# Exam 1

**Dot product**

$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

$\vec{A}\cdot\vec{B} = (a_1b_1 + a_2b_2)$

**Cross product**

$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\ \hat{n}$

Area of parallelogram is equal to half the length of the × prod

**Matrix arithmetic**

1. Addition (A + B)
   - add corresponding elements
   - must be same shape
2. Multiplication (AB = C)
   - multiply each element of row i by column j, then add
3. Inversion ($A^{-1}$)
   - A → M (minors) → C (cofactors) → J (adjoint) → divide by $|A|^{-1}$
   - M: just determinant of the minors
   - C: checkerboard signs
   - J: flip around top left – bottom right diagonal

**One or many solutions**

If |A| is nonzero, the matrix has exactly one solution.
If |A| = 0, the matrix has *either* 0 or ∞ solutions.

**Lines in parametric**

With $P_0$ ($x_0$, $y_0$, $z_0$) and $\vec{v}$ (a, b, c):

$x = at + x_0$
$y = bt + y_0$
$z = ct + z_0$

**Planes in parametric**

With $P_0$ ($x_0$, $y_0$, $z_0$) and $\vec{v}$ (a, b, c):

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

because $\vec{N}\cdot\vec{v} = 0$

Given $P_1$, $P_2$, $P_3$:

Find $P_1P_2, P_1P_3$

Take × prod to get $\vec{N}$

Given surface ax + by + cz, $\vec{N} = a\hat{i} + b\hat{j} + c\hat{k}$

**More parametric**

$\mathbf{v}(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

Intersection of curve/surface: sub curve eqn into surface
Angle between two planes:

Find $N_1, N_2$

Take · prod

**Product rule for · and ×**

$$\frac{d(\vec{A}\cdot\vec{B})}{dt} = \vec{A}\cdot\frac{d\vec{B}}{dt}+\vec{B}\cdot\frac{d\vec{A}}{dt}$$

$$\frac{d(\vec{A}\times\vec{B})}{dt} = \vec{A}\times\frac{d\vec{B}}{dt}+\vec{B}\times\frac{d\vec{A}}{dt}$$

# Exam 2

**Tangent plane**

$$z - z_0 = \frac{\partial z}{\partial x}(x-x_0)+\frac{\partial z}{\partial y}(y-y_0)$$

$$\nabla f_{(x_0,y_0,z_0)}[(x-x_0)+(y-y_0)+(z-z_0)] = \frac{\partial f}{\partial x}(x-x_0)+\frac{\partial f}{\partial y}(y-y_0)+\frac{\partial f}{\partial z}(z-z_0)$$

**Normal vector**

if z = f(x,y), $\frac{\partial z}{\partial x}\hat{i}+\frac{\partial z}{\partial y}\hat{j}-\hat{k}$

$\nabla f_{(x_0,y_0,z_0)}$

**Max/min**

Find partials, set to zero (critical points)

$d = f_{xx}f_{yy} - f_{xy}^2$ d – saddle

d + $f_{xx}$ + min

$f_{xx}$ – max

**Chain rule**

z = f(u,v) $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{du}{dx}+\frac{\partial f}{\partial v}\frac{dv}{dx}$

u = u(x)
v = v(x)

**Gradient**

$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}$ magnitude ⇒ steepness, always points uphill

**Directional derivative**

$\vec{\nabla} f \cdot \frac{\vec{R}}{|\vec{R}|}$ $\Delta z \approx \frac{dz}{ds} \Delta s$

**Lagrange multipliers**

f(x,y), restraint g(x,y)

$L(x, y, \lambda) = f(x, y) - \lambda(x, y)$

take $\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial \lambda}$, set equal to zero, solve system

## Exam 3

**Double integrals**

$$\int_a^b \int_c^d f(x, y) dy dx$$

Do inner ∫ as usual, treat outer variable as constant

Do outer

**Polar coordinates**

$$\int_\theta \int_r f(r, \theta) r dr d\theta$$

**Integration applications**

Mass: $\iint_R \delta dA$

Average of f over R: $\frac{\iint_R f(x, y) dA}{area}$

Center of mass: $\overline{x} = \frac{\iint_R x \delta dA}{mass}$

Moment of inertia: $\iint_R (dist_{axis})^2 \delta dA$

**Work: line integrals**

$\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy$

Paramaterize x and y as functions of t $\Rightarrow \int_{t_0}^{t_1} M \frac{dx}{dt} + N \frac{dy}{dt} dt$

**Gradient fields**

For a gradient field, $\int_C \vec{F} \cdot d\vec{r} = f(end) - f(start)$

Gradient field test: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

To find the potential function, integrate Mdx and Ndy.

**Green's theorem**

$$\oint_C Mdx + Ndy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$

If $N_x - M_y = 1$, $\oint_C \vec{F} \cdot d\vec{r} = area$

**Flux**

$$\oint_C -Ndx + Mdy = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\, dA$$

# Exam 4

**Triple integrals**

$$\iiint_D f(x,y,z) = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dzdydx$$

get outer and middle variables from shadow

**Cylindrical coordinates**

$$\int_\theta \int_r \int_z f(r,\theta,z)\,rdzdrd\theta$$

$y = r\cos\theta$
$x = r\sin\theta$
$z = z$

**Spherical coordinates**

$$\int_\theta \int_\varphi \int_\rho f(\rho,\varphi,\theta)\rho^2 \sin\varphi\, d\rho d\varphi d\theta$$

$x = \rho\sin\varphi\cos\theta$
$y = \rho\sin\varphi\sin\theta$
$z = \rho\cos\theta$

**Applications of 3D integrals**

Mass: $\iiint_D \delta\, dV$

Average of f over D: $\dfrac{\iiint_D f(x,y,z)\, dV}{\text{volume } D}$

Center of mass: $\dfrac{\iiint_D x\, \delta\, dV}{\text{mass}}$

Moment of inertia: $\iiint_D (\text{distance from axis})^2\, \delta\, dV$

Gravitational attraction: $G \iiint_D \delta \cos\varphi \sin\varphi\, dV$

**Surface integrals**

Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, ds$.

1. Inspection. $\mathbf{F} \cdot \mathbf{n}$ is constant.
   $\iint_S \mathbf{F} \cdot \mathbf{n}\, ds = (\mathbf{F} \cdot \mathbf{n})(\text{area } s)$
2. Cylindrical/spherical coords.
   $\int_z\int_\theta\ \mathbf{F} \cdot \mathbf{n}\, (\text{radius})\, dz d\theta$ $\qquad \mathbf{n}_{cyl} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{\text{radius}}$
   $\int_\theta\int_\varphi\ \mathbf{F} \cdot \mathbf{n}\, (\text{radius})^2 \sin\varphi\, d\varphi d\theta$ $\qquad \mathbf{n}_{sph} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\text{radius}}$
3. Rectangular case. S is $z = f(x,y)$.

$\iint_S \mathbf{F} \cdot \mathbf{n}\, ds = \iint_{shadow} \mathbf{F} \cdot (-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k})\, dA$

**Divergence theorem**

S is closed surface oriented outward

$\S\S\ \mathbf{F} \cdot d\mathbf{s} = \iiint_D (\text{div } \mathbf{F})\, dV$ *Flux

$\text{div } \mathbf{F} = M_x + N_y + P_z = \nabla \cdot \mathbf{F}$

**Stokes' theorem**

$\S\mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\, ds$ *Work

**Properties of div, curl, grad**

1. div(curl **F**) = 0

   $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

2. curl(gradient field) = 0

   gradient field test: $\nabla \times \mathbf{F} = 0$