tau_2 x_2^a(t) \geq 0$$

From (6.63), Trajectory X^b at time t cannot be in a region where the surpluses of both part types are negative. Thus X^b has to satisfy one of the following three conditions

1. $x_1^b(t) \geq 0 \quad \text{and} \quad x_2^b(t) \geq 0$
2. $x_1^b(t) \geq 0 \quad \text{and} \quad x_2^b(t) \leq 0$
3. $x_1^b(t) \leq 0 \quad \text{and} \quad x_2^b(t) \geq 0$

Subcase 1: $x_1^b(t) \geq 0 \quad \text{and} \quad x_2^b(t) \geq 0$

Using (6.59) and (6.60) the instantaneous cost at time t for Policy b can be expressed as:

$$g^b(\boldsymbol{x}^b(t)) = c^+ \tau_1 x_1^b(t) + c^+ x_2^b(t) \tau_2$$

Using (6.63), we get

$$g^b(\boldsymbol{x}^b(t)) = c^+ \tau_1 x_1^a(t) + c^+ x_2^a(t) \tau_2 = g^a(\boldsymbol{x}^a(t)) \tag{6.64}$$

Subcase 2: $x_1^b(t) \geq 0 \quad \text{and} \quad x_2^b(t) \leq 0$

Using (6.59) and (6.60) the instantaneous cost at time t for Policy b can be expressed as:

$$g^b(\boldsymbol{x}^b(t)) = c^+\tau_1 x_1^b(t) - c^- x_2^b \tau_2(t)$$

Since $x_2^b(t) \leq 0$, we get,

$$g^b(\boldsymbol{x}^b(t)) \geq c^+\tau_1 x_1^b(t) + c^+ x_2^b \tau_2(t)$$

Using (6.63), we get

$$g^b(\boldsymbol{x}^b(t)) \geq c^+\tau_1 x_1^a(t) + c^+ x_2^a \tau_2(t) = g^a(\boldsymbol{x}^a(t)) \tag{6.65}$$

Subcase 3: $x_1^b(t) \leq 0 \quad \text{and} \quad x_2^b(t) \geq 0$
Using (6.59) and (6.60) the instantaneous cost at time t for Policy b can be expressed as:

$$g^b(\boldsymbol{x}^b(t)) = -c^-\tau_1 x_1^b(t) + c^+ x_2^b \tau_2(t)$$

Since $x_1^b \leq 0$, we get

$$g^b(\boldsymbol{x}^b(t)) \geq c^+\tau_1 x_1^b(t) + c^+ x_2^b \tau_2(t)$$

Using (6.63), we get

$$g^b(\boldsymbol{x}^b(t)) \geq c^+\tau_1 x_1^a(t) + c^+ x_2^a \tau_2(t) = g^a(\boldsymbol{x}^a(t)) \tag{6.66}$$

From (6.64) -(6.66), we get

$$g^a(\boldsymbol{x}^a(t)) \leq g^b(\boldsymbol{x}^b(t)) \quad \text{for all } t \text{ such that } \; x_1^a(t) \geq 0, x_2^a(t) \geq 0$$

Using similar arguments, we can prove the theorem for **Case** 2.

We now find an expression for the optimal hedging point $\boldsymbol{z}$ for this system.

Let the system be operated with a hedging point policy that satisfies conditions (6.61) and (6.62) in Theorem 13. Consider the stochastic process

$$y(t) = \tau_1 x_1(t) + \tau_2 x_2(t)$$

As seen in Theorem 13, the surplus vector never leaves the $1-2$ boundary. Moreover, since the $1-2$ boundary passes through the origin, therefore the surplus of the two part types are both either positive or negative or zero at any time. The definition of $y(t)$ therefore implies that

$$
\begin{aligned}
y(t) > 0 &\iff x_1(t) > 0, x_2(t) > 0 \\
y(t) < 0 &\iff x_1(t) < 0, x_2(t) < 0 \\
y(t) = 0 &\iff x_1(t) = 0, x_2(t) = 0
\end{aligned}
$$

Thus the behavior of process y is exactly the same as that of the single-part-type system described in Chapter 3 with time dependent failures, exponential failure and repair times, demand $d = \tau_1 d_1 + \tau_2 d_2$, maximum production rate $\mu = 1$ and instantaneous cost function $c(y)$ given by:

$$
\begin{aligned}
c(y) &= c^+ y \quad for \quad y \geq 0 \\
&\quad -c^- y \quad for \quad y < 0
\end{aligned}
$$

From Chapter 3, we know that the optimal policy is a hedging point policy for y. Let z^* denote the optimal hedging point for y. z^* is given by (3.21). Let $\boldsymbol{z} = (z_1, z_2)$ denote the hedging point in terms of the surplus vector $\boldsymbol{x}$. Using the defintion of the process $y(t)$ we have,

$$
z^* = \tau_1 z_1 + \tau_2 z_2 \tag{6.67}
$$

Also, from Theorem 13, the hedging points of the two part types satisfy (6.61). Solving (6.67) and (6.61) we get the hedging point $\boldsymbol{z}$. This leads to the following corollary.

Corollary 7 *For a two-part type system satisfying (6.59) and (6.60) the optimal hedging point is given by*

$$
\mathbf{z} = \left(\frac{d_1 z^*}{\tau_1 d_1 + \tau_2 d_2}, \frac{d_2 z^*}{\tau_1 d_2 + \tau_2 d_1}\right)
$$

where

$$
\begin{aligned}
z^* &= \frac{ln(kb(1+\frac{c^-}{c^+}))}{b} \quad if \quad \frac{c^+}{c^+ + c^-} < kb \\
&= 0 \quad if \quad \frac{c^+}{c^+ + c^-} > kb
\end{aligned}
$$

where

$$b = \frac{r}{\tau_1 d_1 + \tau_2 d_2} - \frac{p}{1 - (\tau_1 d_1 + \tau_2 d_2)}$$

$$k = \frac{p}{b(r+p)(1 - (\tau_1 d_1 + \tau_2 d_2))}$$

6.6 Conjecture on complete characterization of an optimal policy

In this section, we use the results of the Section 6.4 to develop a conjecture on the structure of the optimal policy for two-part-type systems with general values of parameters.

We first provide some intuition for the results of Theorems 10 and Theorem 11. Let Z_1 denote the optimal hedging point for Part Type 1 in isolation. In other words, Z_1 represents the optimum level of surplus of Part Type 1 to be built up to hedge against future machine failures if Part Type 2 were not present. Whenever $x_1 < Z_1$, the optimal policy with Part Type 1 in isolation devotes all the capacity of the machine to Part Type 1.

For a two-part type system, the capacity at the machine must be shared between the part types. When both part types are backlogged, then Theorem 10 implies that the optimal allocation of capacity is to allocate all the available capacity to Part Type 1 since this leads to the largest reduction in backlog costs per unit time.

When the Part Type 1 surplus reaches 0, building any additional surplus for Part Type 1 leads to two types of short term costs: inventory costs of Part Type 1; and lost savings in the backlog cost of Part Type 2 that would have accrued had the machine produced Part Type 2 instead of Part Type 1. On the other hand, the benefit of having additional surplus is that it hedges against future machine failures and therefore saves potential backlog costs in Part Type 1. The result of this tradeoff for $x_2 < 0$ is given by Theorem 11. That is, the optimal policy builds up the surplus of Part Type 1 to a level z_1^m, and then maintains it there while devoting the remaining capacity to Part Type 2. For $x_2 < 0$, the line $x_1 = z_1^m$ represents the surface where the marginal gain in the long term average cost of Part Type 1 balances the marginal costs. Equation (3.21) in Chapter 3 yields the value of z_1^m. The optimal hedging value for $g_2^- > 0$ from (3.21) is greater than or equal to that for $g_2^- = 0$. That is,

$$z_1^m \leq Z_1.$$

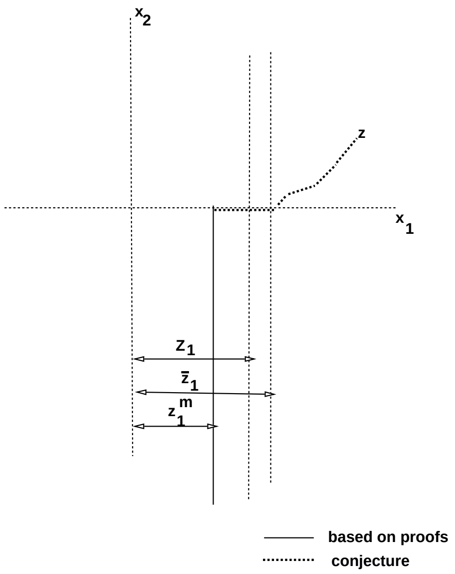

Figure 6-13: Conjecture on the $1-2$ boundary for optimal policy

We now extend the intuition gained from the tradeoffs between the various costs to further characterize an optimal policy, Once x_2 reaches zero, the tradeoff in Theorem 11 is no longer valid. Any additional capacity devoted to building a positive surplus of Part Type 2 now leads to a short term surplus cost instead of reducing in the backlog costs capacity tradeoff. This discontinuity in the tradeoff suggests that once x_2 reaches 0, the optimal policy maintains it at 0 for some time while building up Part Type 1 surplus. This causes the trajectory to move along the x_1 axis until the Part type 1 surplus reaches some value, $\bar{z}_1$.

For positive values of x_1 and x_2 the tradeoff is complicated because it involves analyzing the effect of additional surplus on the the long term expected average cost of each part type. This suggests a $1-2$ boundary that could be a curve rather than a straight line in this region. This makes it impossible to determine it analytically.

The above analysis leads us to the following conjecture on the complete characterization of the optimal policy.

Conjecture 3 *The optimal policy for a two-part type system has a hedging point* $\boldsymbol{z} \geq 0$ *and a* $1-2$ *boundary that is illustrated in Figure 6-13 where*

$$z_1^m < Z_1 < z_1$$

The intuitive argument for $Z_1 < z_1$ is as follows. Z_1 represents the optimal amount of hedging required for Part Type 1 in isolation. From the perspective of Part Type 1, the presence of Part Type 2 effectively reduces the available capacity of the machine. The hedging level required for the Part Type 1 must now be greater than if Part Type 1 were in isolation to compensate for this additional loss in capacity.

This conjecture has been validated in numerical experiments reported in Liberopolous (1992).

6.7 Extension to multiple part types

In this section, we focus on some extensions of the results of the preceeding sections to the mutilple part type system described in Chapter 2. Without loss of generality, let the N part types be numbered such that

$$\frac{g_1^-}{\tau_1} \geq \frac{g_2^-}{\tau_2} \geq \ldots \geq \frac{g_N^-}{\tau_N}$$

It is straightforward to extend the arguments in the proof of Theorem 7 to show that

Theorem 14 *Any hedging point policy with a finite hedging point is stable if the demand satisfies:*

$$\sum_{i=1}^{N} \frac{d_i}{\mu_i} < \frac{r}{r+p}$$

Using arguments similar to those in Theorem 9 leads to the following theorem.

Theorem 15 *The optimal hedging point policy is such that*

$$\sum_{i=1}^{N} \tau_i u_i(\boldsymbol{x}, 1) = 1 \tag{6.68}$$

for all points $\boldsymbol{x} = (x_1, x_2, \ldots x_n)$ in the non-transient surplus space, for which at least one of the components of $\boldsymbol{x}$ is strictly negative.

We now consider the extension of Theorem 10. The arguments in the proof of Theorem 10 can be used on any pair of part types in the N-part-type system which leads to the following theorem.

Theorem 16 *The optimal hedging point policy for a N-part-type system is such that*

$$u_i = \frac{1 - \sum_{j=1}^{i-1} \tau_j u_j}{\tau_i} \quad \textit{if} \;\; x_i < 0. \tag{6.69}$$

It is interesting to observe that if the machine were to be reliable, this theorem defines the optimal transient policy and thus generalizes the result of Connolly (1992) to a N-part-type system. For such a system, since there are no machine failures, the optimal hedging point is zero. The production control problem determines the minimum cost control policy that would guide the surplus vector to zero from a given initial value. It is easy to show that if the initial surplus vector has a positive component of some part type, that part type is not produced until its surplus reaches zero. On the other hand, if the surplus of some part type is initially negative or becomes negative at some instant, then its production rate is chosen according to equation (6.69).

For the stochastic system considered here, Theorem 16 states that the optimal policy does not produce Part Type i if any one of the Part Types $1, \ldots, i-1$ has a negative surplus.

Let $\boldsymbol{x}$ denote the surplus vector in the non-transient surplus space such that all its elements are negative. The optimal control at $\boldsymbol{x}$ devotes all capacity at the machine to Part Type 1 until its surplus reaches zero. At that instant there is a tradeoff between reducing the backlog of Part Type 2 and producing a surplus of Part Type 1 to hedge against future failures. Part Types $3 \ldots N$ are not produced as long as $x_1 < 0$ or $x_2 < 0$. The tradeoff between the Part Type 1 and 2 is therefore similar to the case in a two part type system for $x_2 \leq 0$. This implies that Theorem 11 is valid for the N part type system and provides a characterization of the optimal policy for $x_2 < 0$.

When x_2 reaches zero, then the tradeoff gets more complicated and is difficult to characterize analytically. But the structure of the optimum solution explained in Conjecture 3 suggests the optimal policy consists of a sequence of *partial hedging points*, z_1^1, (z_1^2, z_2^2), (z_1^3, z_2^3, z_3^3), ..., of increasing dimensionality. z_1^1 is the same as z_1^m in Theorem 12.

Thus, when x_2 becomes positive, the optimal policy splits the available capacity at the machine among Part Types 1 and 2 until such time when x_1 reaches some value z_1^2 and x_2 reaches a value z_2^2. The surpluses of the first two part types are maintained at these levels respectively and the remaining capacity is then devoted to Part Type 3 until its surplus reaches zero. ($i.e$, the trajectory reaches $(z_1^2, z_2^2, 0)$) Then, the available capacity at the machine is divided among Part types 1, 2 and 3 until their surplus reaches some other level (z_1^3, z_2^3, z_3^3). The first three part types are maintained at this level and the remaining capacity is devoted to Part Type 4. This process continues until the surplus reaches the hedging point $\boldsymbol{z}$ for the system.

6.8 Summary

In this chapter, we have derived structural results pertaining to the optimal hedging point policy for a two-part type system. It was shown that any hedging point policy with a finite hedging point leads to a stable behavior of the surplus for a feasible demand. Two results were then established on the optimal hedging point policy, one based on sample path arguments and the other based on average cost analysis. These results led to a partial characterization of the optimal $1-2$ boundary for general parameter values and a complete characterization of the optimal policy for special parameter values.

Further work is needed to find the hedging point in the two-part type system. Bounds on the hedging point can be found by decomposing the two-part type system into two one-part type systems (Kimemia and Gershwin, 1982). Another approach is to assume a $1-2$ boundary that satisfies Conjecture 3 and in addition is a straight line with some slope in the positive surplus space. This unknown slope along with the hedging point $\boldsymbol{z}$ and the parameter $\hat{z}_1$ can then be determined so as to minimize the total surplus and backlog costs over a typical failure repair cycle. Future research directions in multiple-part-type systems include validating the conjectures in this paper and developing heuristic methods to determine both the hedging point and the dynamic capacity allocation among the various part types.

Chapter 7

Summary of contributions and future research directions

In this chapter, we summarize the contributions of this thesis. These contributions include new methodology for synthesizing dynamic scheduling policies in multi-part-type manufacturing systems as well as results from using these methodologies. These results include an evaluation of the differential cost function for hedging point policies and a partial characterization of optimal hedging point policies. We also identify areas for future research.

7.1 Structure of the optimal policies

In this thesis, we have investigated some models for production scheduling in an unreliable environment. In Chapter 3, the basic dynamic programming model and the Hamilton-Jacobi-Bellman equations were identified. It was shown that the optimal policy is a *hedging point policy.* This policy divided the surplus space into regions of piecewise constant production rates. If the system has sufficient capacity in the long run, the surplus from any starting state is guided to the *hedging point.* The hedging point represents the optimum amount of surplus to be maintained to compensate for future machine failures.

7.2 Operation dependent failures

Chapter 3 investigated a variant of this model for a single part type system with operation dependent failures of the form (3.1). Methods for the evaluation of the differential cost function and the probability distribution of the surplus for a hedging point policy were studied. It was shown that there exists a hedging point policy that satisfies the optimality conditions in (3.10) for $\beta = 0$ and $\beta = 1$ For $\beta > 1$, it was shown that exists no hedging point policy that would satisfy the optimality conditions in (3.10). These results are the same as those derived in Vakili and Hu (1991). While their results were for a discounted cost objective, our results pertain to an overtaking or ergodic optimality objective. In addition, it was shown in Chapter 3 that the case $\beta < 1$ is ill-defined because the optimum policy in this case approaches a hedging point policy in the limit, though the hedging point policy itself is not optimal.

Further research is needed to prove that (3.10) is indeed the sufficient condition for optimality for the average cost criterion. The structure of the optimal policy for $\beta > 1$ is another possible research direction.

7.3 General repair time distribution

Chapter 4 we investigated hedging point policies for single-part-type systems with general repair time distributions. We derived conditions on system parameters for which hedging point policies are stable and lead to a stationary distribution for the surplus. Conditions on the surplus distribution under the best hedging point policy were derived. These conditions led to necessary and sufficient conditions for a zero hedging point to be the optimal. A solution to the differential cost function was obtained over the negative surplus space for a zero hedging point. It was shown that this solution is the same as that of a deterministic system with an averaged capacity set (AD system). An integro-differential equation was derived for the solution for the positive surplus space. The solution for the negative surplus space was used as a boundary condition in the two-part type system of Chapter 6.

Future research areas include obtaining the numerical solution to the HJB equation over the positive surplus space for a zero hedging point and verify the HJB equations using the solution over the entire surplus space. Further research is also needed to investigate the behavior of the differential cost for non-zero hedging points. Even with a non-zero hedging point, the surplus behavior at points very far from the hedging point is similar to that of the AD system. Thus for such points, we expect the solution of the stochastic and AD systems to be similar.

7.4 Solution of the Bellman equation

Chapter 5 described a method for solving the Bellman equations for a particular policy in a two-part-type system. The policy considered (the OODS policy) is overtaking optimal for an identical deterministic system but with an identical machine. This policy has a zero hedging point and three regions each corresponding to one vertex of the capacity set.

It was shown that the structure of the HJB equations leads to a further division of surplus space into subregions such that in each subregion the differential cost function ($W(x, \alpha)$) is twice continuously differentiable. The boundary conditions to the HJB equations in each subregion were derived from the following properties of the differential cost

1. $W(\boldsymbol{x}, \alpha)$ is continuous at region and subregion boundaries.

2. $W(\boldsymbol{x}, \alpha)$ is continuously differentiable at subregion boundaries.

The continuity of $W(\boldsymbol{x}, 1)$ and $W(\boldsymbol{x}, 0)$ was shown to be equivalent to the continuity of $\frac{\partial W(\boldsymbol{x},\alpha)}{\partial \boldsymbol{x}} \cdot \boldsymbol{u}$ over the entire surplus space since all other terms in the HJB equation are continuous.

The basic insight in the evaluation method was the sequential evaluation scheme in which the solution was first obtained on one subregion boundary and then propagated to other subregions. The HJB equation along that subregion boundary was shown to be the same as that in the single-part-type system of Chapter 4 for $x < 0$. The differential cost from Chapter 4 therefore yields the differential cost along the boundary. This solution was then propagated to other regions using continuity conditions on the differentiation cost and its derivatives at subregion boundaries. The insight from the sequential evaluation procedure has potential applications in constructing numerical solutions for larger systems.

Simple expressions for the differential cost function were obtained in some subregions. It was shown that the quadratic terms in these expressions are the differential costs in these subregions for a related deterministic system. This deterministic system has a capacity set that is an average of the capacity set in the two machine states of the stochastic system. Necessary conditions were derived for the OODS policy to be optimal. The solution in other regions was obtained by numerically solving two one-dimensional integro-differential equations.

Further research is necessary to determine whether analytical expressions can be obtained over the entire surplus space. Such a solution will be useful in generating a complete set of necessary conditions for the optimality of the OODS policy. Another research direction is the extension of this solution to other policies in particular those that do not have a zero hedging point.

7.5 Structural properties

In Chapter 6, we restricted our attention to the class of hedging point policies. Conditions under which a hedging point policy leads to a stable behavior of the surplus were derived.

Using sample path arguments, it was shown that any optimal hedging point does not idle the machine when at least one of the two part types has a negative surplus. Further it was shown that any optimal hedging policy follows the "$c\mu$" rule over some region of the surplus space. A comparison of the average cost between trajectories generated by different hedging point policies was used to obtain a partial characterization of the switching curve between the two part types. This method was extended to obtain a complete characterization of the hedging point and the switching curve for systems where the parameters had special values. The application of these results for the design of heuristics for systems with more part types was discussed

Further research is needed to investigate further properties of the switching curve and the hedging point. One such property of the switching curve is a monotonicity property (Veatch, 1992) A proof of this property will reduce the number of different cases to be considered for a comparison of trajectories. The estimation of the hedging point based on an analysis of a single failure repair cycle also needs to be investigated. Development of heuristic rules based on the intuition gained from the two-part-type systems and an evaluation of their performance constitutes another area for research.

Appendix A

Calculation of parameters for the OODS policy

A.1 Calculation of moments of random variable t'

In this section, we describe a method to calculate the first and second moments of the random variable t' defined in Section 5.6. t' is the length of time it takes for a trajectory originating at time t_0 from $x = (x_1, x_1)$ in Region 1c with $\alpha(t_0) = 1$ to reach the negative x_2 axis.

Consider the surplus behavior of Part Type 1 during the interval $[t_0, t_0 + t']$. If there were no machine failures until the trajectory reaches the boundary, then the value of t' is equal to $\frac{x_1}{\mu_1 - d_1}$. The case where the machine fails before the boundary is reached is illustrated in Figure A-1.

In the figure, the first machine failure occurs at at time T_1. Let $x_1(T_1)$ denote the value of Part Type 1 surplus at time T_1. After time T_1 the surplus of Part Type 1 decreases until the machine is repaired. When the machine gets repaired, the surplus of Part Type 1 increases until either the surplus reaches 0 or the machine fails, whichever happens earlier. If the machine failure happens earlier, then the surplus again decreases until the next repair. In either case, there exists some future time instant at which Part Type 1 surplus is again equal to $x_1(T_1)$. Let t_1 denote the interval of time between T_1 and such an instant. Random variable t_1 has the same distribution as the random variable t^r defined in Section 5.5. After time $T_1 + t_1$, the surplus increases at a rate $\mu_1 - d_1$ until the machine next fails after a further time T_2 which leads to a repetition of the above scenario. This behavior continues until the

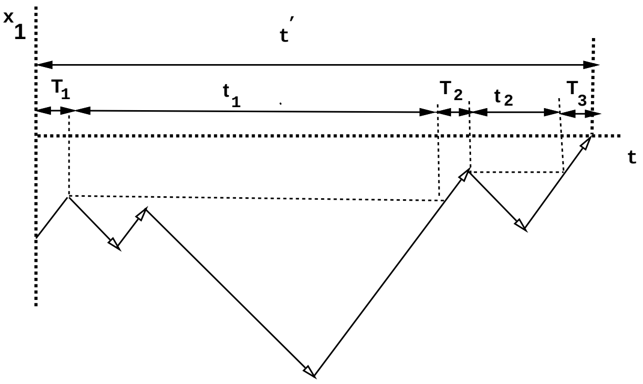

Figure A-1: Expression for random variable $t^{'}$

surplus eventually reaches a value of 0 at time $t^{'}$. It follows that $t^{'}$ can be expressed as a random sum of independent identically distributed random variables t_j as:

$$t^{'} = \sum_{j=1}^{n} t_j + \sum_{j=1}^{n+1} T_j \tag{A.1}$$

where each t_i has the same distribution as t^r in Section 5.5 and n is a random variable that corresponds to the number of arrivals of a Poisson process with rate p during an interval of length, $\frac{x_1}{\mu_1 - d_1}$. From Medhi (1992), we have

$$E\{n\} = p\frac{x_1}{\mu_1 - d_1} \tag{A.2}$$

$$E\{n\}^2 = p\frac{x_1}{\mu_1 - d_1}(1 + p\frac{x_1}{\mu_1 - d_1}) \tag{A.3}$$

The definition of each T_j implies that:

$$\sum_{j=1}^{n+1} T_j = \frac{x_1}{\mu_1 - d_1} \tag{A.4}$$

Substituting in (A.1) yields

$$t^{'} = \frac{x_1}{\mu_1 - d_1} + \sum_{j=1}^{n} t_j \tag{A.5}$$

Taking the expected value of both sides of (A.5) yields:

$$E\{t^{'}\} = \frac{x_1}{\mu_1 - d_1} + E\{n\}E\{t^r\}$$

Substituting for $E\{n\}$ from (A.2) we get

$$E\{t^{'}\} = \frac{x_1}{\mu_1 - d_1}(1 + pE\{t^r\}) \tag{A.6}$$

From (5.29) that

$$E\{t^r\} = \frac{\mu_1}{-pd_1 + r(\mu_1 - d_1)}$$

Substituting this in (A.6) yields after some manipulation,

$$E\{t^{'}\} = \frac{x_1}{\bar{\mu}_1 - d_1} \tag{A.7}$$

It is interesting that this expression is the same as the value for $t^{'}$ in the corresponding AD system described in Chapter 5. Squaring both sides in (A.5) gives

$$(t^{'})^2 = (\frac{x_1}{\mu_1 - d_1})^2 + 2\left(\sum_{j=1}^{n} t_j\right)\left(\frac{x_1}{\mu_1 - d_1}\right) + \left(\sum_{j=1}^{n}(t_j)^2\right)$$

Taking expected values of both sides,

$$E\{(t^{'})^2\} = \frac{x_1^2}{(\mu_1 - d_1)^2} + 2E\{n\}E\{t^r\} + E\{(\sum_{j=1}^{n} t_j)^2\}$$

The last term in the above expression can be evaluated from the formula for the second moment of a random sum of independent identically distributed random variables (Medhi, 1982). Therefore

$$E\{(\sum_{j=1}^{n} t_j)^2\} = E\{n\}\left[E\{(t^r)^2\} - (E\{t^r\})^2\right] + E\{n^2\}(E\{t^r\})^2$$

Substituting for the first and second moments of n from (A.2) and (A.3), the second moment of $t^{'}$ can now be written as:

$$\begin{aligned} E\{(t^{'})^2\} &= \frac{x_1^2}{(\mu_1 - d_1)^2} + 2p\frac{x_1}{\mu_1 - d_1}E\{t^r\} + \frac{px_1}{(\mu_1 - d_1)}\left(E\{(t^r)^2\} - (E\{t^r\})^2\right) \\ &\quad + E\{t^r\}\frac{px_1}{(\mu_1 - d_1)}\left(1 + \frac{px_1}{(\mu_1 - d_1)}\right) \end{aligned} \tag{A.8}$$

where

$$E\{t^r\} = \frac{\mu_1}{-pd_1 + r(\mu_1 - d_1)}$$

and from Corollary 3 in Chapter 4, the second moment of t^r is given by

$$E\{(t^r)^2\} = \frac{2\mu_1^2(\mu_1 - d_1)}{(r(\mu_1 - d_1) - pd_1)^3}$$

It can be noted that $E\{t^{'}\}$ is linear in x_1 while $E\{t^{'}\}^2$ contains only quadratic and linear terms in x_1.

A.2 Average cost of OODS policy

In this section, we describe a method to calculate the average cost $\bar{J}$ of the two-part type system under the OODS policy.

It was shown in Chapter 6 that for a feasible demand the stochastic process $(\boldsymbol{x}(t), \alpha(t))$ is a recurrent process and has a stationary distribution. The non-transient surplus region of this process for a OODS policy is contained in a cone with vertex at $\mathbf{0}$ and having as its extreme rays, the direction vector $(-d_1, -d_2)$ and the negative x_2 axis.

In this non-transient region, the control for Part Type 1 depends only on the Part Type 1 surplus. Thus Part Type 1 behaves like the single part type of [2]. The average cost of Part Type 1 denoted by $\bar{J}_1$ is therefore given by (Gershwin, 1993)

$$\bar{J}_1 = \frac{g_1^- p\mu_1 d_1}{(r+p)(r(\mu_1 - d_1) - pd_1)}$$

In the non-transient surplus region, Part Type 2 is produced only when the surplus is on the $1c - 2b$ boundary at a rate $u_2^b = \mu_2(1 - \frac{\mu_1}{d_1})$. In Chapter 5 it was shown that the behavior of Part Type 2 is same as that in the single-part-type in Chapter 4 with general repair time t^r having a maximum production rate u_2^b. It was also shown that the first and second moments of t^r are given by Corollary 3.

Using Corollary 4 in Chapter 4, the average value of Part Type 2 surplus is therefore given by:

$$E\{x_2\} = \frac{p\mu_2 d_2(1 - \frac{\mu_1}{d_1})E\{(t^r)^2\}}{2(1 + pE\{t^r\})(\mu_2(1 - \frac{\mu_1}{d_1}) - d_2 - pd_2 E\{t^r\})} \tag{A.9}$$

Under the OODS policy, the surplus for Part Type 2 is never positive in the non-transient surplus space. Thus the average cost of Part Type 2 surplus $\bar{J}_2$ is given by

$$\bar{J}_2 = g_2^- \limsup_{T\to\infty} \int_0^T x(t)dt$$

which yields

$$\bar{J}_2 = g_2^- E\{x_2\} \tag{A.10}$$

From (A.9) and (A.10) we get,

$$\bar{J}_2 = \frac{g_2^- p\mu_2 d_2(1-\frac{\mu_1}{d_1})E\{(t^r)^2\}}{2(1+pE\{t^r\})(\mu_2(1-\frac{\mu_1}{d_1})-d_2-pd_2E\{t^r\})^2}$$

The average cost for the system is the sum of the average costs of the two part types. Thus

$$\bar{J} = \frac{g_1^- p\mu_1 d_1}{(r+p)(r(\mu_1-d_1)-pd_1)} + \frac{g_2^- p\mu_2 d_2(1-\frac{\mu_1}{d_1})E\{(t^r)^2\}}{2(1+pE\{t^r\})(\mu_2(1-\frac{\mu_1}{d_1})-d_2-pd_2E\{t^r\})^2} \tag{A.11}$$

A.3 Dynkin's equation

In this section, we reproduce Dynkin's equation in the form given in Bielecki and Kumar (1988). In this form, Dynkin's equation yields an expression for the expected cost incurred by an optimal policy over an interval $[0, S]$ in terms of the average cost $\bar{J}$ and the differential cost $W(\boldsymbol{x}, \alpha)$ of the policy. Specifically Dynkin's equation states that

$$E\{\int_0 Sg(\boldsymbol{x}(s))ds\} = \bar{J}S + E\{W(\boldsymbol{x}, \alpha(S))\} - W(\boldsymbol{x}(0), \alpha(0)) \tag{A.12}$$

Appendix B

Canonical form of Bellman equation

B.1 Bellman equation in Region $2a$

Let x_1, x_2 denote the given coordinate system and $W(x_1, x_2)$ be some given function. In the following we use subscripts to denote derivatives. For example W_{x_1} represent the partial derivative of W with respect to x_1 and $W_{x_1 x_1}$ denote the partial derivative of W_{x_1} with respect to x_1. Consider a new coordinate system given by

$$\begin{aligned} \xi &= \xi(x_1, x_2) \\ \eta &= \eta(x_1, x_2) \end{aligned}$$

The conversion of derivatives between the two coordinate system is given by (see Myint-U, 1987)

$$\begin{aligned} W_{x_1} &= W_\xi \xi_{x_1} + W_\eta \eta_{x_1} \\ W_{x_2} &= W_\xi \xi_{x_2} + W_\eta \eta_{x_2} \end{aligned}$$

Let the hyperbolic equation in the x_1, x_2 system be given by

$$AW_{x_1 x_1} + BW_{x_1 x_2} + CW_{x_2 x_2} + DW_{x_1} + EW_{x_2} + FW = G$$

where A, B, C, D, E and F are constants while G is some function of x_1 and x_2. Then the partial differential equation in the (ξ, η) coordinate system is given by (see Myint-U, 1987)

$$A^\star \hat{W}_{\xi\xi} + B^\star \hat{W}_{\xi\eta} + C^\star \hat{W}_{\eta\eta} + D^\star \hat{W}_\xi + E^\star \hat{W}_\eta + F^\star \hat{W} = G^\star$$

where

$$\begin{aligned}
A^\star &= A\xi_{x_1}^2 + B\xi_{x_1}\xi_{x_2} + C\xi_{x_1}^2 \\
B^\star &= 2A\xi_{x_1}\eta_x + B(\xi_{x_1}\eta_{x_2} + \xi_{x_2}\eta_{x_1}) + 2C\xi_{x_2}\eta_{x_2} \\
C^\star &= A\eta_{x_1}^2 + B\eta_{x_1}\eta_{x_2} + C\eta_{x_2}^2 \\
D^\star &= A\xi_{x_1x_1} + B\xi_{x_1x_2} + C\xi_{x_2x_2} + D\xi_{x_1} + E\xi_{x_2} \\
E^\star &= A\eta_{x_1x_1} + B\eta_{x_1x_2} + C\eta_{x_1x_2} + D\eta_{x_1} + E\eta_{x_2} \\
F^\star &= F \\
G^\star &= G(\xi, \eta)
\end{aligned}$$

In Region $2a$ (see Section 5.8) the (ξ, η) coordinate system is

$$\begin{aligned}
\xi &= x_2 + \frac{(\mu_2 - d_2)}{d_1} x_1 \\
\eta &= x_2 - \frac{d_2}{d_1} x_1
\end{aligned}$$

Thus,

$$\begin{aligned}
W_{x_1} &= W_\xi + W_\eta \\
W_{x_2} &= -\frac{d_2}{d_1} W_\eta + \frac{\mu_2 - d_2}{d_1} W_\xi
\end{aligned} \tag{B.1}$$

The coefficients of the partial differential equation in the (ξ, η) coordinate system are given by

$$\begin{aligned}
A &= -d_1^2 \\
B &= d_1(\mu_2 - 2d_2) \\
C &= d_2(\mu_2 - d_2) \\
D &= -(r + p)d_1
\end{aligned}$$

$$
\begin{aligned}
E &= -d_2 p + r(\mu_2 - d_2) \\
F &= 0 \\
G &= J^{\star}(r+p) - (g_1^{+}x_1 - g_2^{-}x_2)(r+p) - d_1 g_1^{+} + d_2 g_2^{-}
\end{aligned}
$$

Thus the coefficients in the (ξ, η) coordinate system are given by:

$$
\begin{aligned}
A^{\star} &= 0 \\
B^{\star} &= \mu_2^2 \\
C^{\star} &= 0 \\
D^{\star} &= -p\mu_2 \\
E^{\star} &= r\mu_2 \\
F^{\star} &= 0 \\
G^{\star} &= \bar{J}(r+p) - (\frac{g_1^{+}d_1 - g_2^{-}d_2}{\mu_2})\xi + (\frac{g_1^{+}d_1 + g_2^{-}(\mu_2 - d_2)}{\mu_2})\eta - d_1 g_1^{+} + d_2 g_2^{-}
\end{aligned}
$$

In other words, the partial differential equation can be expressed as:

$$
\hat{W}_{\xi\eta} = \frac{p}{\mu_2}^2 \hat{W}_{\xi} - \frac{r}{\mu_2}^2 \hat{W}_{\eta} + \frac{1}{\mu_2^2} G^{\star}(\xi, \eta) \tag{B.2}
$$

From [20], this partial differential equation (B.2) can be reduced to its canonical form by changing the dependent variable to $U(\xi, \eta)$ where

$$
U(\xi, \eta) = \hat{W}(\xi, \eta) e^{\frac{r}{\mu_2}\xi - \frac{p}{\mu_2}\eta} \tag{B.3}
$$

The canonical form of (B.2) is given by:

$$
U_{\xi,\eta} = -\frac{pr}{\mu_2^2} U_{\xi\eta} + G(\xi, \eta)
$$

where

$$
\begin{aligned}
G(\xi, \eta) \;\; = \;\; & \frac{e^{\frac{r\xi - p\eta}{\mu_2}}}{\mu_2^2} \Bigg[\bar{J}(r+p) - (\frac{g_1^{+}d_1 - g_2^{-}d_2}{\mu_2})\xi \\
& + (\frac{g_1^{+}d_1 + g_2^{-}(\mu_2 - d_2)}{\mu_2})\eta - d_1 g_1^{+} + d_2 g_2^{-} \Bigg]
\end{aligned}
$$

B.2 $0b - 2a$ boundary condition

In this section, we express the boundary conditions along the $0b - 2a$ boundary in terms of the ξ, η coordinate system and the dependent variable $U(\xi, \eta)$. The notation used in this chapter is the same as that in Section 5.8 of Chapter 5.

Consider the boundary condition specified in (5.83) in Chapter 5. It can be recalled that this equation states that

$$\lim_{x_1 \to 0^+} \frac{d}{dx_1} W^{0b-2a}(x_1) = 0 \tag{B.4}$$

From the definition of (ξ, η) it follows that

$$x_1 = \frac{d_1}{\mu_2}(\xi - \eta) \tag{B.5}$$

$$x_2 = \frac{d_2}{\mu_2}\xi + (1 - \frac{d_2}{\mu_2})\eta \tag{B.6}$$

The $0b - 2a$ boundary is represented by $x_2 = 0$ in the old coordinate system and by $\eta = -\frac{d_2}{\mu_2 - d_2}\xi$ in the new coordinate system. Using (B.5) along this boundary yields

$$x_1 = \frac{d_1}{\mu_2}(\xi + \frac{d_2}{\mu_2 - d_2}\xi)$$

which on simplification gives

$$x_1 = \frac{d_1}{\mu_2 - d_2}\xi \tag{B.7}$$

Thus

$$W^{0b-2a}(x_1) = W^{0b-2a}(\frac{d_1}{\mu_2 - d_2}\xi) \tag{B.8}$$

Using the chain rule of differentiation for the right side of the above expression yields

$$\frac{dW^{0b-2a}(x_1)}{dx_1} = \left[\frac{d\hat{W}^{0b-2a}(\xi)}{d\xi}\right]\frac{d\xi}{dx_1}$$

Using (B.7) gives

$$\frac{d}{dx_1}W^{0b-2a}(x_1) = \left[\frac{d}{d\xi}\hat{W}^{0b-2a}(\xi)\right]\frac{(\mu_2 - d_2)}{d_1} \tag{B.9}$$

Equation (B.9) implies that the boundary condition B.4) in the new coordinate system becomes

$$\frac{d}{d\xi}W^{0b-2a}(\xi) = 0 \tag{B.10}$$

Let $\phi(\xi)$ be defined as:

$$\phi(\xi) = \hat{W}^{0b-2a}(\xi)$$

and let

$$\phi^{'}(\xi) = \frac{d}{d\xi}\phi(\xi)$$

Along the $0b-2a$ boundary, equation (B.3) becomes

$$U^{0b-2a}(\xi) = \phi(\xi)e^{\frac{\xi}{\mu_2}(r+\frac{pd_2}{\mu_2-d_2})}$$

Differentiating both sides with respect to ξ yields:

$$\frac{d}{d\xi}U^{0b-2a}(\xi) = \phi^{'}(\xi)e^{\frac{\xi}{\mu_2}(r+\frac{pd_2}{\mu_2-d_2})} + \left(\frac{r}{\mu_2} + \frac{pd_2}{\mu_2(\mu_2-d_2)}\right)U^{0b-2a}(\xi) \tag{B.11}$$

Differentiating both sides of (B.3) with respect to η and rearranging terms we get:

$$\frac{\partial \hat{W}(\xi,\eta)}{\partial \eta} = e^{-(\frac{r\xi}{\mu_2}-\frac{p\eta}{\mu_2})}\frac{\partial U(\xi,\eta)}{\partial \eta} + \frac{p}{\mu_2}U(\xi,\eta)e^{-(\frac{r\xi}{\mu_2}-\frac{p\eta}{\mu_2})} \tag{B.12}$$

Taking the limit of both sides in (B.12) as $\eta \to -\frac{d_2}{\mu_2-d_2}\xi$ yields

$$\lim_{\eta\to-\frac{d_2}{\mu_2-d_2}\xi\ (\xi,\eta)\in \text{ Region } 2a} \frac{\partial \hat{W}(\xi,\eta)}{\partial \eta}$$
$$= \left[\lim_{\eta\to-\frac{d_2}{\mu_2-d_2}\ (\xi,\eta)\in \text{ Region } 2a} \frac{\partial U(\xi,\eta)}{\partial \eta}\ \frac{p}{\mu_2}U^{0b-2a}(\xi)\right] e^{-\frac{\xi}{\mu_2}(r+\frac{pd_2}{\mu_2-d_2})} \tag{B.13}$$

It can be recalled from (5.81) that a boundary condition along the $0b-2a$ boundary is given by:

$$\lim_{x_2\to0\quad \boldsymbol{x}\in \text{ Region } 2a}\left[(\mu_2-d_2)\frac{\partial W((x_1,x_2),1)}{\partial x_2} - d_1\frac{\partial W((x_1,x_2),1)}{\partial x_1}\right] = -d_1\frac{d}{dx_1}W^{0b-2a}(x_1,1)$$

Using equations (B.1) and (B.9) to express the right and left sides of this equation, the above equation is equivalent to

$$\lim_{\eta \to -\frac{d_2}{\mu_2 - d_2}\ (\xi,\eta)\in \ \text{Region}2a} \mu_2 \frac{\partial \hat{W}(\xi,\eta)}{\partial \eta} = (\mu_2 - d_2)\phi^{'}(\xi) \tag{B.14}$$

From (B.13) and (B.14), we get

$$(\mu_2 - d_2)\phi^{'}(\xi) = \mu_2 e^{-\frac{\xi}{\mu_2}(r + \frac{pd_2}{\mu_2 - d_2})} \left[\lim_{\eta \to -\frac{\xi d_2}{d_1}\ (\xi,\eta)\in \text{ Region } 2a} \frac{\partial U(\xi,\eta)}{\partial \eta} + pU^{0b-2a}(\xi) \right]$$

which on simplification yields

$$\phi^{'}(\xi) e^{\frac{\xi}{\mu_2}(r + \frac{pd_2}{\mu_2 - d_2})} = \frac{\mu_2}{\mu_2 - d_2} \left[\lim_{\eta \to -\frac{\xi d_2}{d_1}\ (\xi,\eta)\in \text{ Region } 2a} \frac{\partial U(\xi,\eta)}{\partial \eta} \right] + \frac{p}{\mu_2 - d_2} U^{0b-2a}(\xi) \tag{B.15}$$

Substituting for the left side of (B.15) using (B.11) and rearranging terms we get

$$\begin{aligned} &\frac{\mu_2}{(\mu_2 - d_2)} \lim_{\eta \to -\frac{\xi d_2}{d_1}\ (\xi,\eta)\in \text{ Region } 2a} \frac{\partial U(\xi,\eta)}{\partial \eta} \\ = \ &\frac{dU^{0b-2a}(\xi)}{d\xi} - \left[\frac{p}{\mu_2 - d_2} - \frac{r}{\mu_2} + \frac{pd_2}{\mu_2(\mu_2 - d_2)} \right] U^{0b-2a}(\xi) \end{aligned}$$

which gives

$$\lim_{\eta \to -\frac{\xi d_2}{d_1}\ (\xi,\eta)\in \text{ Region } 2a} \frac{\partial U(\xi,\eta)}{\partial \eta} = \frac{\mu_2 - d_2}{\mu_2} \frac{d}{d\xi} U^{0b-2a}(\xi) - \left[\frac{p}{\mu_2} - \frac{r(\mu_2 - d_2)}{\mu_2^2} + \frac{pd_2}{\mu_2^2} \right] U^{0b-2a}(\xi)$$

Let

$$k_1 = \frac{\mu_2 - d_2}{\mu_2}$$

and

$$k_2 = -\frac{p}{\mu_2} + \frac{r(\mu_2 - d_2)}{\mu_2^2} - \frac{pd_2}{\mu_2^2}$$

Therefore,

$$\lim_{\eta \to -\frac{\xi d_2}{d_1}\ (\xi,\eta)\in \text{ Region } 2a} \frac{\partial U(\xi,\eta)}{\partial \eta} = k_1 \frac{d}{d\xi} U^{0b-2a}(\xi) + k_2 U^{0b-2a}(\xi) \tag{B.16}$$

In Region $1a$, the constants k_3 and k_4 replace k_1 and k_2. Using the symmetry with

Region $2b$, these constants are defined as

$$k_3 = \frac{\mu_1 - d_1}{\mu_1}$$

and

$$k_4 = -\frac{p}{\mu_1} + \frac{r(\mu_1 - d_1)}{\mu_1^2} - \frac{pd_1}{\mu_1^2}$$

Appendix C

Derivation of Necessary Optimality Conditions

C.1 Optimality conditions in Region $2b$

In this section, we derive conditions on the coefficients for which the function $L(x_1, x_2)$ defined as

$$L(x_1, x_2) = (l_1 x_1 + l_2 x_2 + l_0)e^{ax_1} + f_1 x_1 + f_2 x_2 + f_0 \tag{C.1}$$

takes non-positive values at all points belonging to a region that is defined by:

$$\begin{aligned} x_1 + \frac{d_1}{\mu_2 - d_2} x_2 &\leq 0 \\ x_1 &\geq 0 \end{aligned} \tag{C.2}$$

This is equivalent to finding conditions on the coefficients for which this maximum value of the following non-linear program is always non-positive.

$$\max \quad L(x_1, x_2) \tag{C.3}$$

subject to

$$\begin{aligned} x_1 + \frac{d_1}{\mu_2 - d_2} x_2 &\leq 0 \\ x_1 &\geq 0 \end{aligned}$$

The feasible region of this non-linear program is a cone with its vertex at the origin that has the two lines $x_1 + \frac{d_1}{\mu_2 - d_2} x_2 = 0$ and $x_1 = 0$ as its extreme rays.

There are three possibilities for any point where the objective function attains its maximum value. Such a point could lie on on of these extreme rays or in the interior of the region. We consider each case separately and write down necessary and sufficient conditions for a point to be an optimum solution in each case. We then specify that the value of the objective function at any point that satisfies these conditions be non-positive. Let ∇L and $\nabla^2 L$ denote the gradient and the Hessian matrix of L. That is,

$$\nabla L = \left(\left[a l_1 x_1 + a l_2 x_2 + (l_1 + a l_3) \right] e^{a x_1} + f_1, l_2 e^{a x_1} + f_2 \right) \tag{C.4}$$

and

$$\nabla^2 L = \begin{bmatrix} [a^2 l_1 x_1 + a^2 l_2 x_2 + (a l_1 + a^2 l_3)] e^{a x_1} & a l_2 e^{a x_1} \\ a l_2 e^{a x_1} & 0 \end{bmatrix} \tag{C.5}$$

- Case 1: An optimum solution lies on the line $x_1 = 0$.

  In this case, the objective function becomes

  $$L(0, x_2) = (l_2 + f_2) x_2 + (l_0 + f_0)$$

  Over the feasible region, x_2 is non-positive. Sufficient conditions for a point $(0, x_2^*)$ to be have a finite maximum (Luenberger, 1973) are

  $$l_2 + f_2 \geq 0 \tag{C.6}$$

  If this condition is satisfied, then a maximum value of the objective function is always at $(0, 0)$. This value is less than zero if and only if

  $$l_0 + f_0 \leq 0 \tag{C.7}$$

- Case 2: An optimum solution is a point other than $(0, 0)$ on the line $x_2 = -\frac{\mu_2 - d_2}{-d_1} x_1$.

In this case, the objective function becomes

$$L_1(x_1) = L(x_1, -\frac{\mu_2 - d_2}{-d_1} x_1)$$

Therefore

$$L_1(x_1) = \left[(l_1 - \frac{\mu_2 - d_2}{d_1} l_2) x_1 + l_0 \right] e^{a x_1} + (f_1 - \frac{\mu_2 - d_2}{d_1} f_2) x_1 + f_0$$

Over the feasible region we have $x_1 > 0$. Sufficient conditions for a point x_1^* to be have a finite maximum are that the first derivative at x_1^* be equal to zero and the second derivative at x_1^* be non-postitive. This gives

$$\left[(l_1 - \frac{\mu_2 - d_2}{d_1} l_2) a x_1^* + (a l_0 + (l_1 - \frac{\mu_2 - d_2}{d_1})) \right] e^{a x_1^*} + f_1 - \frac{\mu_2 - d_2}{d_1} f_2 = 0 \quad \text{(C.8)}$$

and

$$\left[(l_1 - \frac{\mu_2 - d_2}{d_1} l_2) a^2 x_1^* + (a^2 l_0 + a(l_1 - \frac{\mu_2 - d_2}{d_1})) \right] e^{a x_1^*} \leq 0 \qquad \text{(C.9)}$$

Therefore, if there exists a point $x_1^* > 0$ that satisfies (C.8) and (C.9) then we the maximum value is less than zero if and only if

$$L_1(x_1^*) \leq 0 \qquad \text{(C.10)}$$

- Case 3: An optimum solution (x_1^*, x_2^*) lies in the interior of the feasible region. In this case, the first derivative conditions become

$$\nabla L(x_1^*, x_2^*) = (0, 0) \qquad \text{(C.11)}$$

Solving this for (x_1^*, x_2^*) using (C.4) yields

$$\begin{aligned} x_1^* &= \frac{1}{a} \ln \left(\frac{-f_2}{l_2} \right) \\ x_2^* &= \frac{\frac{f_1 l_2}{f_2} - (l_1 + a l_3) - l_1 \ln \left(\frac{-f_2}{l_2} \right)}{a l_2} \end{aligned} \qquad \text{(C.12)}$$

If this point (x_1^*, x_2^*) also satifies the feasibility conditions (C.2) and the second derivative condition

$$\nabla^2 L(x_1^*, x_2^*) \leq 0 \tag{C.13}$$

where $\nabla^2 L$ is defined in (C.5), then this point is indeed a local maximum.

We want the value of the objective function at such a point to to be non-positive. Therefore we want the coefficients to satisfy

$$L(x_1^*, x_2^*) \leq 0 \tag{C.14}$$

Now we derive conditions on the parameters for which (5.156) and (5.158) in Chapter 5 are satisfied.

A comparison of (5.156) and (C.1) gives $a = -\frac{r+p}{d_1}$, $l_1 = l_2 = 0$, $l_0 = k_2^2 b$, $f_0 = b_2^{2b}$, $f_1 = a_{12}^{2b}$, $f_2 = a_{22}^{2b}$. Thus

$$\nabla^2 L = \begin{bmatrix} k_2^{2b}(\frac{r+p}{d_1})^2 e^{-\frac{r+p}{d_1}x_1} & 0 \\ 0 & 0 \end{bmatrix}$$

From (5.56) it can be verified that $k_2^{2b} \leq 0$. Therefore L is a concave function and the second derivative conditions (C.9) and (C.13) are always satisfied. Since $a_{22}^{2b} > 0$ therefore (C.6) is satisfied. For the same reason (C.11) is never satisfied which means there is no interior optimum for (C.3). Equation (5.54) implies that (C.7) is satisfied for all parameter values. Therefore the only condition that the parameters need to satisfy is (C.10). This condition becomes

$$k_2^{2b} e^{-\frac{(r+p)}{d_1}x_1^*} + b_2^{2b} + \left(a_{12}^{2b} - \frac{(\mu_2 - d_2)}{d_1} a_{22}^{2b} \right) \leq 0 \tag{C.15}$$

if x_1^* given by (C.8) is positive. This value of x_1^* is

$$x_1^* = \frac{1}{a} \ln \left[\frac{-(\mu_2 - d_2)a_{22}^{2b} + d_1 a_{12}^{2b}}{k_2^{2b}(r+p)} \right] \tag{C.16}$$

Consider condition (5.158). A comparison of this condition with (C.1) gives

$$\begin{aligned} a &= -\frac{r+p}{d_1} \\ l_1 &= \frac{r+p}{d_1}\mu_1 k_1^{2b}, \end{aligned}$$

$$
\begin{aligned}
l_2 &= \frac{r+p}{d_1}\mu_1 k_2^{2b}, \\
l_0 &= \mu_2 k_2^{2b} - \mu_1 k_1^{2b} - \frac{r+p}{d_1}\mu_1 k_0^{2b}, \\
f_0 &= -\mu_1 b_1^{2b} + \mu_2 b_2^{2b}, \\
f_1 &= \mu_2 a_{12}^{2b} - \mu_1 a_{11}^{2b}, \\
f_2 &= \mu_2 a_{22}^{2b} - \mu_1 a_{12}^{2b}.
\end{aligned}
$$

In this case, the conditions on the parameters can be summarized as follows:

1. If the coefficients of $M(x_1, x_2)$ are such that (C.6) is satisfied, then the coefficients must be such that (C.7) is also satisfied.

2. If the coefficients are such that there exists a stricly positive solution to (C.8) and (C.9) then the coefficients must also satisfy (C.10).

3. If there exists a solution to (C.12) that satisfies the feasibility conditions given by (C.2) and second derivative conditions given by (C.13) then (C.14) must also be satisfied.

In this section, we have derived conditions on the parameters of the system in Chapter 5 for which (5.158) is always satisfied in Region $2b$.

C.2 Optimality conditions in Region $1b$

In this section, we derive conditions on the coefficients for which the function $M(x_1, x_2)$ defined as

$$M(x_1, x_2) = m_0 e^{\hat{a}x_2} + h_1 x_1 + h_2 x_2 + h_0 \tag{C.17}$$

takes only non-positive values at all points belonging to a region that is defined by:

$$
\begin{aligned}
x_1 + \frac{\mu_1 - d_1}{d_2} x_2 &\leq 0 \\
x_2 &\geq 0
\end{aligned} \tag{C.18}
$$

In particular we are interested in the case where the function M is the function given by (5.163). The constraint set in (C.18) represents Region $1b$ in Chapter 5. A comparison of (5.163) and (C.17) gives

$$\begin{aligned}
\hat{a} &= -\frac{(r+p)}{d_2} \\
m_0 &= -\mu_2 \frac{(r+p)}{d_2} k_0^{1b} \\
h_0 &= \mu_2 b_2^{1b} \\
h_1 &= \mu_2 a_{12}^{1b} - \mu_1 a_{11}^{1b} \\
h_2 &= \mu_2 a_{22}^{1b} - \mu_1 a_{12}^{1b}
\end{aligned}$$

As in Section C.1, the coefficients of M must be such that the optimal value of the following non-linear program is non-positive.

$$\max \quad M(x_1, x_2) \tag{C.19}$$

subject to

$$\begin{aligned}
x_1 + \frac{\mu_1 - d_1}{d_2} x_2 &\leq 0 \\
x_2 &\geq 0
\end{aligned}$$

The feasible region is a cone with its vertex at the origin. This cone has the two lines $x_1 + \frac{\mu_1 - d_1}{d_2} x_2 = 0$ and $x_2 = 0$ as its extreme rays.

Let ∇M and $\nabla^2 M$ denote the gradient and the Hessian matrix of M. That is,

$$\nabla M = (h_1, \hat{a} e^{a x_2} + h_2) \tag{C.20}$$

and

$$\nabla^2 M = \begin{bmatrix} 0 & 0 \\ 0 & \hat{a}^2 m_0 e^{\hat{a} x_2} \end{bmatrix} \tag{C.21}$$

As in Section C.1, we consider three cases. The first two correspond to an optimal solution lying on one of the two extreme rays. The third case analyzes an interior solution. The three cases are explained below.

- Case 1: An optimum solution lies on the line $x_2 = 0$.

In this case, the objective function becomes

$$M(x_1, 0) = h_1 x_1 + (m_0 + h_0)$$

Over the feasible region, x_1 is non-positive. Sufficient conditions for a point $(0, x_1^*)$ to be have a non-positive maximum (Luenberger, 1973) are

$$h_1 \geq 0 \tag{C.22}$$

and

$$m_0 + h_0 \leq 0 \tag{C.23}$$

Substituting for h_1 in (C.22) yields

$$\mu_2 a_{12}^{1b} - \mu_1 a_{11}^{1b} \geq 0 \tag{C.24}$$

Substituting for m_0 and h_0 in (C.23) and using (5.71) we can verify that (C.23) is always satisfied.

- Case 2: An optimum solution is a point other than $(0,0)$ on the line $x_1 = -\frac{\mu_1 - d_1}{-d_2} x_2$.

  In this case, the objective function becomes

$$M_1(x_2) = M(-\frac{\mu_1 - d_1}{-d_2} x_2, x_2)$$

Therefore

$$M_1(x_2) = m_0 e^{\hat{a} x_2} + (-\frac{\mu_1 - d_1}{d_2} h_1 + h_2) x_2 + h_0$$

Over the feasible region we have $x_2 > 0$. Sufficient conditions for a point x_2^* to be have a finite maximum are that the first derivative at x_2^* be equal to zero and the second derivative at x_2^* be non-postitive. This gives

$$m_0 \hat{a} e^{\hat{a} x_2} + (-\frac{\mu_1 - d_1}{d_2} h_1 + h_2) = 0 \tag{C.25}$$

and

$$m_0 \leq 0 \tag{C.26}$$

Let x_2^* be a solution to (C.25). Therefore,

$$x_2^* = \frac{1}{a} \ln \left(\frac{(-\frac{\mu_1 - d_1}{d_2} h_1 + h_2)}{m_0} \right) \tag{C.27}$$

If the coefficients of M are such that (C.26) is satisfied and $x_2^* > 0$ then we want the maximum value of M to be non-postive. That is

$$M_1(x_2^*) \leq 0 \tag{C.28}$$

This gives

$$m_0 e^{\hat{a} x_2^*} + (-\frac{\mu_1 - d_1}{d_2} h_1 + h_2) x_2^* + h_0$$

This is a condition in terms of the parameters of the system in Chapter 5.

- Case 3: The optimum is in the interior of the feasible region. A necessary condition for an interior point to be an optimum solution to (C.19) is that the gradient vector, ∇M, be zero at that point. But from (C.20), this is not possible for $h_1 > 0$. Therefore, the optimum solution is never in the interior of the feasible region.

In this section, we have derived conditions on the parameters of the system in Chapter 5 for which (5.163) is always satisfied in Region $1b$.

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