Reliable Architectures for Networks under Stress

Guy E. Weichenberg, Vincent W. S. Chan and Muriel Médard
Laboratory for Information and Decision Systems,
Massachusetts Institute of Technology

***Abstract*— In this paper, we consider the task of designing a physical network topology that meets a high level of reliability with unreliable network elements. Our simple network model is one in which nodes are invulnerable and links are subject to failure in a statistically independent fashion. Our reliability metrics are all- and two-terminal reliability. In our treatment of the problem, we bring together previous contributions in the field and introduce additional insights that allow us to design networks which meet prescribed reliability levels. We address both the case when links are very reliable, and the often neglected case, where links are very unreliable. We focus on Harary graphs as candidate topologies, as they have been shown to possess many attractive reliability properties, and develop new results for this family of graphs.**

***Index Terms*— network reliability, network design, Harary graphs**

I. Introduction and motivation

Network reliability — the notion of connectedness of network nodes in the face of component failures — is an important consideration in network design for obvious reasons. The network reliability synthesis problem considered herein is the design of a network which achieves a prescribed level of "reliability", while minimizing the number of components used.

A large portion of previous contributions to the research area of network reliability are of theoretical nature with little immediate applicability to the design of real communication networks. In addition, existing results in the field are generally fragmented and a cohesive methodology for planning a network, based on different reliability metrics, has yet to emerge. This work is a step towards bridging the gap between theory and practice by providing design tools which are of immediate value in the planning of networks.

In addition, most reliability studies to date have focused on the analysis and design of networks when links are very reliable. However, the design of networks when links are unreliable, which is addressed in this paper, should not be overlooked for several reasons. In situations where the probability that a network is connected is quite small, some degree of connectedness in the network could still allow for important functions to be carried out, such as relaying emergency signals in times of distress. Another reason is that even small probabilities of connectedness could allow for acceptable expected times to failure for emergency functions or procedures to be carried out should a network come under stress.

The ideas presented in this work are applicable to the design of local-area networks (LANs) and metropolitan-area networks (MANs), where communication link costs are inexpensive enough to permit an artificial topology to be imposed on a set of network nodes. In addition, the model we will be using in this paper, where links are vulnerable and nodes are invulnerable, is particularly relevant to optical networks where the electronics in nodes are significantly more reliable than the optics in communication links.

Recently, network reliability metrics have been broadened to include some measure of performance, such as throughput or delay, since for many networks a more meaningful measure than connectedness is the degree to which network performance is degraded [1]. Connectedness measures, however, remain useful in situations where network performance is considered satisfactory as long as the network remains connected, or when the network's ability to provide a minimal level of service is of interest. In addition, connectedness is the relevant metric in many military applications, where capacities of network components are over-designed, such that connectedness of nodes ensures acceptable network performance.

Most of the necessary background, including definitions, notation and relevant work completed in the field, are covered in the following section. Additional background will be provided throughout the work when necessary. Section III outlines the modelling assumptions employed in this work. In section IV, we present bounding techniques which are valuable in the design of reliable networks. Section V specializes the techniques in section IV to Harary graphs, and introduces new results for this family of graphs. Finally, section VI concludes this work.

II. Graph theory background

Graph theory is generally used as a framework for modelling and analysis in network reliability studies. By exploiting the richness of graph theory, researchers have identified a myriad of metrics to define and assess the reliability of networks. These criteria can be broadly categorized as either deterministic or probabilistic reliability metrics. Let $K \subseteq N$ be the set of nodes in the graph underlying a network among which communication is of interest. Then, a *k-terminal* reliability metric reflects the difficulty in disrupting communication among any two nodes $s, d \in K$. When $|K| = n$, this is called an *all-terminal metric*, and when $|K| = 2$, it is called a *two-terminal metric*. Before delving into a discussion of these

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different metrics, we present some necessary graph theoretic background and notation.

A. Definitions and notation

In this work, networks will be modelled as undirected graphs. An *undirected graph* G is an ordered pair of sets (N, E), where the elements of N are nodes and the elements of E are edges. Edges in a graph will correspond to links in a network, and nodes in a graph will correspond to nodes in a network. An *edge* is an unordered pair of distinct nodes. The sizes of sets N and E are denoted by n and e, respectively. Two nodes are *adjacent* if they are elements of an edge. An edge is *incident* at its end nodes. The *incidence matrix* $\mathbf{A}$ of an undirected graph is the $n \times e$ matrix (each row corresponds to a node and each column to an edge) with the (i, j)th entry defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if edge } j \text{ is incident at node } i, \\ 0, & \text{otherwise.} \end{cases}$$

The size of the set of edges incident at node i is its *degree* and is denoted by d_i. The smallest degree of all nodes in a graph is denoted by δ, and the largest degree is denoted by Δ. If $\Delta = \delta$ then the graph is *regular* of degree Δ. If a graph is not regular but $\delta = \lfloor 2e/n \rfloor$, then the graph is *almost-regular*. The n-node graph which has all of its nodes adjacent is the *complete graph* K_n. Graph $G' = (N', E')$ is a *subgraph* of $G = (N, E)$ if $N' \subseteq N$ and $E' \subseteq E$ and if the endpoints of all edges in E' lie in N'.

A *path* is a sequence of distinct nodes such that consecutive nodes share an edge. Any two paths are *edge-disjoint* if they have no edges in common and *node-disjoint* if they have no nodes in common apart from the end nodes. The maximum length of any shortest path between any two nodes in a graph G is the *diameter* $k(G)$ of the graph. If we modify the definition of a path such that the first and last nodes in the sequence are identical, then we have the definition of a *cycle*.

Two distinct nodes are *connected* if there exists a path between the nodes. An undirected graph is *connected* if there exists a path between every pair of distinct nodes. A (minimal) set of edges in a graph whose removal disconnects the graph is a *(prime) edge cutset*. A (minimal) set of nodes which has the same property is a *(prime) node cutset*. The minimum cardinality of an edge cutset is the *edge connectivity* or *cohesion* $\lambda(G)$. The minimum cardinality of a node cutset is the *node connectivity* or *connectivity* $\chi(G)$. Analogous two-terminal metrics are the edge-connectivity $\lambda_{sd}(G)$ and node-connectivity $\chi_{sd}(G)$ with respect to a pair of nodes s and d. The two-terminal edge (respectively, node) connectivity of a graph is the minimum number of edges (respectively, nodes) whose removal disconnects the node pair.

An undirected graph G is a *tree* if it is connected and has no cycles. Another property of a tree is that it has $n-1$ edges. Given a connected, undirected graph $G = (N, E)$, let E' be a subset of E such that $T = (N, E')$ is a tree. T is called a *spanning tree* of G. We denote the number of spanning trees in G by $t(G)$. Clearly, the deletion of any edge in a tree results in the disconnection of the tree. As a result, prime edge cutsets of the graph G can be formed from one edge of a spanning tree of G and some of the edges not in this spanning tree.

B. Deterministic metrics

Two rudimentary, deterministic, all-terminal reliability criteria are the cohesion and connectivity of the graph underlying a network. An n-node, e-edge graph having maximum cohesion is a *max-*λ graph. Similarly, an n-node, e-edge graph having maximum connectivity is a *max-*χ graph. The following result relates connectivity and cohesion to the basic parameters of a graph [2]:

Theorem 1

$$\chi \leq \lambda \leq \delta \leq \frac{1}{n} \sum_{i=1}^{n} d_i = 2e/n.$$

Harary has shown [3] that the bounds in Theorem 1 can be achieved, through the construction of *Harary graphs*. Harary graphs are discussed below, in addition to the more general family of *circulant* graphs to which they belong. Since these bounds can be achieved, we see that any max-χ graph is necessarily max-λ, although the converse is not true in general.

More refined deterministic criteria for network reliability can also be defined, such as the number of edge or node cutsets of order λ or χ in a *max-*λ or *max-*χ graph, respectively. A graph is *super-*λ if it is max-λ and every edge disconnecting set of order λ isolates a point of degree λ. Similarly, a graph is *super-*χ if it is max-χ and every node disconnecting set of order χ isolates a point of degree χ.

An alternative measure of a graph's ability to remain connected is the number of spanning trees it possesses. The characterization of graphs with a maximum number of trees has been solved for sparse graphs when the number of edges is at most $n+3$, and for dense graphs when the number of edges is at most $n/2$ less than that of the complete graph K_n [4]–[6]. In addition, for the remaining cases where at most n edges are missing from K_n, Kel'mans, Petingi, Boesch, Suffel, Gilbert and Myrvold have described graphs with a maximum number of trees, assuming that they are almost regular [7]–[10].

C. Probabilistic metrics

Deterministic reliability metrics do not provide adequate measure of the susceptibility of networks to disconnection because these metrics do not account for the reliability of network components. Probabilistic reliability criteria, on the other hand, require knowledge of deterministic network properties, in addition to the reliability of network components, and thus yield a more meaningful measure of network reliability. For this reason, this work will primarily be concerned with probabilistic reliability criteria.

Probabilistic reliability metrics require the concept of a probabilistic graph. A *probabilistic graph* is an undirected graph $G = (N, E)$ where each node in N has an associated probability of being in an operational state and likewise for each edge in E. In probabilistic reliability analyses, networks under stress are modelled as probabilistic graphs.

Most approaches to probabilistic reliability analysis have focused on the probability that a subset of nodes in a network are connected. Thus, the all-terminal reliability of a probabilistic graph can be defined as the probability that any two nodes in the graph have an operating path connecting them. If links fail in a statistically independent fashion with probability p, then the all-terminal reliability $P_c(G,p)$ is given by:

$$P_c(G,p) = \sum_{i=n-1}^{e} A_i(1-p)^i p^{e-i} \quad (1)$$

$$= 1 - \sum_{i=\lambda}^{e} C_i p^i (1-p)^{e-i} \quad (2)$$

where A_i denotes the number of connected subgraphs with i edges, and C_i denotes the number of edge cutsets of cardinality i. For values of p sufficiently close to zero, $P_c(G,p)$ can be accurately approximated by $1 - C_\lambda p^\lambda (1-p)^{e-\lambda}$. In this case, an optimally reliable graph — one that achieves the maximum $P_c(G,p)$ over all graphs with the same number of nodes and edges — has a minimum number of cutsets of size $\lambda = \lfloor 2e/n \rfloor$. Therefore, in this regime of p, optimally reliable graphs are super-λ graphs. For values of p sufficiently close to unity, $P_c(G,p)$ can be accurately approximated by the first term in (1), $A_{n-1}(1-p)^{n-1}p^{e-n+1}$, where $A_{n-1} = t(G)$. Therefore, for values of p sufficiently close to unity, an optimally reliable graph has a maximum number of spanning trees.

The two-terminal reliability of a probabilistic graph is the probability that a given pair of nodes, s and d, have an operating path connecting them:

$$P_c^{sd}(G,p) = \sum_{i=w_{sd}}^{e} A_i^{sd}(1-p)^i p^{e-i} \quad (3)$$

$$= 1 - \sum_{i=\lambda_{sd}}^{e} C_i^{sd} p^i (1-p)^{e-i} \quad (4)$$

where w_{sd} is the shortest path length between nodes s and d, A_i^{sd} is the number of subgraphs with i edges that connect nodes s and d, λ_{sd} is the minimum number of edge failures required to disconnect nodes s and d, and C_i^{sd} is the number of cutsets with respect to nodes s and d of cardinality i. If we wish to maximize $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ when p is small, then it is apparent from (4) that the property of super-λ is a necessary condition. This is because $\lambda = \min_{s,d}[\lambda_{sd}]$, and for super-$\lambda$ graphs, $C_{\lambda_{sd}}^{sd}$ attains the minimum bound of two.

D. Harary graphs and circulants

As previously mentioned, Harary graphs, first presented in [3], achieve the bounds presented in Theorem 1. This result implies that Harary graphs also achieve the maximum value of $\min_{s,d}[\lambda_{sd}]$ and $\min_{s,d}[\chi_{sd}]$ over all graphs with n nodes and e edges. In a $H(n,\Delta)$ Harary graph where Δ is even, each node i, $0 \le i \le n-1$, is adjacent to nodes $i \pm 1, i \pm 2, \ldots, i \pm \lfloor \Delta/2 \rfloor (\text{mod } n)$; and if Δ is odd, then each node $i = 1, \ldots, \lfloor (n-1)/2 \rfloor$ is also adjacent to node $i + \lfloor n/2 \rfloor$. See Figure 1 for an example of a Harary graph. Harary graphs have the following properties [11]:

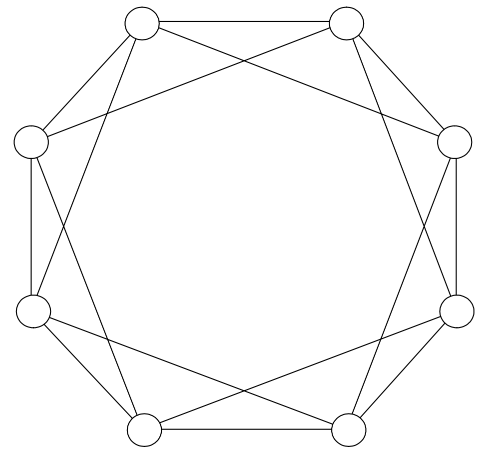

Fig. 1. The $H(8,4)$ Harary graph.

- $H(n,\Delta)$ has $e = \lceil nk/2 \rceil$, $\chi = \lambda = \Delta$;
- $H(n,\Delta)$ is regular of degree Δ, unless n and Δ are both odd;
- $H(n,\Delta)$ has one node of degree $\Delta+1$ and $n-1$ nodes of degree Δ if n and Δ are both odd.

Harary graphs belong to a more general family of graphs known as *circulants*. The circulant graph $C_n\langle a_1, a_2, \ldots, a_h\rangle$, or more compactly, $C_n\langle a_i\rangle$, where $0 < a_1 < a_2 < \ldots < a_h < (n+1)/2$, has $i \pm a_1, i \pm a_2, \ldots, i \pm a_h (\text{mod } n)$ adjacent to each node i. Owing to a theorem by Mader [12], which proves that every connected node-symmetric[1] graph has $\lambda = \Delta$, all connected circulants are max-λ. Furthermore, in [13], Boesch and Wang prove the following result:

Theorem 2 *The only circulants which are not super-λ are the cycles and the graphs $C_{2m}\langle 2, 4, \ldots, m-1, m\rangle$ with $m \ge 3$, and m an odd integer.*

In [14], Wang and Yang derive the following useful result for the number of spanning trees in circulant graphs:

Theorem 3 *The number of spanning trees in the degree Δ circulant graph $G = C_n\langle a_1, a_2, \ldots, a_h\rangle$ is:*

$$t(G) = \begin{cases} \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h}\sin^2(a_j i\pi/n)\right], & \text{if } \Delta \text{ is even,} \\ \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h-1}\sin^2(a_j i\pi/n) - (-1)^i + 1\right], & \\ \quad \text{if } \Delta \text{ is odd.} & \end{cases}$$

In [13], Boesch and Wang examine the diameter properties of circulants and derive lower diameter bounds for the family of graphs. In [15], the same authors determined that even

[1]Two nodes u and v in a graph are *similar* if there is an automorphism which maps u onto v. A graph in which all nodes are similar is *node-symmetric*.

degree Harary graphs possess the fewest number of edge cutsets of cutset cardinality i, when $\lambda \leq i \leq 2\Delta - 3$. Each cutset in the above range of cardinalities was shown to isolate a single node in the Harary graph.

III. Network model

As mentioned in the introductory section, networks will be modelled as probabilistic graphs. In addition, we assume the following about the the graphs underlying the networks considered in the remainder of this paper:

- Nodes are invulnerable;
- Edges fail in a statistically independent fashion with probability p;
- Edge capacities are assumed to be sufficiently large to carry any possible network flow;
- Once an edge fails it cannot be repaired.

IV. Bounds on probabilistic reliability metrics

In this section, we introduce new and simple techniques to bound the probability of connection of a network and the probability of connection of a node pair in a network, which are useful in carrying out any network design methodology. The quality of these bounds are illustrated for the ten node, degree three Harary graph in Figures 2, 3, and 4. In the discussion that follows, we assume that all graphs are Δ regular and have maximum connectivity.

A. All-terminal reliability when $p \approx 0$

In this subsection, we derive upper and lower bounds for the probability that graph G is connected $P_c(G,p)$. The general approach we follow is based on enumeration of prime failure events. We define a *prime failure event* as an event in which a subset of nodes become disconnected from the rest of the graph through the failure of the minimal number of edges. Clearly, prime failure events are only a subset of all possible graph disconnection events, since graph disconnection can also occur when more than the minimal number of edges fail. Therefore, in order to obtain an upper bound for $P_c(G,p)$, we subtract from unity the probabilities of the mutually exclusive prime failure events:

$$P_c(G,p) \leq 1 - \sum_{i=\lambda}^{e} B_i p^i (1-p)^{e-i} \tag{5}$$

where B_i is the number of prime failure events of cardinality i. To obtain a lower bound for $P_c(G,p)$, we note that any failure scenario requires that at least one of the prime failure events occur. Therefore, we obtain a lower bound for $P_c(G,p)$ by subtracting from unity the union bound of the prime failure events:

$$P_c(G,p) \geq 1 - \sum_{i=\lambda}^{e} B_i p^i. \tag{6}$$

It now remains to determine the coefficients B_i. If the graph under consideration is either trivially small, or simple and symmetric as is the case with Harary networks, then closed form, analytic solutions or bounds are obtainable; otherwise, one must resort to more general techniques.

We now introduce a technique to determine the coefficients B_i for general graphs. It is known [16] that a vector representation of the prime failure events of a graph can be expressed in two ways as the modulo two sum of a subset of rows of a graph's incidence matrix. Specifically, a prime failure event partitions a network into two subsets of nodes. Therefore, we can obtain a prime failure event by adding modulo two the rows that correspond to each of the nodes in one of the partitions. Conversely, it can be shown that the modulo two sum of any proper subset of rows of a graph's incidence matrix yields a prime failure event. Therefore, we can find all prime failure events of a graph by summing modulo two the rows of the $2^{n-1} - 1$ subsets of the rows the incidence matrix which yield distinct partitions of the network[2]. The B_i coefficients are determined by simply counting the number of prime failure events obtained which have cardinality i.

Another approach to upper bounding $P_c(G,p)$ when $p \approx 0$ is to compute a lower bound on the first few terms of the summation in (2) and to then subtract these terms from unity. As discussed in section II-D, Boesch and Wang demonstrated in [15] that even degree Harary graphs possess the fewest number of edge cutsets of cardinality i, when $\lambda \leq i \leq 2\Delta - 3$. The number of cutsets of cardinality i achieved by Harary graphs in this range is $n\binom{e-\Delta}{i-\Delta}$. This expression is thus a lower bound achievable by any Δ regular graph with n nodes. Using this result, we obtain the following upper bound for $P_c(G,p)$ for any Δ regular graph with n nodes:

$$P_c(G,p) \leq 1 - \sum_{i=\lambda}^{2\Delta-3} n\binom{e-\Delta}{i-\Delta} p^i (1-p)^{e-i}.$$

B. Two-terminal reliability when $p \approx 0$

If instead of the probability that graph $G = (N, E)$ is connected $P_c(G,p)$, we desire the probability that nodes $s, d \in N$ are connected $P_c^{sd}(G,p)$, we can use an approach similar to that of section IV-A to obtain the following bounds:

$$1 - \sum_{i=\lambda_{sd}}^{e} B_i^{sd} p^i \leq P_c^{sd}(G,p) \leq 1 - \sum_{i=\lambda_{sd}}^{e} B_i^{sd} p^i (1-p)^{e-i} \tag{7}$$

where B_i^{sd} is the number of prime failure events with respect to nodes s and d of cardinality i, and λ_{sd} is the minimum number of edge failures required to disconnect nodes s and d.

In order to determine the coefficients B_i^{sd}, we use an approach similar to that of §IV-A. Since we are only interested in prime failure events of G which disconnect nodes s and d, we add modulo two to the row corresponding to s all possible subsets of the remaining rows of the incidence matrix, except for the row corresponding to d. Clearly, there are 2^{n-2} such possible subsets. This will provide us with a binary

[2] Note that if we sum modulo two the rows of all 2^n possible subsets, then we are counting every partitioning scenario twice, including the null and complete partitions.

vector representation of all possible prime failure events which disconnect s and d.

In a similar manner to section IV-A, we can upper bound $P_c^{sd}(G,p)$ by lower bounding the first few terms in the summation of (4). We lower bound C_i^{sd} for $\lambda \leq i \leq 2\Delta - 3$ for any Δ regular graph with n nodes, and obtain the following upper bound for $P_c^{sd}(G,p)$:

$$P_c^{sd}(G,p) \leq 1 - \sum_{i=\lambda}^{2\Delta-3} 2\binom{e-\Delta}{i-\Delta} p^i (1-p)^{e-i}.$$

C. All-terminal reliability when $p \approx 1$

We approach the task of bounding $P_c(G,p)$ in the regime of $p \approx 1$ in an analogous fashion to section IV-A. The events of interest here, however, are the existence of spanning trees rather than prime failure events. A lower bound for $P_c(G,p)$ is obtained by summing the events that correspond to a spanning tree existing *and* the remaining links in the network being inoperative:

$$P_c(G,p) \geq t(G)(1-p)^{n-1} p^{e-n+1}. \tag{8}$$

An upper bound for $P_c(G,p)$ can be obtained be invoking the union bound on the spanning tree events:

$$P_c(G,p) \leq t(G)(1-p)^{n-1}. \tag{9}$$

It now remains to determine $t(G)$. Fortunately, this is a well studied problem, and $t(G)$ is known [17] to be the determinant of an $(n-1)\times(n-1)$ matrix $\mathbf{T}(\mathbf{G})$ whose $(i,j)^{\text{th}}$ entry is defined as follows:

$$t_{ij} = \begin{cases} d_i, & \text{if } i = j, \\ -1, & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

D. Two-terminal reliability when $p \approx 1$

When $p \approx 1$, most of the links in a network have failed and the underlying graph has relatively few edges. In such sparsely connected graphs, the disconnection of nodes s and d is nearly equivalent to a set of edge-disjoint paths between s and d all having failed. To be precise, the disconnection of nodes s and d actually implies the failure of a set of Δ edge-disjoint paths between s and d, but the converse is not necessarily true. This is because each of the edge-disjoint paths can fail but there may still exist a path between s and d through the use of segments of the failed disjoint paths. Hence, we can lower bound $P_c^{sd}(G,p)$ as follows:

$$\begin{aligned} P_c^{sd}(G,p) &\geq 1 - \text{Pr}(\Delta \text{ edge-disjoint paths fail}) \\ &= 1 - \prod_{i=1}^{\Delta} \text{Pr(path } i \text{ fails)} \\ &= 1 - \prod_{i=1}^{\Delta} \left[1 - (1-p)^{l_i}\right] \end{aligned} \tag{10}$$

where l_i is the length of the ith edge-disjoint path, and the second and third lines follow from the independence of edge failures.

The value of $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ when $p \approx 1$ corresponds to a node pair with shortest path length equal to the graph diameter $k(G)$. A simple lower bound for $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ is $(1-p)^{k(G)}$, which is just the probability that the shortest path between the most distant node pair is available:

$$(1-p)^{k(G)} \leq \min_{s,d}\left[P_c^{sd}(G,p)\right]. \tag{11}$$

A tighter lower bound for $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ can be derived using (10) if the lengths or an upper bound on the lengths of the edge-disjoint paths joining the most distant node pair is available.

V. Analysis of Harary graphs

In this section, we specialize the results of the previous section to the family of Harary graphs. Our reason for focusing on Harary graphs is that they possess good reliability properties in the $p \approx 0$ regime. Specifically, we showed in section II-C that when $p \approx 0$, a necessary condition for $P_c(G,p)$ and $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ to be maximized is that G must be super-λ. Among super-λ graphs, even degree Harary graphs are especially good when p is small, since they achieve the fewest number of cutsets of cardinality i, when $\lambda \leq i \leq 2\Delta - 3$. Admittedly, however, when $p \approx 1$ Harary graphs are suboptimal graphs as they possess relatively few spanning trees and large diameters. In fact, it is easy to find circulant graphs with the same number of nodes and edges which possess more spanning trees and smaller diameters than the corresponding Harary graphs. Nonetheless, we justify our attention to Harary graphs in this work since highly reliable links (i.e. $p \approx 0$) are more realistic than highly unreliable links (i.e. $p \approx 1$) in a network.

Before beginning our analysis of Harary graphs, we prove an intuitive and useful theorem regarding this family of graphs.

Theorem 4 *Consider a Harary graph $H(n,\Delta)$, where Δ is even. Partition the n nodes into a subset of j nodes S_j and a subset of $n-j$ nodes S_{n-j}, where we assume that $j \leq n-j$. Then, the minimum number of edges joining S_j to S_{n-j} occurs when the j nodes in S_j (and hence, the $n-j$ nodes in S_{n-j}) are consecutively numbered (modulo n).*

To prove the theorem, we need the following lemma:

Lemma 1 *Partition the n nodes of the $H(n,\Delta)$ Harary graph into a subset of $j \leq n-j$ nodes S_j, and a subset of $n-j$ nodes S_{n-j}, such that the nodes in S_j (and hence, the $n-j$ nodes in S_{n-j}) are consecutively numbered (modulo n). Then, the number of edges joining S_j to S_{n-j} is:*

$$\begin{array}{ll} \Delta, & \textit{if } j = 1, \\ j\Delta - 2\binom{j}{2}, & \textit{if } 2 \leq j \leq \lfloor \Delta/2 \rfloor + 1, \\ \lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil, & \textit{otherwise.} \end{array} \tag{12}$$

Proof. The case of $j = 1$ is trivial. When $2 \leq j \leq \lfloor \Delta/2 \rfloor + 1$, a consecutive partition of j nodes allows the nodes in S_j to be fully connected. In this case, the number of edges joining S_j to S_{n-j} follows from the fact that the total number of edge endpoints incident at S_j's nodes is $j\Delta$ and that the total number of edge endpoints in a fully connected subgraph of j nodes is $2\binom{j}{2}$. For the remaining case, when the nodes are consecutively arranged, the nodes at either end of the S_j partition possess $\lceil\Delta/2\rceil$ connections to S_{n-j}, the nodes which are second from either end of the partition possess $\lceil\Delta/2\rceil - 1$ connections to S_{n-j}, and so on. Hence, the total number of edges joining S_j to S_{n-j} is the constant $2\sum_{i=1}^{\lceil\Delta/2\rceil} i = \left(\lceil\Delta/2\rceil^2 + \lceil\Delta/2\rceil\right)$, as required. □

We are now ready to prove Theorem 4:

Proof of Theorem 4. The case of $j = 1$ is trivial. Consider now the case of $2 \leq j \leq \Delta/2+1$. Note that minimizing the number of edges joining S_j to S_{n-j} is equivalent to maximizing the number of internal edges shared by the nodes of one of the partitions. When $2 \leq j \leq \Delta/2+1$, a consecutive partition of j nodes allows the nodes in S_j to be fully connected, yielding the maximum number of internal connections, and hence the minimum number of external edges.

For the remaining case where $\Delta/2 + 2 \leq j \leq n/2$, we carry out the proof by induction. We may use our result for $j = \Delta/2 + 1$ as our base case. Now, assume that a consecutive arrangement of j nodes achieves the minimum number of external edges. Let us now proceed by contradiction by assuming the existence of a partition S'_{j+1} of $j+1$ nodes which achieves a smaller number of external edges than the number achieved by a consecutive arrangement of $j+1$ nodes in Lemma 1.

If we can find a node in S'_{j+1} which contains at least $\Delta/2$ edges to S'_{n-j-1}, then we move this node to S'_{n-j-1}. This creates a partitioning of the graph into j and $n-j$ nodes which achieves fewer edges joining the two partitions than a consecutive arrangement. This would contradict our induction hypothesis, implying that a consecutive arrangement of nodes is optimal.

Now, let us consider the case where there does not exist a node in S'_{j+1} which contains at least $\Delta/2$ edges to S'_{n-j-1}. We proceed by finding a pair of consecutive nodes in the graph such that one of the nodes u belongs to S'_{j+1} and the other node v belongs to S'_{n-j-1}. Examining the window of $\Delta+1$ consecutive nodes centered at u, our assumption that there does not exist a node in S'_{j+1} which has at least $\Delta/2$ edges to S'_{n-j-1} requires that at least $\Delta/2+2$ nodes in this window belong to S'_{j+1}. We now consider the window of $\Delta+1$ consecutive nodes centered at v. Since the window formed by the union of u and v's windows of length $\Delta+1$ has size $\Delta+2$ nodes, there can be at most $\Delta/2$ nodes in this larger window that belong to S'_{n-j-1}. By moving v to S'_{j+1}, we create a partitioning of the graph into $j+2$ and $n-j-2$ nodes which achieves fewer edges joining the two partitions than that of the S'_{j+1} and S'_{n-j-1} partitioning, and hence, fewer than that of a consecutive arrangement of j and $n-j$ nodes. Note that by moving v to S'_{j+1}, we have not created a node in S'_{j+1} which possesses at least $\Delta/2$ edges to the other partition. This is because the $j+1$ nodes initially in S'_{j+1} only gain internal edges by moving v to S'_{j+1}, and v now possesses fewer than $\Delta/2$ edges to the other partition. Thus, we can continue in this way – finding a pair of consecutive nodes in different partitions and moving one node to the other partition, always decreasing the number of edges connecting the partitions, until we have increased the size of our initial partition of j nodes to $n-j$ nodes. At this point, we have created a partitioning of the graph into j and $n-j$ nodes which achieves fewer edges joining the partitions than the partitioning of the graph in our induction hypothesis, which was assumed to be optimal. This is a contradiction, implying that a consecutive arrangement of nodes is optimal. □

A. All-terminal reliability when $p \approx 0$

Every graph disconnection scenario can be viewed as a partitioning of the graph into two subsets of nodes which are disconnected. Now, since a partition of j consecutive nodes minimizes the number of edges joining S_j to S_{n-j} in an even degree Harary graph, the probability that a partition of j nodes becomes disconnected from a partition of S_{n-j} nodes is maximized when the partition of j nodes are consecutive. We can therefore form an upper bound for the probability of graph disconnection (and hence, a lower bound for the probability of graph connection) by upper bounding the probability of S_j and S_{n-j} becoming disconnected by the consecutive case, and then employing a union bound on these events. Furthermore, since the $H(n, 2\lfloor\frac{\Delta}{2}\rfloor)$ Harary graph is a subgraph of the $H(n,\Delta)$ Harary graph, the all-terminal reliability of an odd degree Harary graphs is lower bounded by the all-terminal reliability of the Harary graph with degree one less. Thus, a lower bound for $P_c(G,p)$ for a Harary graph $H(n,\Delta)$ is:

$$\begin{aligned} P_c(G,p) \geq 1 - \Bigg(& np^{\Delta} + \sum_{i=2}^{\lfloor\Delta/2\rfloor+1} \binom{n}{i} p^{i\Delta - 2\binom{i}{2}} \\ & + \sum_{i=\lfloor\Delta/2\rfloor+2}^{\lfloor n/2\rfloor} \binom{n}{i} p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil} \Bigg). \end{aligned} \tag{13}$$

Because cutset failure events were used to derive (13), the bound is tight for p close to zero. We can derive a slightly looser lower bound for $P_c(G,p)$ by bounding some of the terms in (13):

$$\begin{aligned} P_c(G,p) \geq 1 - \Bigg(& np^{\Delta} + \left\lfloor\frac{\Delta}{2}\right\rfloor \binom{n}{\lfloor\frac{\Delta}{2}\rfloor+1} \big(p^{2\Delta-2} \\ & - p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\big) + p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil} \\ & \left[2^{n-1} + \frac{1}{2}\binom{n}{\lfloor n/2\rfloor} - n - 1\right]\Bigg). \end{aligned} \tag{14}$$

The quality of these bounds is illustrated in Figure 2 for the ten node, degree three Harary graph. The bounds plotted are quite tight for values of p less than approximately 0.1. Furthermore, the more useful upper bounds on the probability of disconnection are tighter than the lower bounds. The bounds derived here thus useful tools for the design of Harary networks in the $p \approx 0$ regime.

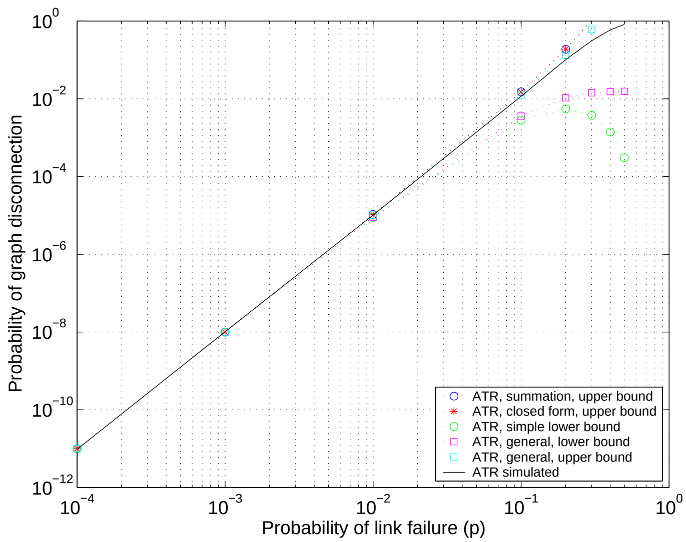

Fig. 2. Probability of graph disconnection versus p for $H(10,3)$. "ATR, simple lower bound" refers to $np^{\Delta}(1-p)^{e-\Delta}$, "ATR, general, lower bound" refers to (5), "ATR, general, upper bound" refers to (6), "ATR, summation, upper bound" refers to (13), and "ATR, closed form, upper bound" refers to (14).

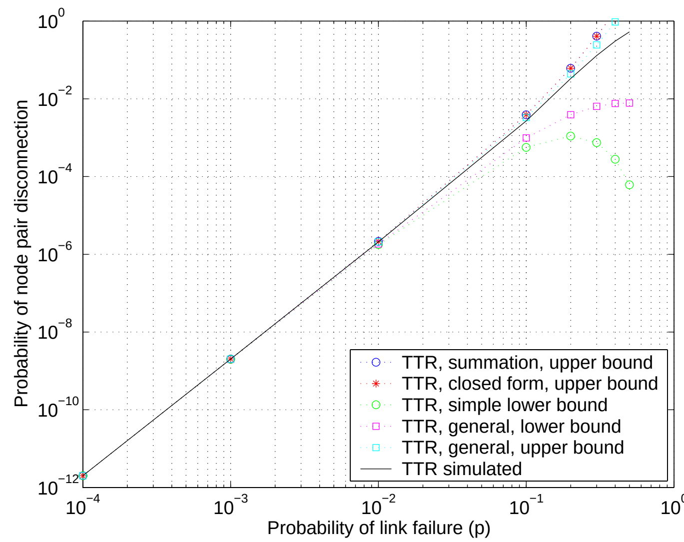

Fig. 3. Worst-case probability of node pair disconnection versus p for $H(10,3)$. "TTR, simple lower bound" refers to $2p^{\Delta}(1-p)^{e-\Delta}$, "TTR, general, lower bound" refers to the right inequality of (7), "TTR, general, upper bound" refers to the left inequality of (7), "TTR, summation, upper bound" refers to (15), and "TTR, closed form, upper bound" refers to (16).

B. Two-terminal reliability when $p \approx 0$

The derivation of a lower bound for the node pair connection probability $P_c^{sd}(G,p)$ is virtually identical to that of $P_c(G,p)$ for p close to zero in section V-A. The difference is that we are only interested in partitions of the network nodes that result in nodes s and d residing in different partitions. Hence, we modify (13) to obtain:

$$P_c^{sd}(G,p) \geq 1 - \left(2p^{\Delta} + 2 \sum_{i=2}^{\Delta/2+1} \binom{n-2}{i-1} p^{i\Delta - 2\binom{i}{2}} + 2 \sum_{i=\Delta/2+2}^{\lfloor n/2 \rfloor} \binom{n-2}{i-1} p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right). \tag{15}$$

In a manner similar to section V-A, we can derive a slightly looser upper bound for $P_c^{sd}(G,p)$:

$$P_c^{sd}(G,p) \geq 1 - \left(2p^{\Delta} + 2 \left\lfloor \frac{\Delta}{2} \right\rfloor \binom{n-2}{\lfloor \frac{\Delta}{2} \rfloor} \left(p^{2\Delta-2} - p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \right) + p^{\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil} \left[2^{n-2} + \binom{n-2}{\lfloor \frac{n-2}{2} \rfloor} - 2 \right] \right). \tag{16}$$

The quality of these bounds is illustrated in Figure 3 for the ten node, degree three Harary graph. As in the all-terminal case, the two-terminal bounds plotted are quite tight for values of p less than approximately 0.1, and the upper bounds on the probability of disconnection are tighter than the lower bounds.

C. All-terminal reliability when $p \approx 1$

For p close to unity, we bound $P_c(G,p)$ using the approach outlined in section IV-C, which requires knowledge of the number of spanning trees in a graph. We specialize Theorem 3 to Harary graphs:

Lemma 2 *The number of spanning trees in the degree Δ Harary graph is:*

$$t(G) = \begin{cases} \frac{1}{n} \prod_{i=1}^{n-1} \left[4 \sum_{j=1}^{h} \sin^2(ji\pi/n) \right], & \text{if } \Delta \text{ is even,} \\ \frac{1}{n} \prod_{i=1}^{n-1} \left[4 \sum_{j=1}^{h-1} \sin^2(ji\pi/n) - (-1)^i + 1 \right], & \text{if } \Delta \text{ is odd.} \end{cases}$$

The quality of these bounds is illustrated in Figure 4 for the ten node, degree three Harary graph. In general, it appears that Harary graphs have fewer spanning trees than many of its circulant counterparts with the same number of nodes and edges. For example, the Harary graph $H(10,4)$ possesses 30250 spanning trees, whereas the circulant $C_{10}\langle 1,3 \rangle$ possesses 40500 spanning trees. For values of p close to 1 this translates to a probability of connection for $H(10,4)$ which is smaller than that of $C_{10}\langle 1,3 \rangle$ by approximately $10250(1-p)^9$.

D. Two-terminal reliability when $p \approx 1$

When the probability of link failure p is close to unity, we bound the probability of node pair connection using the technique outlined in section IV-D. This technique requires knowledge of the edge-disjoint path lengths between nodes s and d. We consider Harary graphs of even degree only, as the case of odd degree is significantly more complex. Let d_{sd} denote the node separation of s and d. Define the parameter h as $\min(d_{sd}, N - d_{sd})$. By inspecting the structure

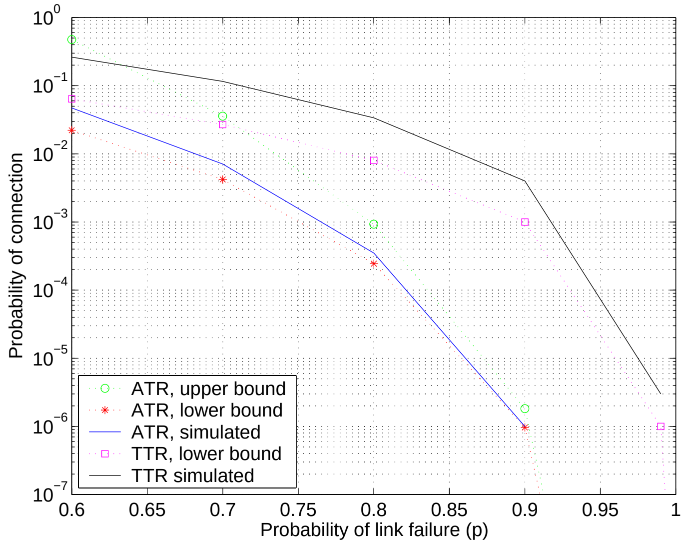

Fig. 4. Probability of graph connection and worst-case probability of node pair connection versus p for $H(10,3)$. "ATR, lower bound" refers to (8), "ATR, upper bound" refers to (9), and "TTR, lower bound" refers to (17).

of even degree Harary graphs, the length of path i for $i = 1, \ldots, \min(h, \Delta/2)$ is found to be:

$$l_i = \left\lceil \frac{h-i+1}{\Delta/2} \right\rceil + 1 - \delta_1(i)$$

where the function $\delta_x(i)$ equals unity when its argument i equals x and is otherwise equal to zero. If $\Delta/2 > h$, then the length of path i for $i = h+1, \ldots, \Delta/2$ is given by:

$$l_i = \left\lceil \frac{i-h}{\Delta/2} \right\rceil + 1.$$

Finally, the length of path i for $i = \Delta/2+1, \ldots, \Delta$ is given by:

$$l_i = \left\lceil \frac{n-h-i+1}{\Delta/2} \right\rceil + 1 - \delta_{\Delta/2+1}(i).$$

These path lengths can now be substituted into (10) to obtain a lower bound for $P_c^{sd}(G,p)$.

When $p \approx 1$, $P_c^{sd}(G,p)$ is minimized for node pairs which are most distantly placed in G. For even degree Harary graphs, such node pairs have indices which differ by $\lceil (n-1)/2 \rceil$. The diameter of even degree Harary graphs is thus $\frac{2}{\Delta}\lceil \frac{n-1}{2} \rceil$. For odd degree Harary graphs[3], most distantly placed nodes can be shown to have indices which differ by $\lceil (n+\Delta-3)/4 \rceil$, with a resulting graph diameter of $\frac{2}{\Delta-1}\left\lceil \frac{n+\Delta-3}{4} \right\rceil$. Thus, using (11), we have the following lower bound for $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ for Harary graphs:

$$(1-p)^{k(G)} \le \min_{s,d}\left[P_c^{sd}(G,p)\right] \tag{17}$$

where,

$$k(G) = \begin{cases} \frac{2}{\Delta}\lceil \frac{n-1}{2} \rceil, & \text{if } \Delta \text{ is even,} \\ \frac{2}{\Delta-1}\left\lceil \frac{n+\Delta-3}{4} \right\rceil, & \text{if } \Delta \text{ is odd.} \end{cases}$$

[3]We restrict our attention to odd degree Harary graphs which are *strictly* regular. These graphs therefore have an even number of nodes.

The quality of this bound is illustrated in Figure 4 for the ten node, degree three Harary graph. Note that as the number of nodes n increases relative to the degree Δ, odd degree Harary graphs possess diameters which are approximately half as large as even degree Harary graphs.

Furthermore, because Harary graphs are defined such that nodes are connected to their nearest neighbors[4], the diameter of Harary graphs are generally larger than graphs with the same number of nodes and edges. For example, the Harary graph $H(30,4)$ has diameter eight, whereas the circulant $C_{30}\langle 4,5 \rangle$ has diameter four. It is interesting to consider the relationship between a graph's diameter and its number of spanning trees. Although a smaller diameter does not necessarily imply a larger number of spanning trees, or vice versa, there does seem to exist an inverse correlation between these properties. The intuition behind this trend is that for the same number of nodes and edges, the nodes of a graph with a larger diameter are generally more distant from one another. The result is that there are fewer combinations of edges of the graph that could form spanning trees since there are more constraints on the edges in order that more distant nodes be connected. Hence, the number of spanning trees generally decreases with diameter when the number of nodes and edges is held constant. Thus, when $p \approx 1$, graphs which have good all-terminal reliability performance generally have good two-terminal reliability performance, and vice versa.

VI. Conclusion

In this paper, general reliability bounds which are useful in the design of communication networks were presented. Our reliability study addressed the often neglected $p \approx 1$ regime, in which network diameter and number of spanning trees were identified as the key figures of merit. Our reliability study was then specialized to Harary graphs, which yielded new results for this family of graphs.

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