18.100A: Typed Lecture Notes

Lecture 7:

Convergent Sequences of Real Numbers

We will do another example of limits that converge:

Example 1

$\lim_{n\to\infty} \frac{1}{n^2+2n+100} = 0$.

Proof: Let $\epsilon > 0$ and choose $M \in \mathbb{N}$ such that $M > \frac{\epsilon^{-1}}{2}$. Then, $\forall n \geq M$,

$$\left|\frac{1}{n^2+2m+100} - 0\right| \leq \frac{1}{n^2+2m+100} \leq \frac{1}{2n} \leq \frac{1}{2M} < \epsilon.$$

■

The fact that we can go from a complicated rational function to one that works for our purposes (namely to prove the sequence converges to 0) is *awesome*.

Example 2

Consider the sequence $x_n = (-1)^n$. This sequence is divergent.

Proof: Let $x \in \mathbb{R}$. We claim $\lim_{n\to\infty}(-1)^n \neq x$. To prove this, we simply need find *an* epsilon that stops the sequence from converging. For instance, consider $\epsilon_0 = \frac{1}{2}$. Then, for $M \in \mathbb{N}$,

$$1 = |(-1)^M - (-1)^{M+1}| \leq |(-1)^M - x| + |(-1)^{M+1} - x|.$$

Thus, either $|(-1)^M - x| \geq \frac{1}{2}$ or $|(-1)^{M+1} - x| \geq \frac{1}{2}$. In either case, this shows that the limit cannot converge to x. ■

Theorem 3

If $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded.

Before we start the proof, let's first talk about the idea of the proof. Let $\epsilon = 1$ such that $|x_n - x| < 1$ for all $n \leq M$ for some $M \in \mathbb{N}$. Then, there are finitely many elements not in the interval $(x-1, x+1)$. We use this to our advantage.

Proof: Suppose that $\lim_{n\to\infty} x_n = x$. Thus, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n \geq M$. Let

$$B = \max\{|x_1|, |x_2|, \ldots, |x_{M-1}|, |x| + 1\}.$$

If $n < M$, then $|x_n| \leq B$ by construction. If $n \geq M$, then

$$|x_n| \leq |x_n - x| + |x| < 1 + |x| \leq B.$$

□

Definition 4 (Monotone)

A sequence $\{x_n\}$ is **monotone increasing** if $\forall n \in \mathbb{N}$, $x_n \leq x_{n+1}$. A sequence $\{x_n\}$ is **monotone decreasing** if $\forall n \in \mathbb{N}$, $x_n \geq x_{n+1}$. If $\{x_n\}$ is either monotone increasing or monotone decreasing, we say $\{x_n\}$ is **monotone** or **monotonic.**

Example 5

For example, $x_n = \frac{1}{n}$ is monotone, $y_n = -\frac{1}{n}$ is monotone increasing, and $(-1)^n$ is neither.

Theorem 6

Let $\{x_n\}$ be a monotone increasing sequence. Then, $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded. Moreover, $\lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$.

Proof: Firstly, we know that if $\{x_n\}$ is convergent then it is bounded by the previous theorem, Now assume that $\{x_n\}$ is bounded. Then, $x := \sup\{x_n \mid n \in \mathbb{N}\}$ exists in $\mathbb{R}$ by the lowest upper bound property of $\mathbb{R}$. We now prove that

$$\lim_{n\to\infty} x_n = x.$$

Let $\epsilon > 0$. Then, $\exists M_0 \in \mathbb{N}$ such that

$$x - \epsilon < x_{M_0} < x$$

since x is the supremum of the set. Let $M = M_0$. Then, $\forall n \geq M$, we have

$$x - \epsilon < x_{M_0} = x_M \leq x_n \leq x < x + \epsilon.$$

Therefore, $|x_n| < |x + \epsilon|$. Therefore, $x_n \to x$. $\square$

Theorem 7

Let $\{x_n\}$ be a monotone decreasing function. Then, $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded. Moreover,

$$\lim_{n\to\infty} x_n = \inf\{x_n \mid n \in \mathbb{N}\}$$

The proof of this is similar to the previous theorem and is thus omitted.

Definition 8 (Subsequence)

Informally, a subsequence is a sequence with entries coming from another given sequence. In other words, let $\{x_n\}$ be a sequence and let $\{n_k\}$ be a strictly increasing sequence of natural numbers. Then the sequence

$$\{x_{n_k}\}_{k=1}^{\infty}$$

is called a **subsequence** of $\{x_n\}$.

Consider the sequence $\{x_n\} = n$ – in other words, the sequence $1, 2, 3, 4, \ldots$. Then, the following are subsequences of x_n :

$$1, 3, 5, 7, 9, 11, \ldots$$
$$2, 4, 6, 8, 10, \ldots$$
$$2, 3, 5, 7, 11, 13, \ldots.$$

The first two are described by $x_{n_k} = x_{2k}$ and $x_{n_k} = x_{2k-1}$ respectively.

Question 9. *How would we describe the third?*

Continuing to let $\{x_n = n\}_n$, the following are *not* subsequences:

$$1, 1, 1, 1, 1, 1, \ldots$$
$$1, 1, 3, 3, 5, 5, \ldots.$$

Now consider the sequence $\{(-1)^n\}$. Then we have the subsequences

$$x_{n_k} = x_{2k-1} \to -1, -1, -1, \ldots$$
$$y_{n_k} = x_{2k} \to 1, 1, 1, \ldots.$$

Theorem 10

If $\{x_n\}$ converges to x, then any subsequence of x_n will converge to x.

Proof: Suppose $\lim_{n\to\infty} x_n = x$. Let $\epsilon > 0$. Then, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$,

$$|x_n - x| < \epsilon.$$

Choose $M = M_0$. If $k \geq M$, then $n_k \geq k \geq M = M_0$. Hence, for all $\epsilon > 0$ there exists an M such that for all $n_k > M$,

$$|x_{n-k} - x| < \epsilon.$$

□

Remark 11. *Notice that this also implies that the sequence $\{(-1)^n\}_n$ is divergent.*

Notation 12 (DNC)

We can denote the statement "a sequence does not converge"/"a sequence is divergent" as "the sequence DNC".