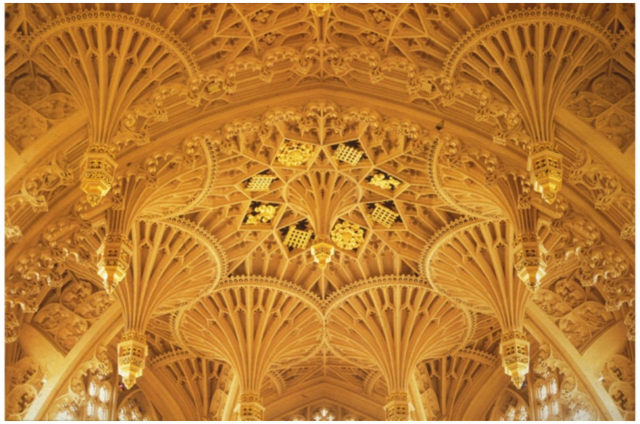
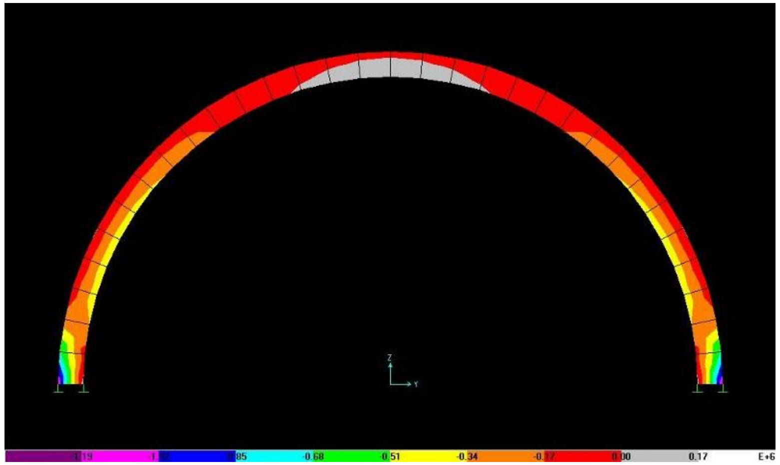
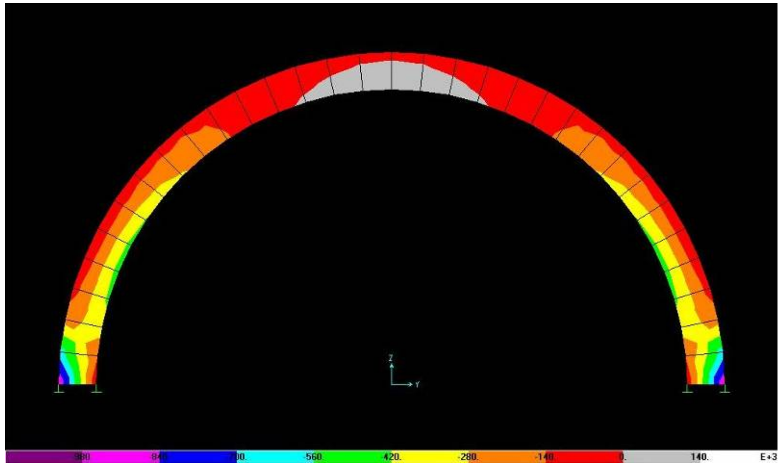
Lower-bound Analysis of Masonry Vaulted Structures

Philippe Block
Prof. John Ochsendorf

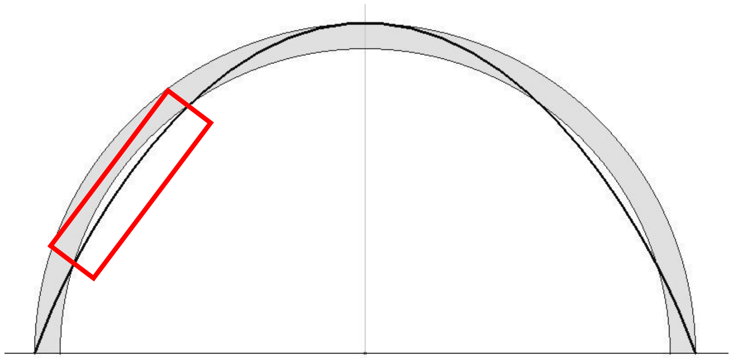
MIT

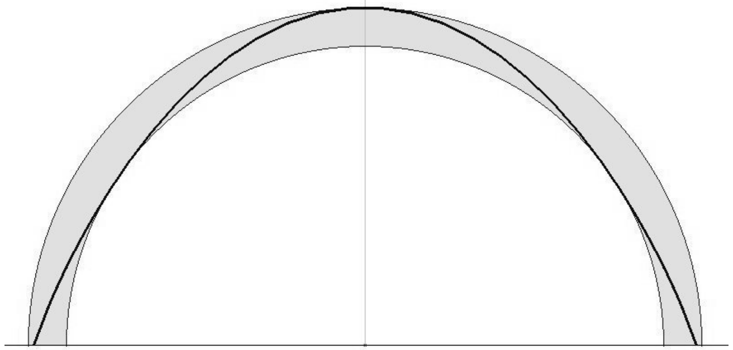



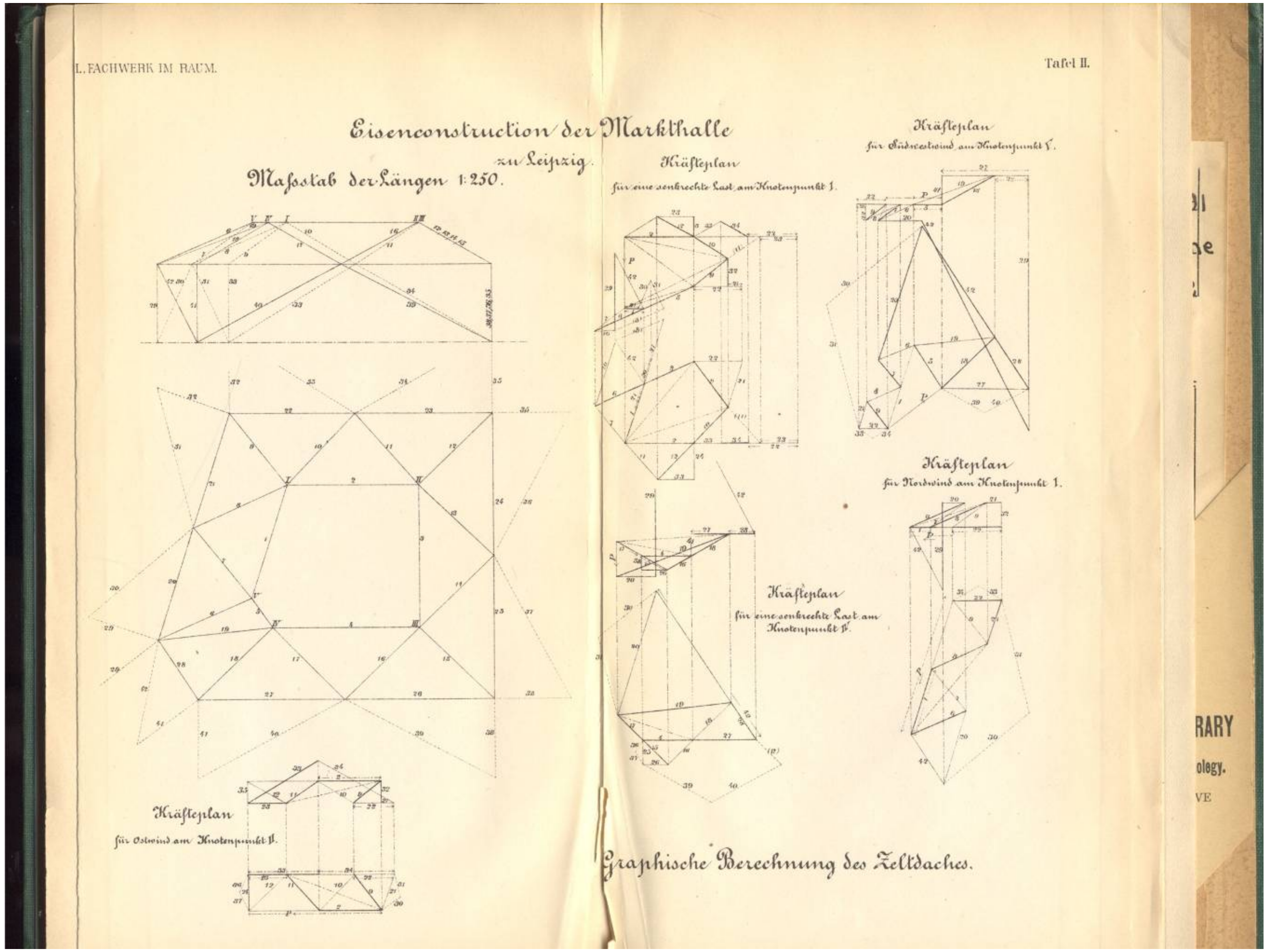


t/R = .08



t/R = .12




I. FACHWERK IM RAUM.
Tafel II.
Eisenconstruction der Markthalle zu Leipzig.
Masstab der Längen 1:250.
Kräfteplan
Graphische Berechnung des Zeltdaches.

(Föppl, 1892)

LOWER-BOUND ANALYSIS OF VAULTED MASONRY STRUCTURES

Philippe Block, John Ochsendorf

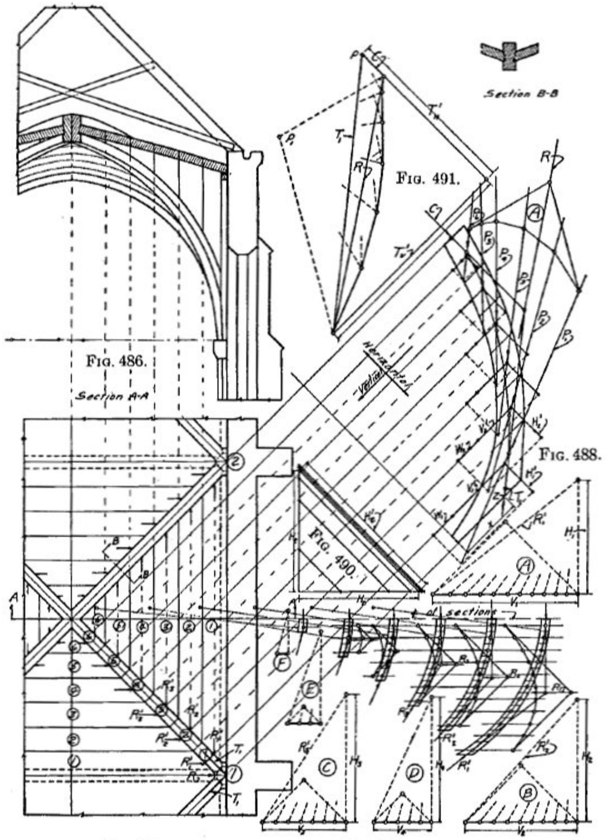
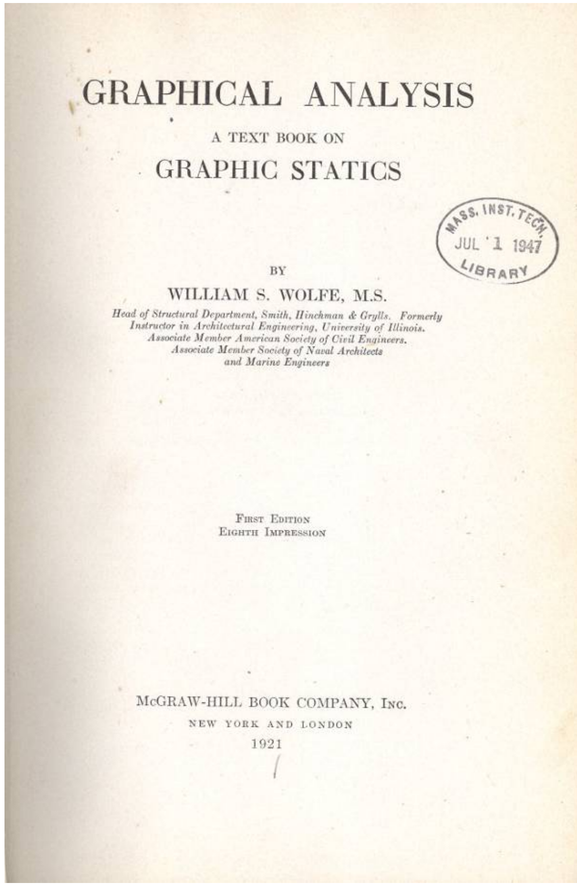


Fig. 487. Fig. 489.



GRAPHICAL ANALYSIS

A TEXT BOOK ON

GRAPHIC STATICS

BY

WILLIAM S. WOLFE, M.S.

Head of Structural Department, Smith, Hinchman & Grylls. Formerly Instructor in Architectural Engineering, University of Illinois. Associate Member American Society of Civil Engineers. Associate Member Society of Naval Architects and Marine Engineers

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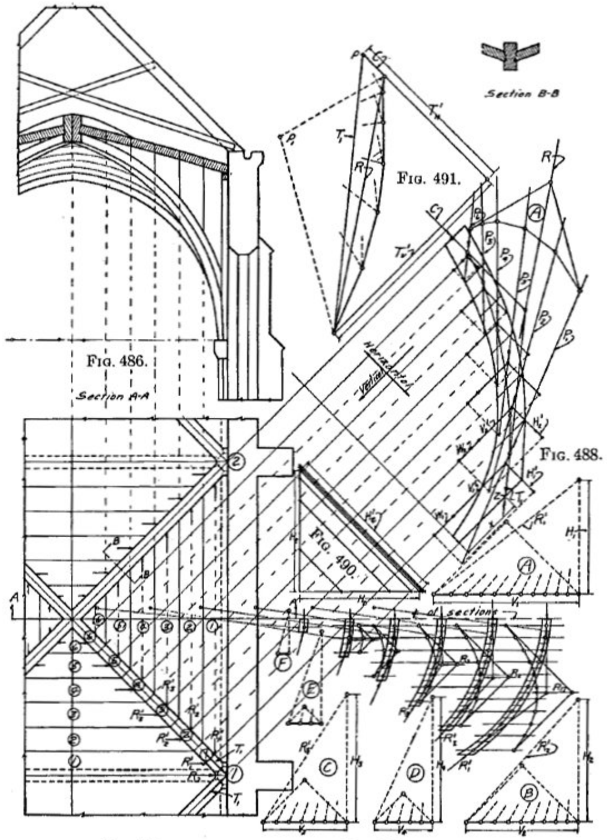
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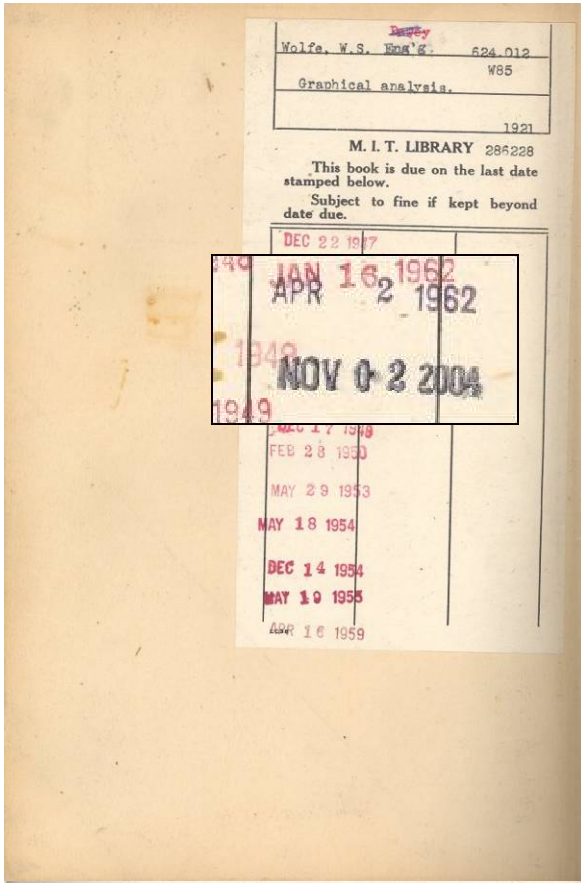
Fig. 491.
Fig. 486.
Section A-A
Fig. 488.
Fig. 490.
Fig. 487.
Fig. 489.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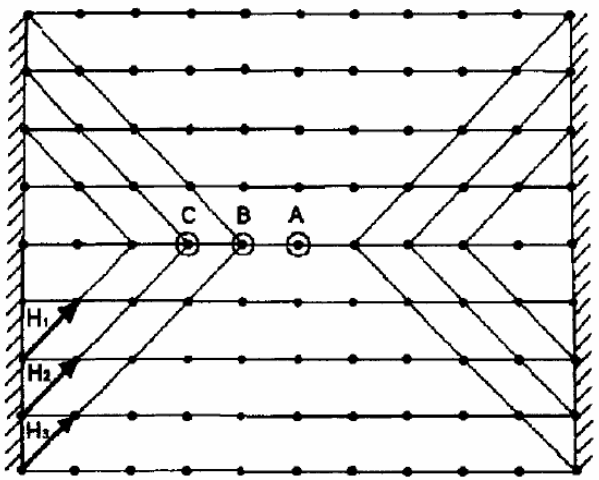
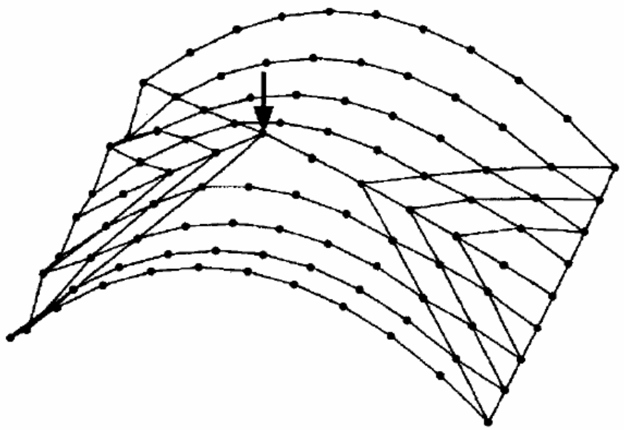
KEY REFERENCE

Force Network Method (D. O'Dwyer, 1999)

> Analysis of 3-D Masonry Structures





INTRODUCTION TO METHODOLOGY

Assumptions for the analysis of masonry structures

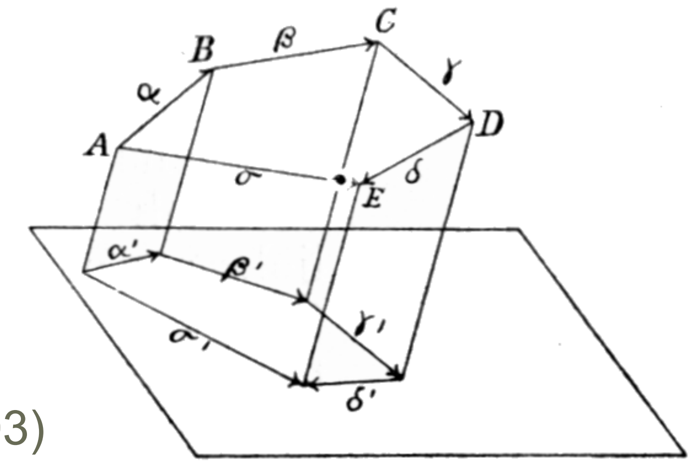
– **Limit analysis (lower bound theorem)**
  "*the vault will stand as long as a thrust network can be found that fits within its section***"**

– **Three main assumptions (introduced by Heyman)**
  - **No tensile strength**
  - **Infinite compressive strength (rigid)**
  - **Sliding does not occur**

INTRODUCTION TO METHODOLOGY

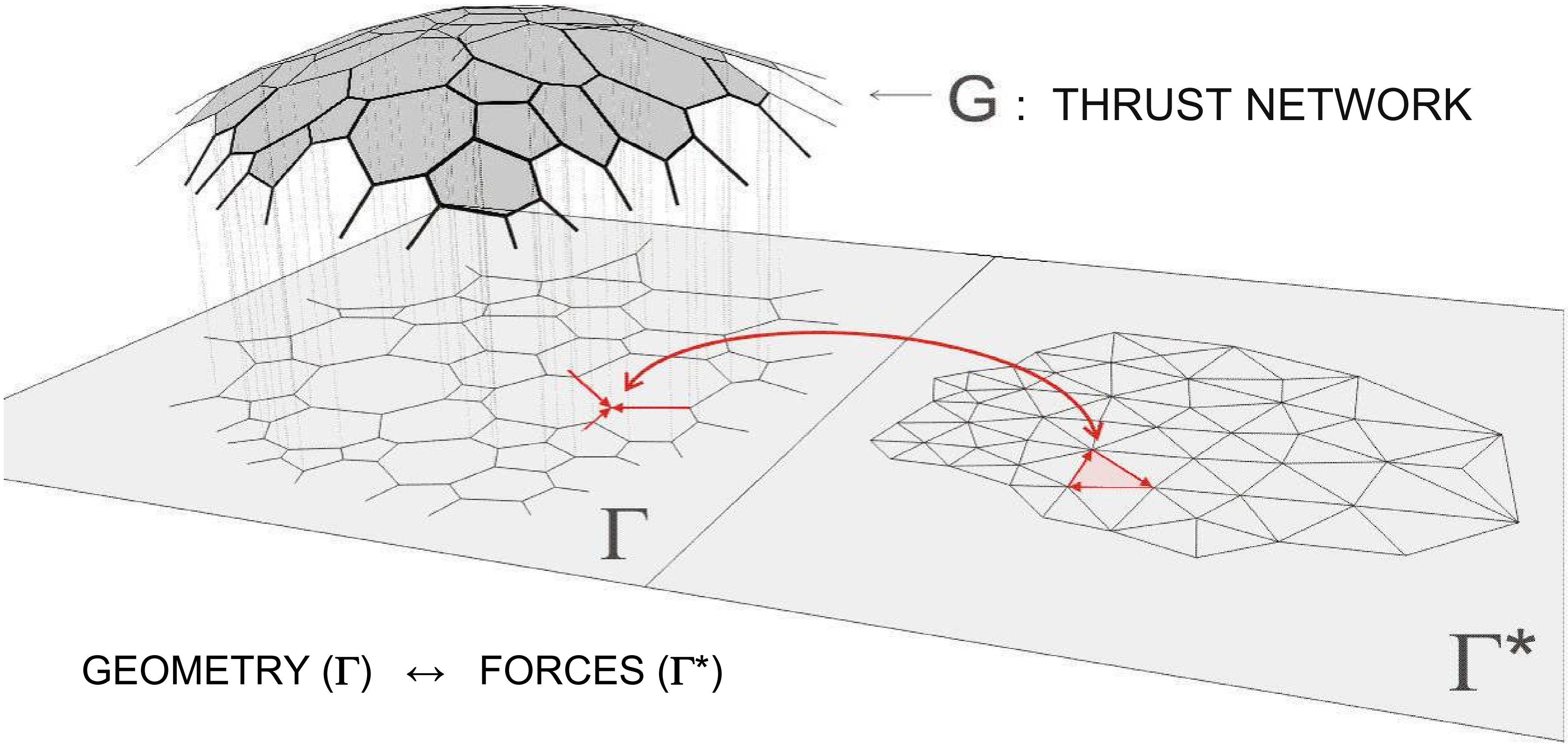
General assumptions

- **Parallel, such as gravitational, loading cases only**
- **Vault does not curl back onto itself**
- **Funicular solution (= compression-only or tension-only) is assumed**



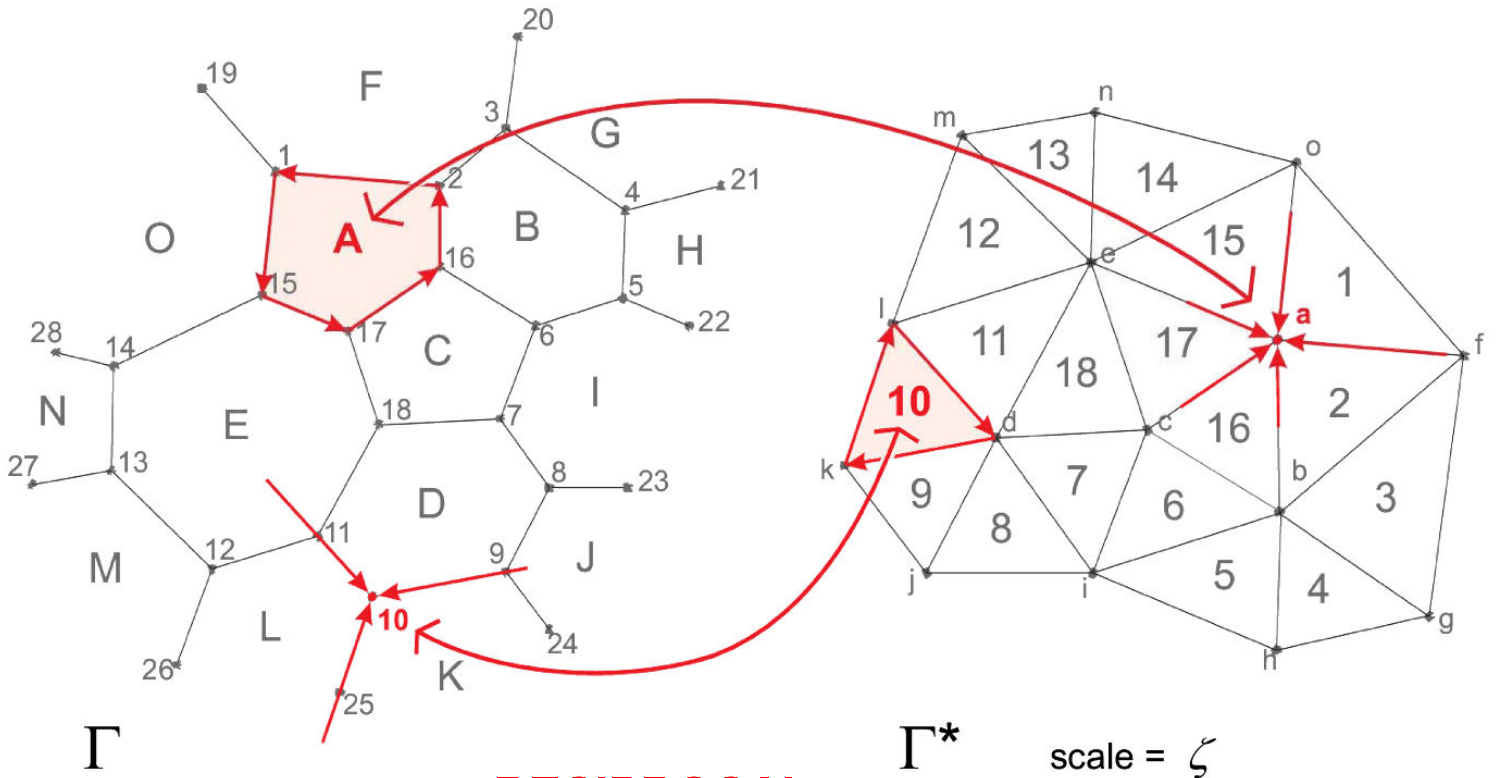
(Henrici, 1903)

THRUST NETWORK METHOD



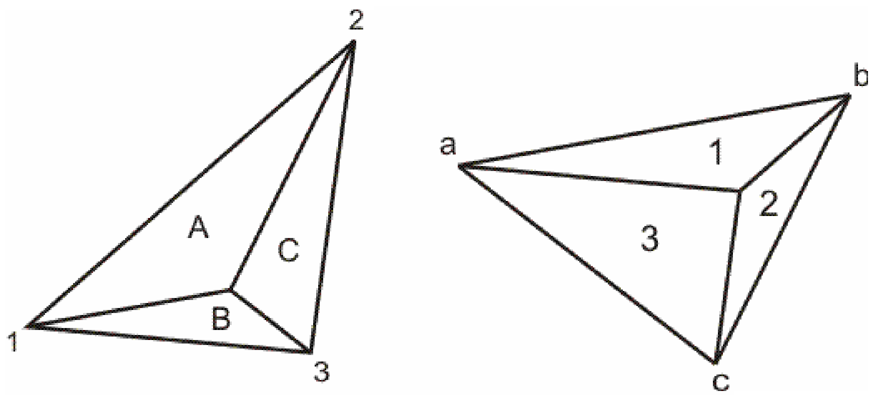
GEOMETRY (Γ) ↔ FORCES (Γ*)

THRUST NETWORK METHOD

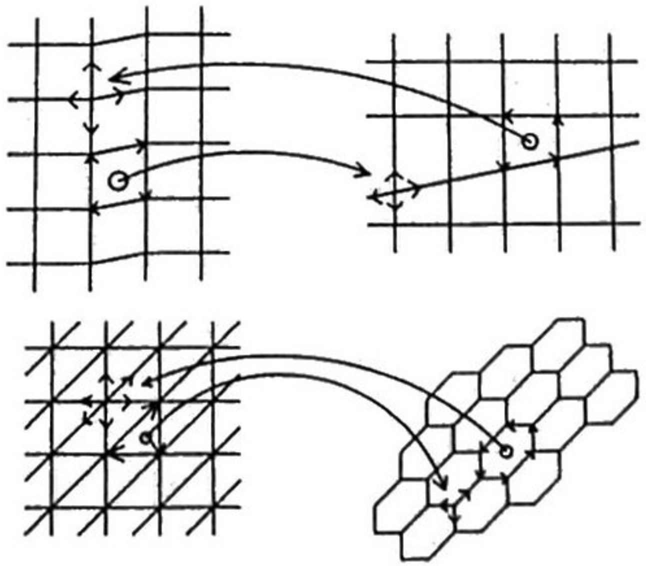


RECIPROCAL RELATIONSHIP

RECIPROCAL DIAGRAMS



(Maxwell, 1864)

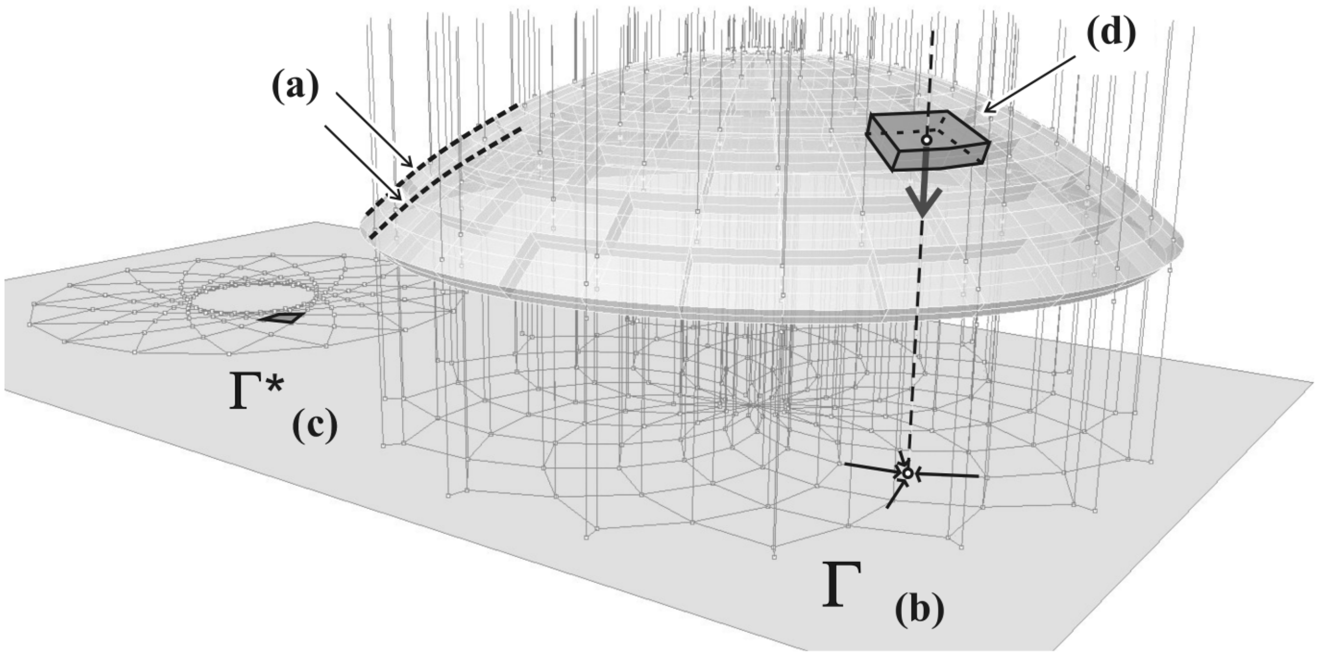


(Williams, 1986)

Maxwell (1864). Two plane figures are reciprocal when they consist of an equal number of lines, so that corresponding lines in the two figures are parallel, and corresponding lines which converge to a point in one figure form a closed polygon in the other.

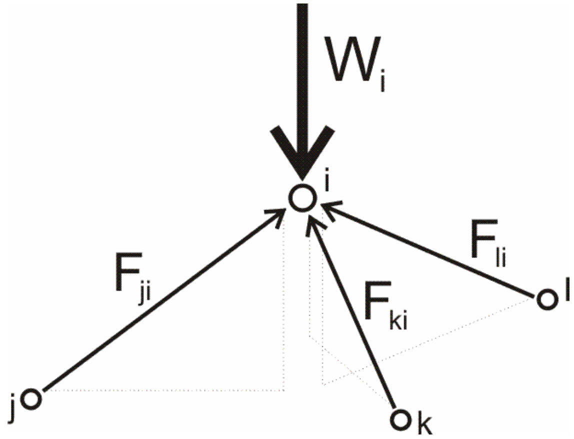
METHODOLOGY (OVERVIEW)

1. Choose a possible thrust network topology Γ
2. Attribute the correct weights per node
3. Compute reciprocal diagram Γ^* from Γ
4. Formulate problem as linear optimization problem to solve for G



METHODOLOGY

1. Nodal equilibrium in thrust network



$$\boxed{F_{ji}^{V} + F_{ki}^{V} + F_{li}^{V} = P_i}$$

$$F_{ji}^{H} \cdot \frac{(z_i - z_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} + F_{ki}^{H} \cdot \frac{(z_i - z_k)}{\sqrt{(x_i - x_k)^2 + (y_i - y_k)^2}} + F_{li}^{H} \cdot \frac{(z_i - z_l)}{\sqrt{(x_i - x_l)^2 + (y_i - y_l)^2}} = P_i$$

2. Using reciprocal grid Γ*

$$\boxed{F_{ji}^{H} = \zeta \cdot H_{i,j}^{*} \qquad F_{ki}^{H} = \zeta \cdot H_{i,k}^{*} \qquad F_{li}^{H} = \zeta \cdot H_{i,l}^{*}}$$

METHODOLOGY

3. Formulation

$$\left(\frac{H^*_{i,j}}{H_{i,j}}+\frac{H^*_{i,k}}{H_{i,k}}+\frac{H^*_{i,l}}{H_{i,l}}\right)\cdot z_i-\frac{H^*_{i,j}}{H_{i,j}}\cdot z_j-\frac{H^*_{i,k}}{H_{i,k}}\cdot z_k-\frac{H^*_{i,l}}{H_{i,l}}\cdot z_l-P_i\cdot r=0$$

where

$H_{i,j}$: length of the branch *i,j* in the primal grid Γ*

$H^*_{i,j}$: length of the branch *i,j* in the dual / reciprocal grid Γ*

$r=\frac{1}{\zeta}$: the scale of the dual grid Γ* (force magnitude)

$$C_i\cdot \boxed{z_i}+C_j\cdot \boxed{z_j}+C_k\cdot \boxed{z_k}+C_l\cdot \boxed{z_l}-P_i\cdot \boxed{r}=0$$

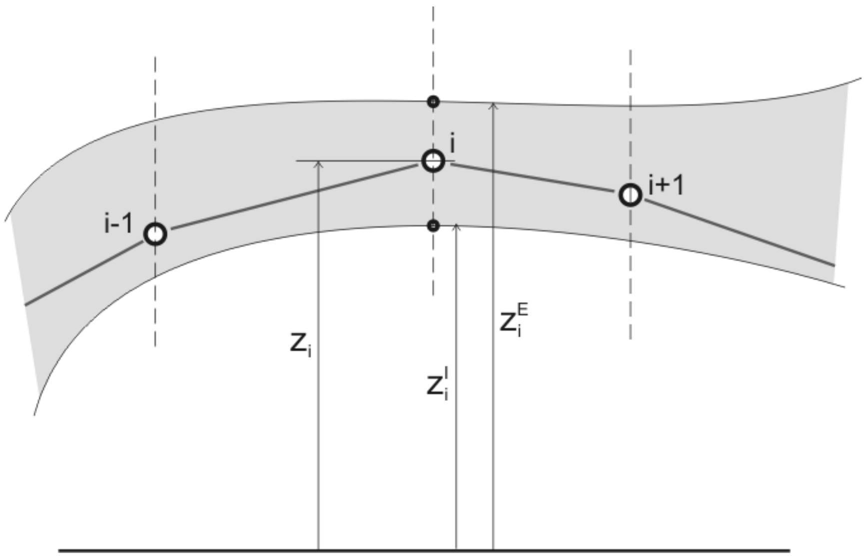
LINEAR

METHODOLOGY

4. Nodal height constraints

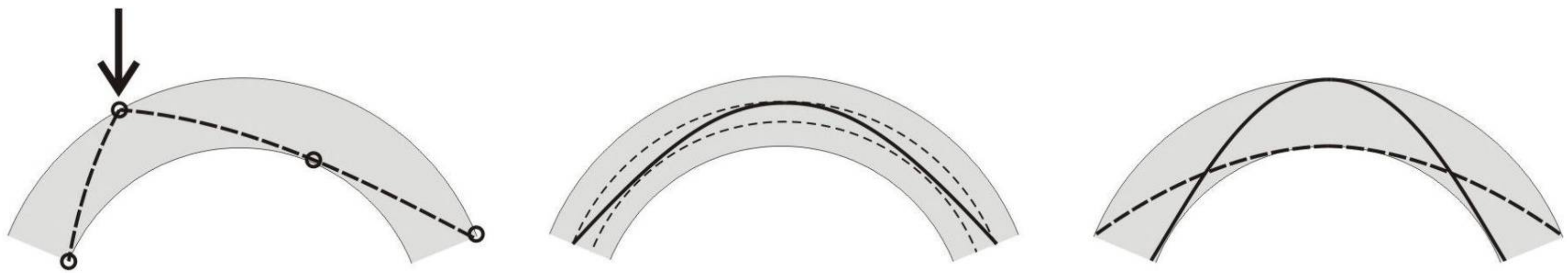
$$\boxed{z_i^I \leq z_i \leq z_i^E}$$

> Envelope of actual vault (or middle third)



METHODOLOGY

5. Objective function



min / max $r = \frac{1}{\zeta}$

METHODOLOGY

7. Overview formulation

– Linear constraints:

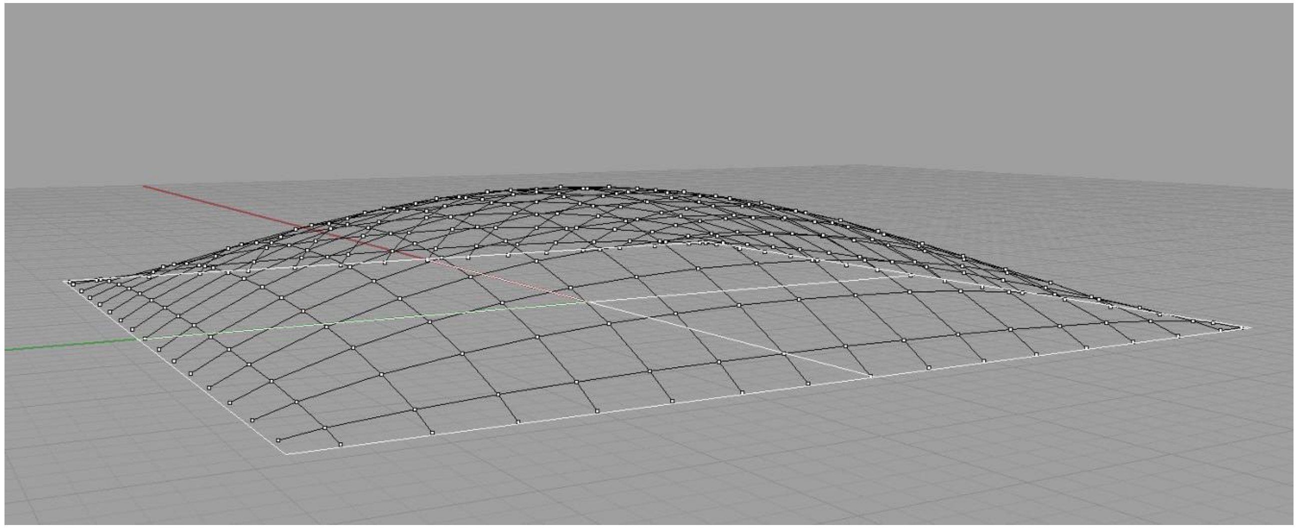
$$\boxed{F_{ji}^{V} + F_{ki}^{V} + F_{li}^{V} = P_i} \qquad \boxed{F_{ji}^{H} = \zeta \cdot H^{*}_{i,j}} \qquad \boxed{z_i^{I} \leq z_i \leq z_i^{E}}$$

$$C_i \cdot \boxed{z_i} + C_j \cdot \boxed{z_j} + C_k \cdot \boxed{z_k} + C_l \cdot \boxed{z_l} - P_i \cdot \boxed{r} = 0 \qquad -\boxed{z_i} \leq -z_i^{I} \quad \& \quad \boxed{z_i} \leq z_i^{E}$$

– Linear objective function: minimizing / maximizing $r = \dfrac{1}{\zeta}$

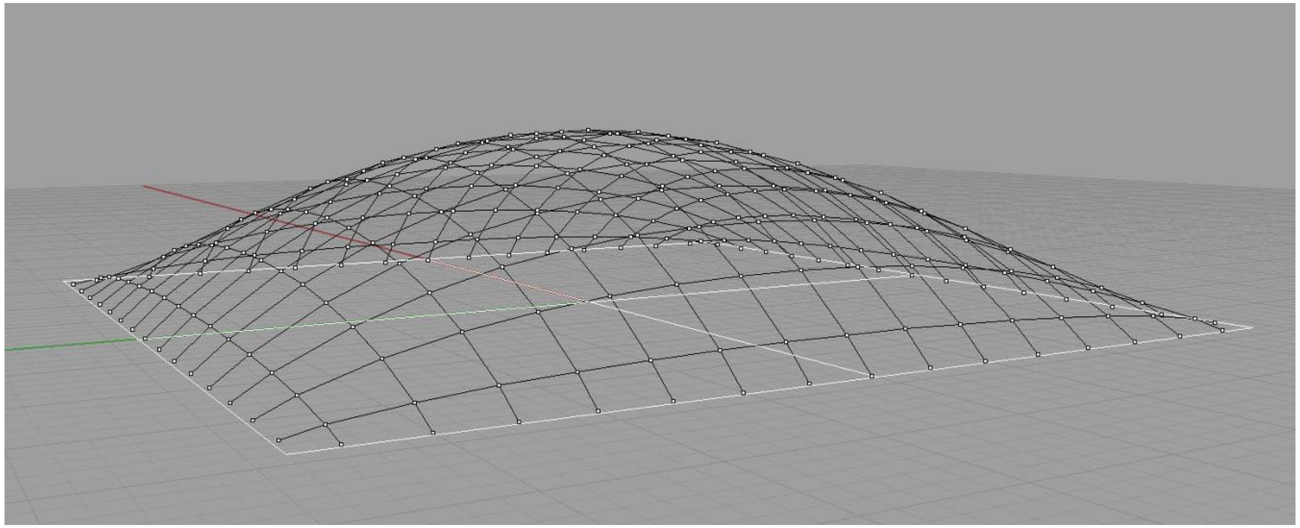
– Solve using **linear optimization** (Simplex method)

INDETERMINACY / PARAMETERS (1)



Δz_{max}	5	7.5	10
ζ	16.643	11.095	8.321

INDETERMINACY / PARAMETERS (1)



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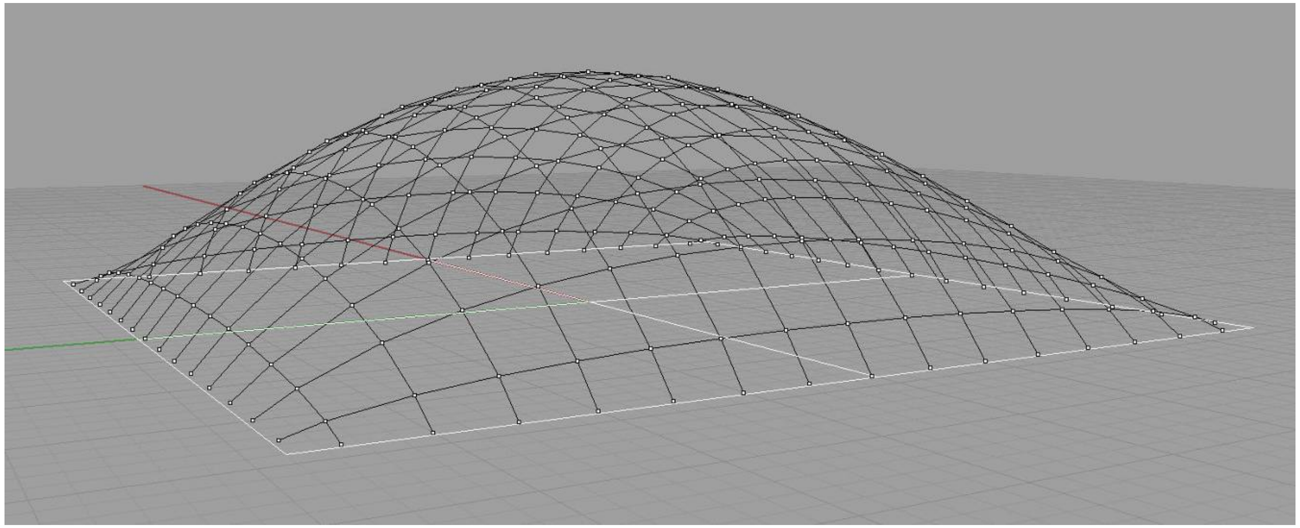
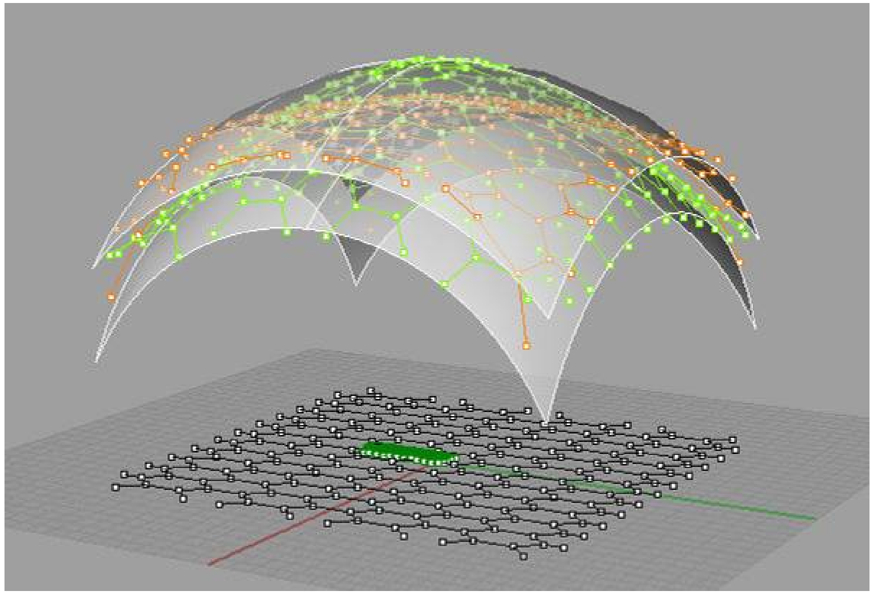
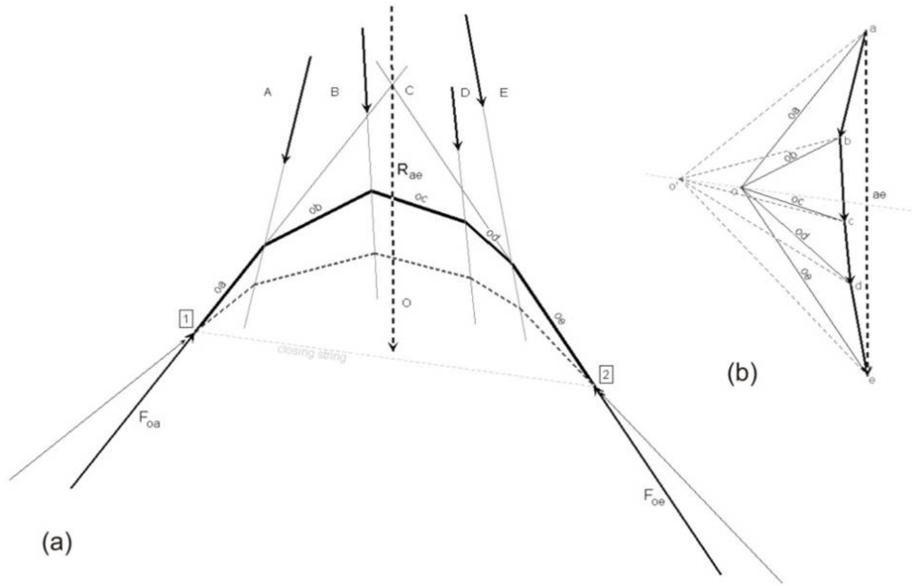


Δz_{max}	5	7.5	10
ζ	16.643	11.095	8.321

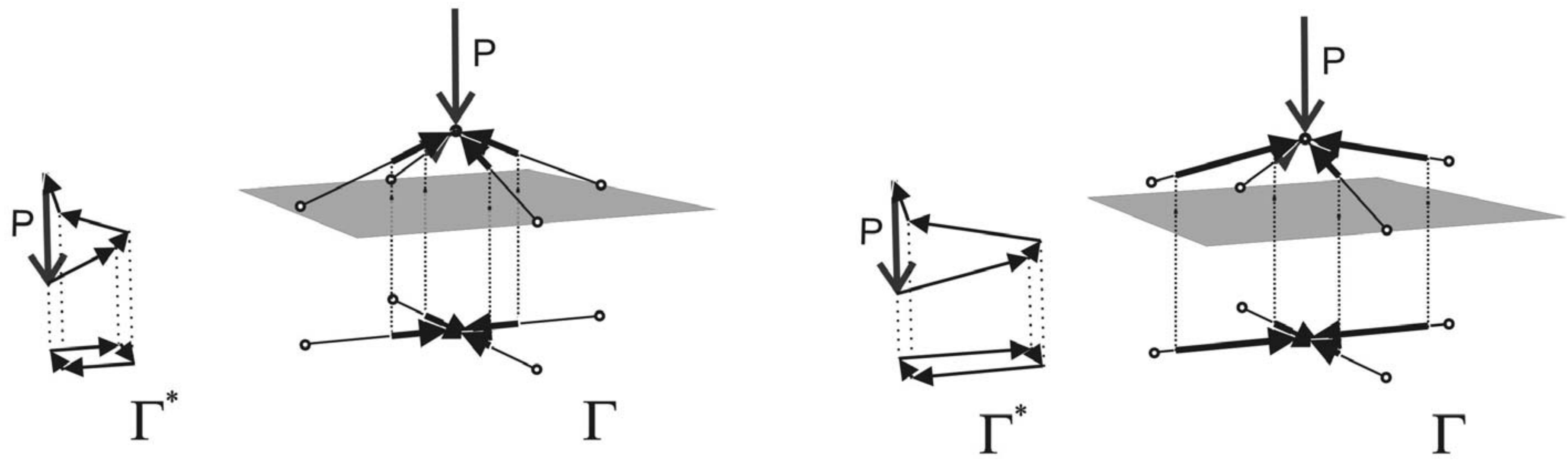
INDETERMINACY / PARAMETERS (2)



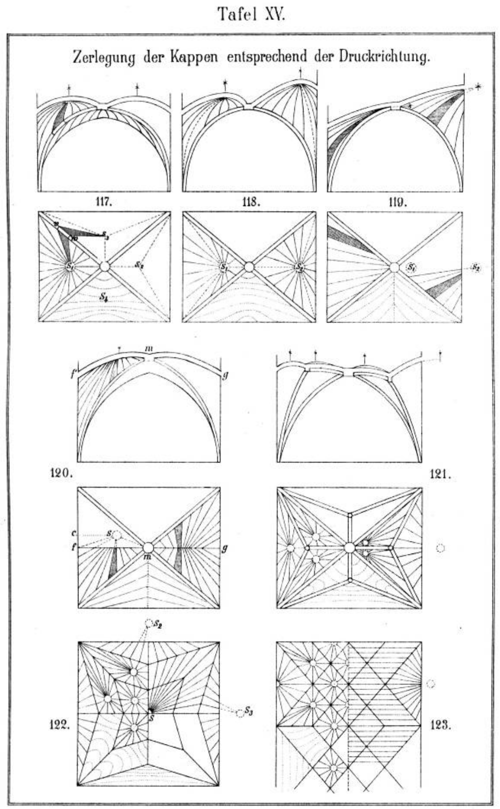


One parameter (i.e. scale ζ) controls infinite number of funicular solutions (equivalent to horizontal thrust in 2-D)

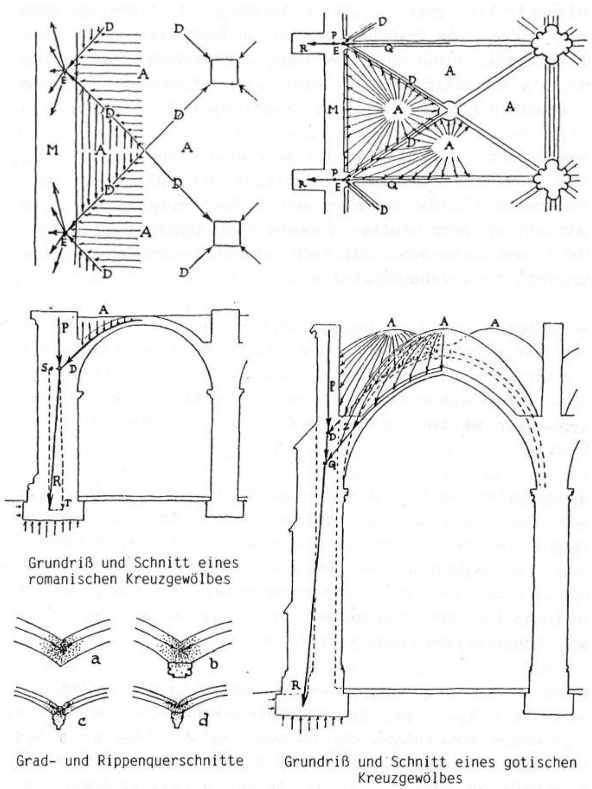
INDETERMINACY / PARAMETERS (3)



FORCE PATTERNS Cross vaults

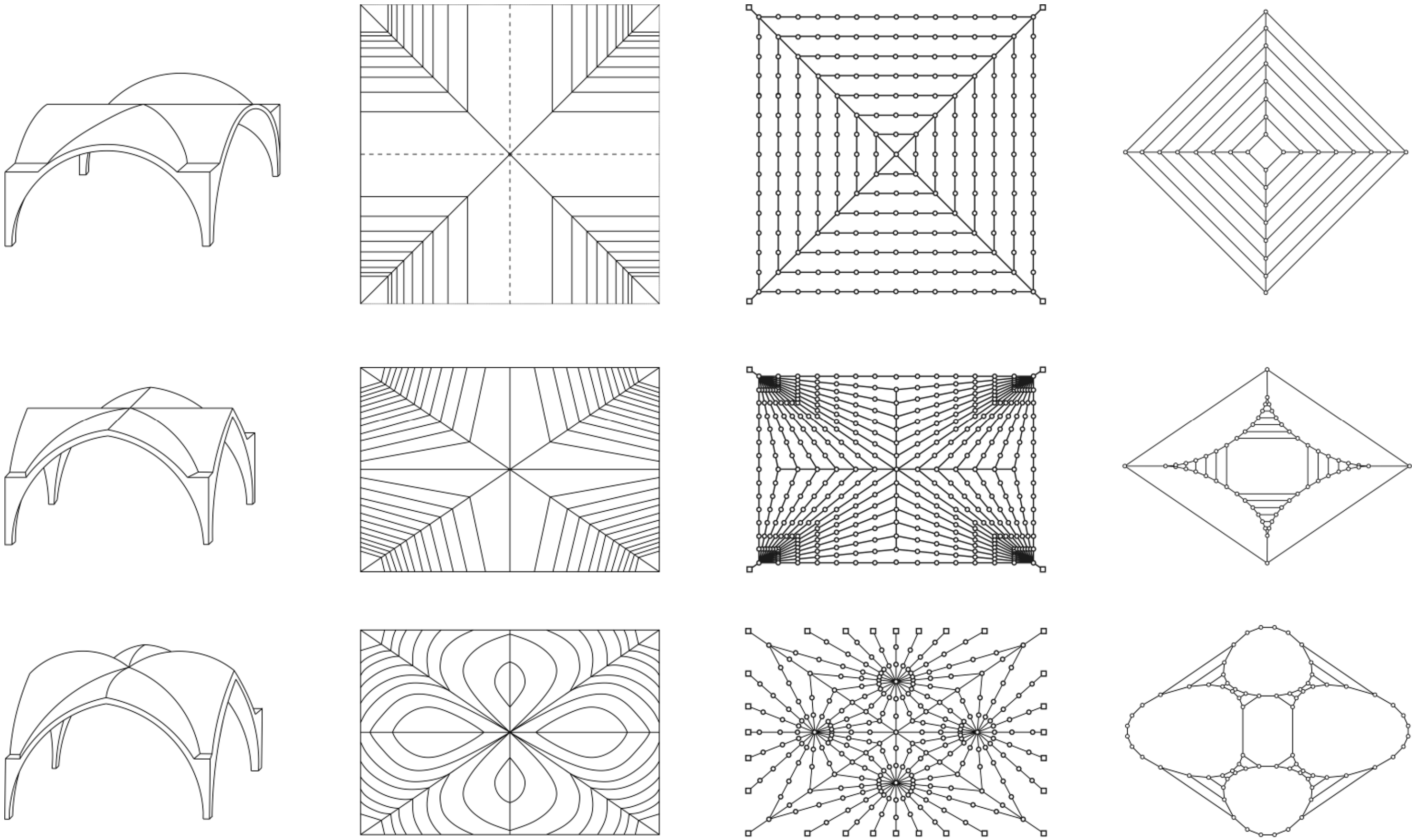


(Ungewitter, 1890)

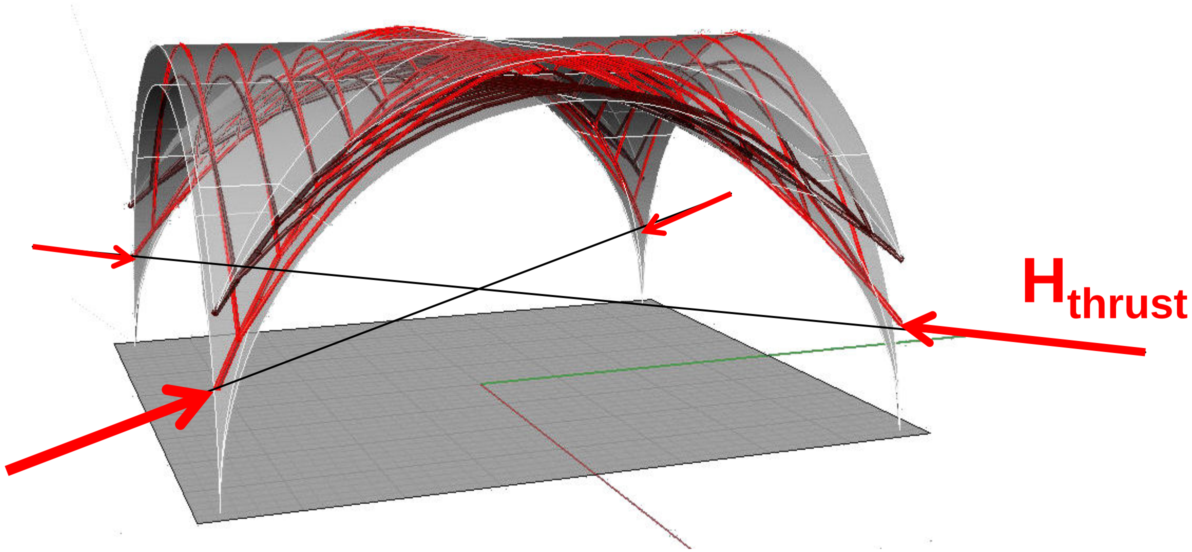


(Rave, 1939)

FORCE PATTERNS Cross vaults



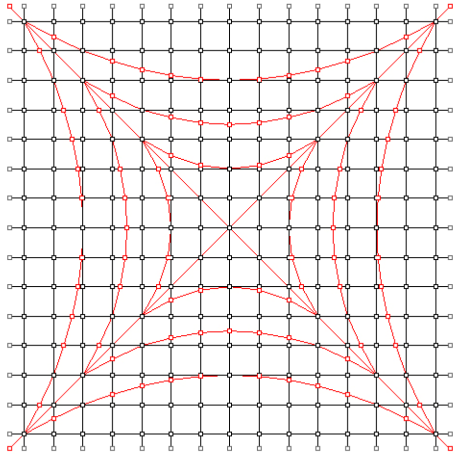
EXAMPLES ANALYSIS Groin vault (pseudo 3-D)

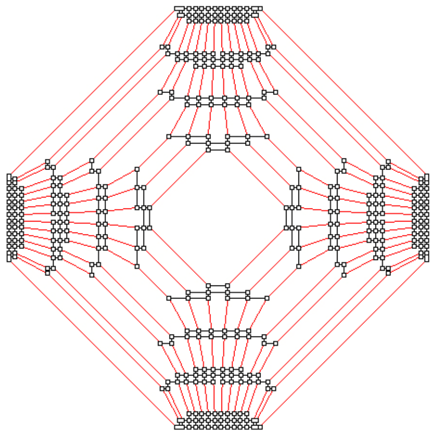


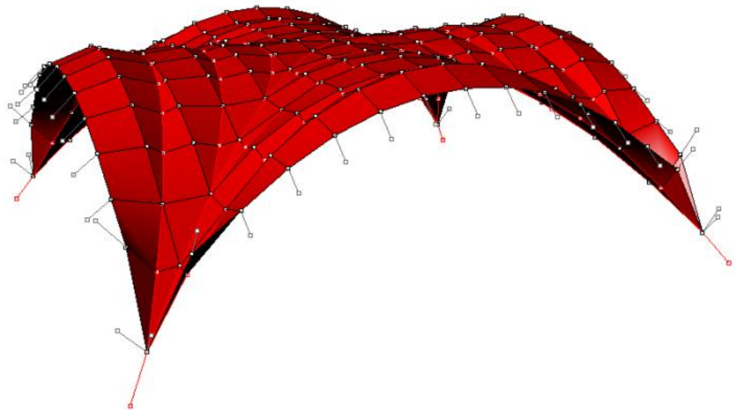
H_{thrust} (*)	
■ (red)	23% (min)
■ (brown)	32% (max)

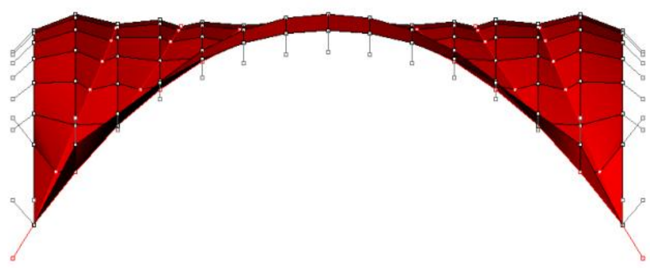
(*) as percentage of the weight of the vault

EXAMPLES ANALYSIS   Groin vault (full 3-D)

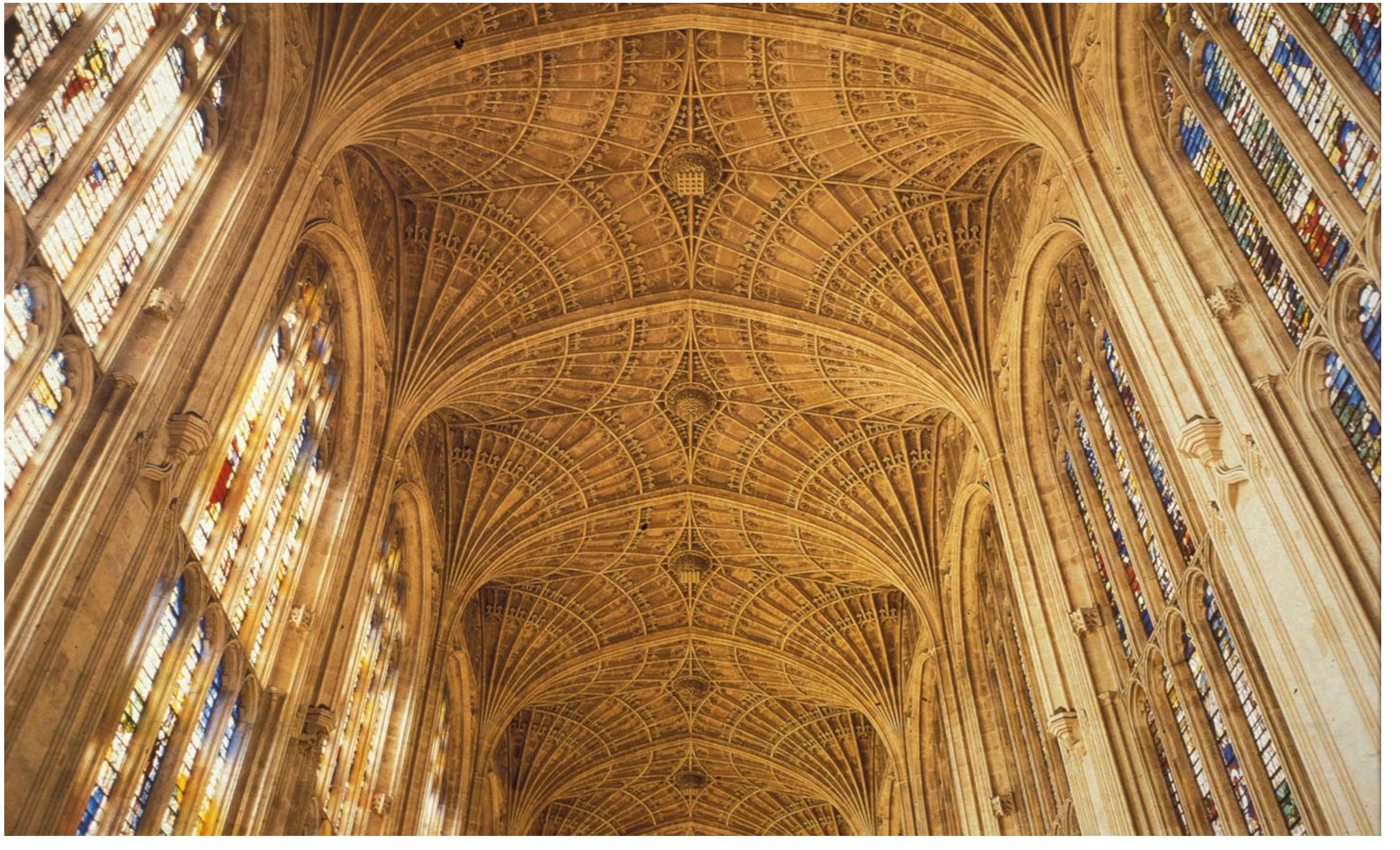








FAN VAULTS Kings College Chapel, Cambridge, UK



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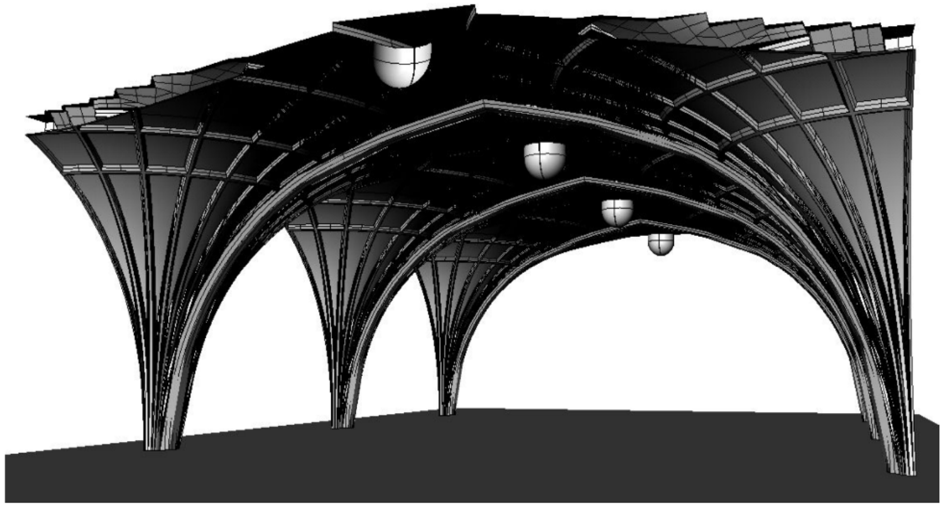

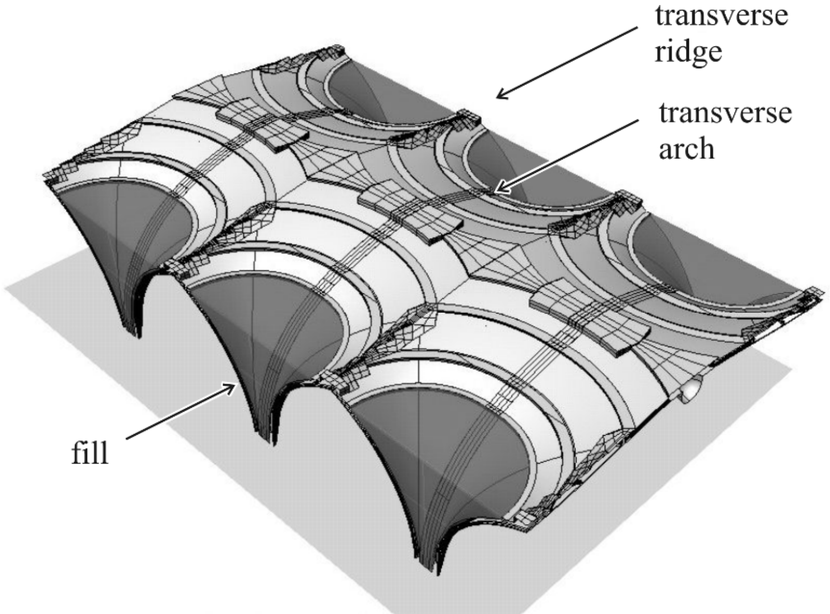

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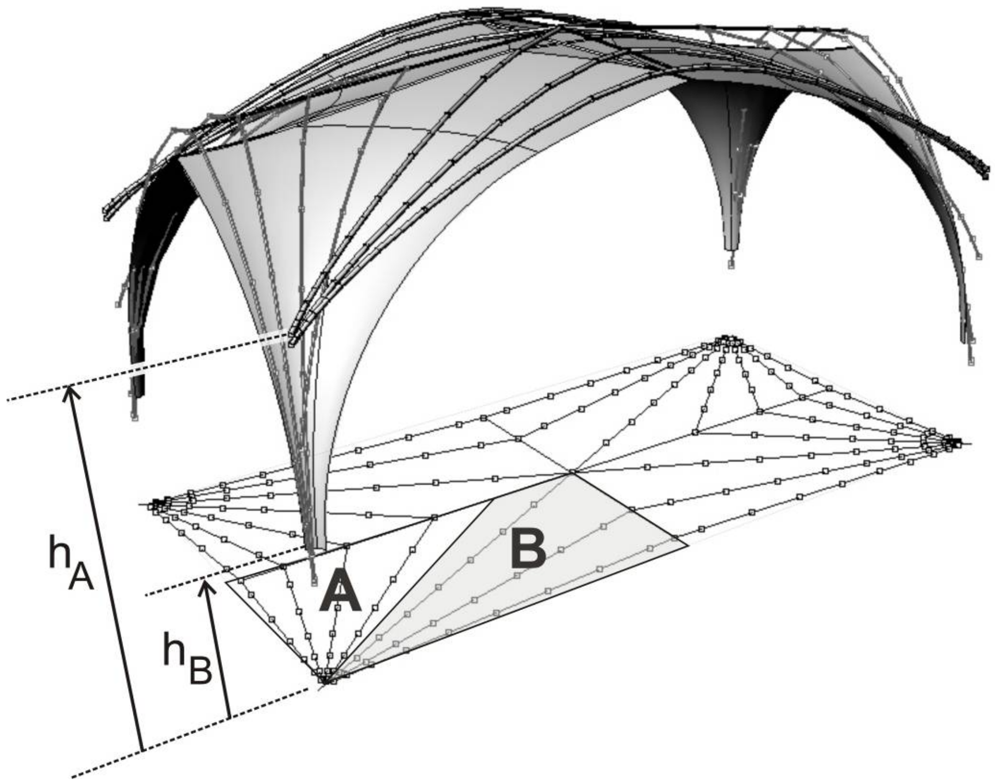
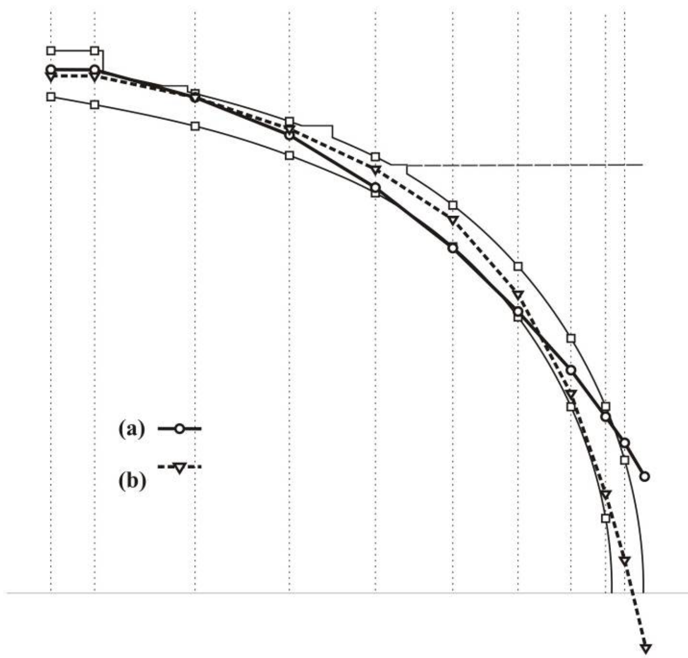
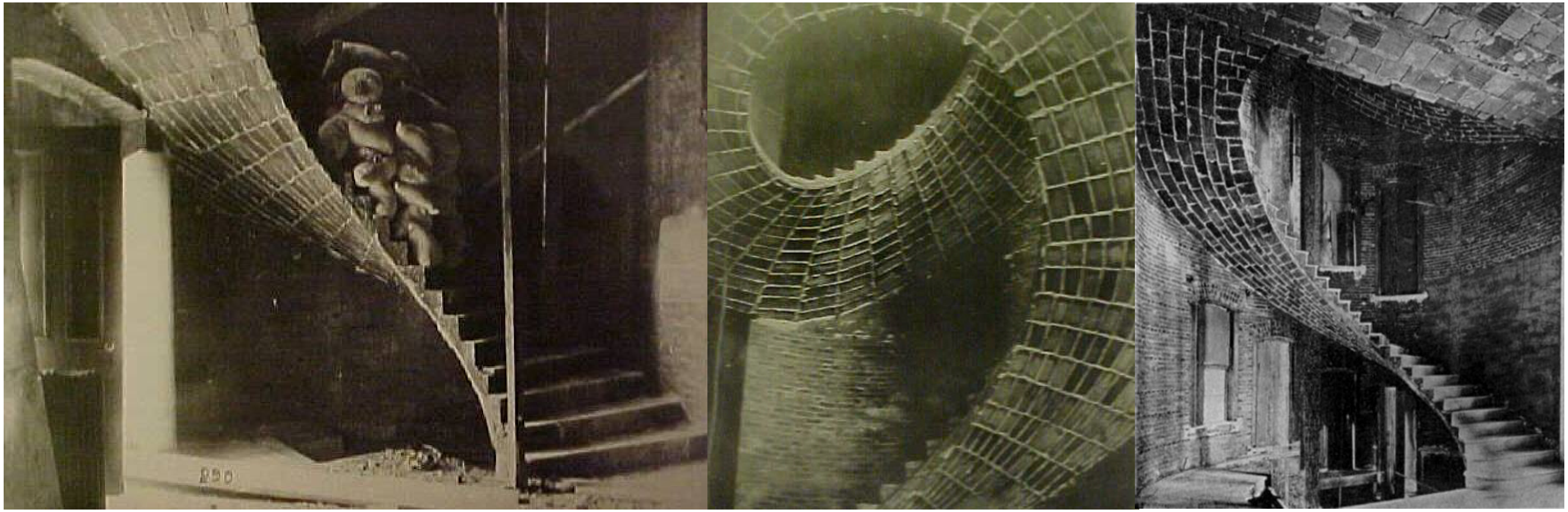


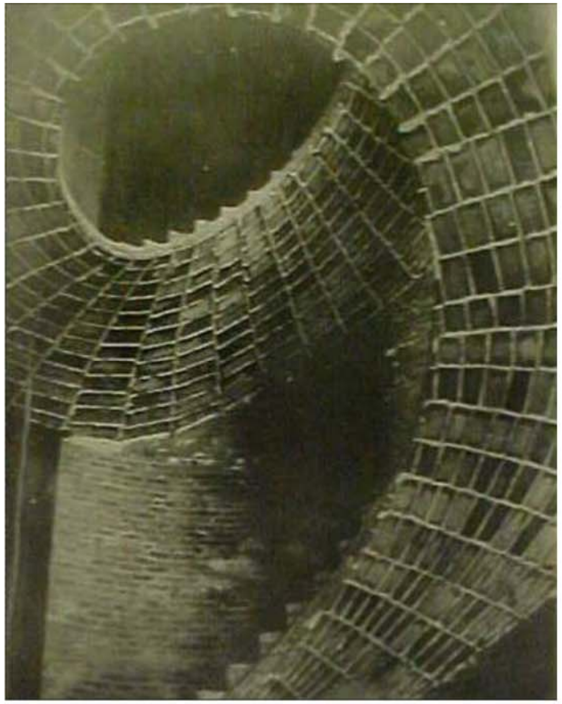


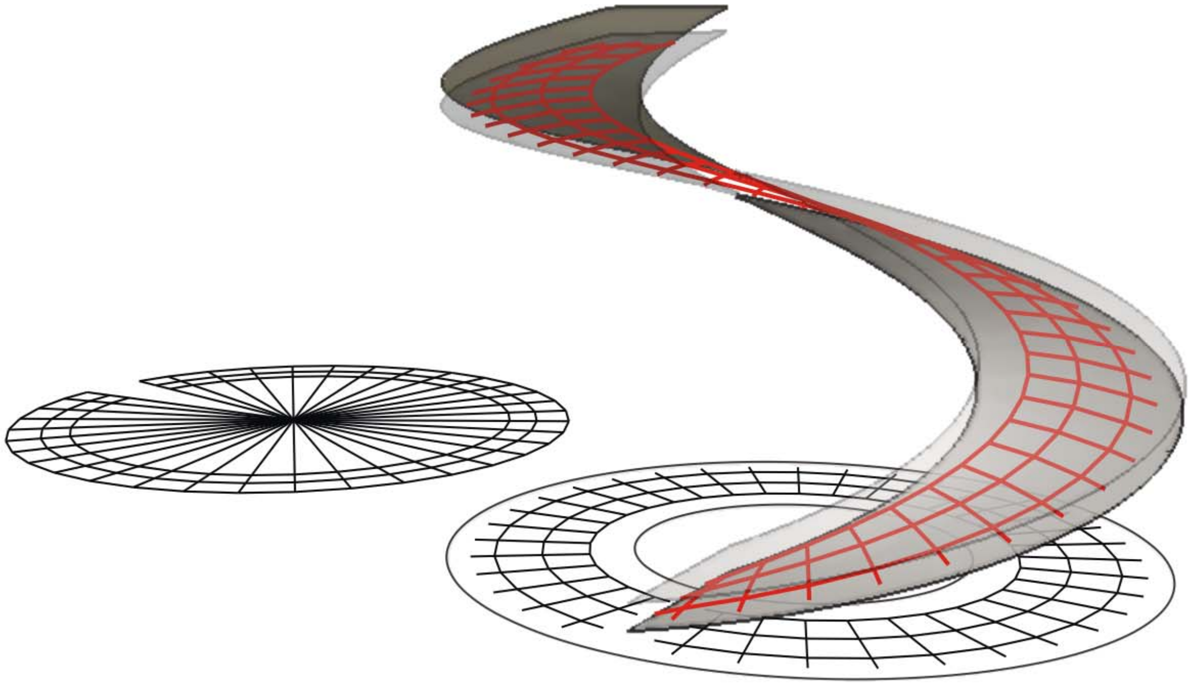
SPIRAL STAIRCASES by the Guastavino Company



Historical spiral staircases in thin brick/tile by the Guastavino Company

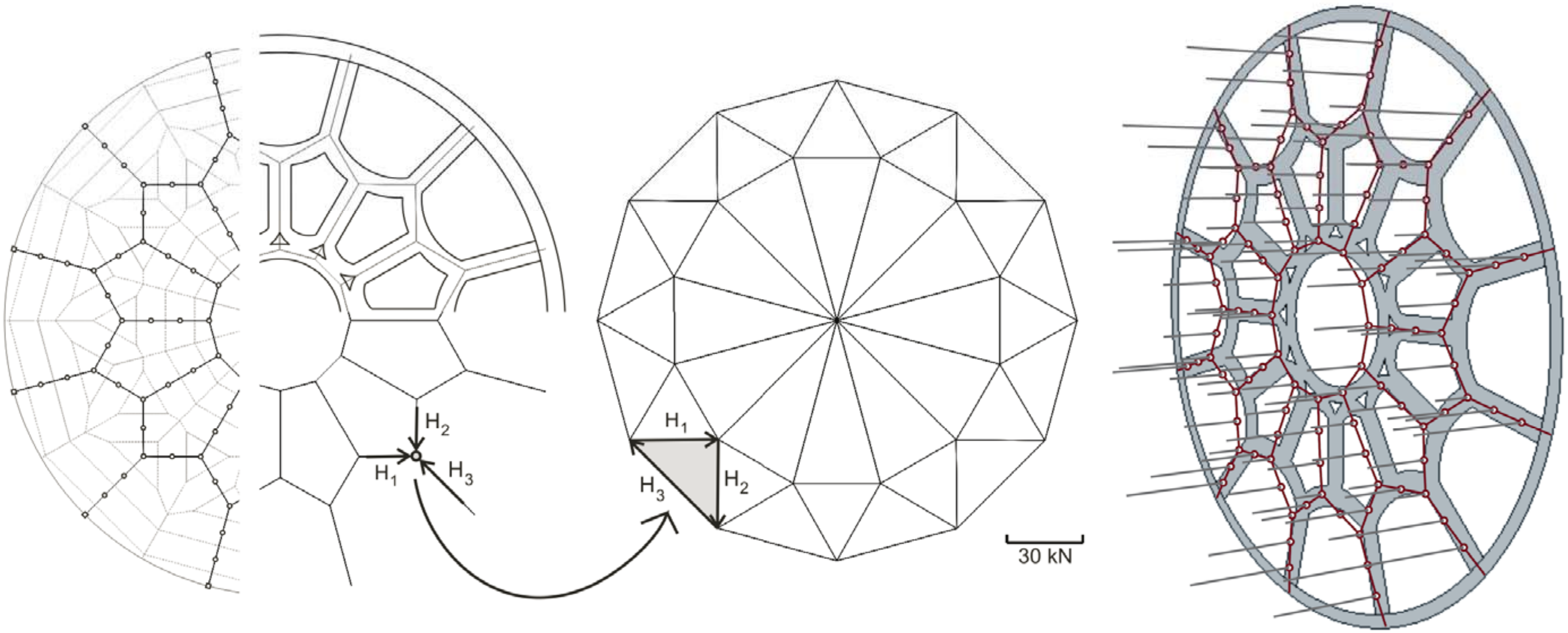
SPIRAL STAIRCASES by the Guastavino Company





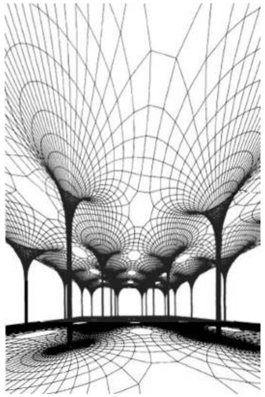
ROSE WINDOWS Mantes-la-Jolie Cathedral, France

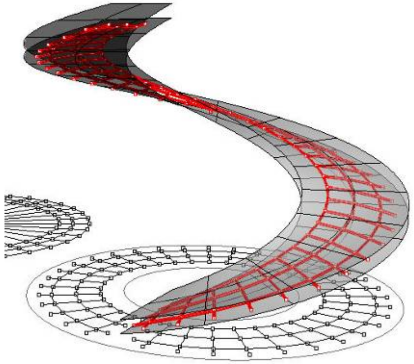
ROSE WINDOWS Mantes-la-Jolie Cathedral, France



KEY CONTRIBUTIONS

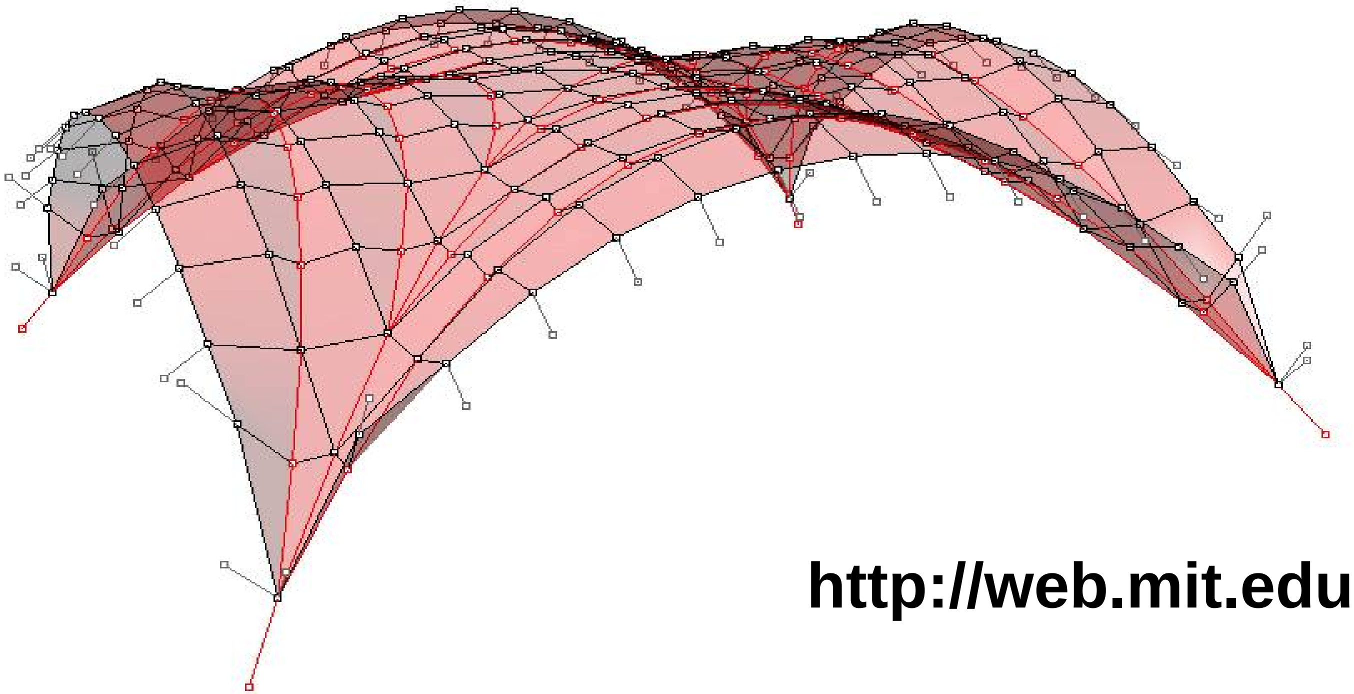
- Three-dimensional equivalent for graphic statics
  - clear graphical feedback of the forces in the system
  - high level of control of different equilibrium solutions
- Fully 3-D method to understand the stability of vaults with complex geometries in unreinforced masonry



Thank you…

Prof. John Ochsendorf, MIT
Prof. Chris Williams, Bath University
Xuan Vinh Doan, MIT



http://web.mit.edu/masonry/