18.100A: Typed Lecture Notes

Lecture 11:
Absolute Convergence and the Comparison Test for Series

Recall 1
Last time we showed that if $\sum x_n$ converges then $\lim_{n\to\infty} x_n = 0$.

Question 2. *Is the converse true? Does* $\lim_{n\to\infty} x_n = 0 \implies \sum x_n$ *converges?*

Theorem 3
The series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

Proof: We will show that there exists a subsequence of $s_m = \sum_{n=1}^{m} \frac{1}{n}$ which is unbounded, which will imply the series diverges. Consider, for $\ell \in \mathbb{N}$,

$$s_{2^\ell} = \sum_{n=1}^{2^\ell} \frac{1}{n}.$$

Then,

$$\begin{aligned} s_{2^\ell} &= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \ldots \left(\frac{1}{2^{\ell-1}+1} + \cdots + \frac{1}{2^\ell}\right) \\ &= 1 + \sum_{\lambda=1}^{\ell} \sum_{n=2^{\lambda-1}+1}^{2^\lambda} \frac{1}{n} \\ &\geq 1 + \sum_{\lambda=1}^{\ell} \sum_{n=2^{\lambda-1}+1}^{2^\lambda} \frac{1}{2^\lambda} \\ &= 1 + \sum_{\lambda=1}^{\ell} \frac{1}{2^\lambda}(2^\lambda - (2^{\lambda-1}+1) + 1) \\ &= 1 + \sum_{\lambda=1}^{\ell} \frac{2^{\lambda-1}}{2^\lambda} \\ &= 1 + \frac{\ell}{2}. \end{aligned}$$

Thus, $\{s_{2^\ell}\}_{\ell=1}^{\infty}$ is unbounded which implies $\{s_{2^\ell}\}$ does not converge. □

Remark 4. *The series* $\sum \frac{1}{n}$ *is called the* ***harmonic series.***

Theorem 5
Let $\alpha \in \mathbb{R}$ and $\sum x_n$ and $\sum y_n$ be convergent series. Then the series $\sum(\alpha x_n + y_n)$ converges and

$$\sum(\alpha x_n + y_n) = \alpha \sum x_n + \sum y_n.$$

Proof: The partial sums satisfy

$$\sum_{n=1}^{m}(\alpha x_n + y_n) = \alpha \sum_{n=1}^{m} x_n + \sum_{n=1}^{m} y_n.$$

By linear properties of limits, it follows that

$$\lim_{m\to\infty} \sum_{n=1}^{m}(\alpha x_n + y_n) = \alpha \sum x_n + \sum y_n.$$

$\square$

Series with non-negative terms are easier to work with than general series as then $\{s_n\}$ is a monotone sequence.

Theorem 6

If $\forall n \in \mathbb{N}\ x_n \geq 0$, then $\sum x_n$ converges if and only if $\{s_m\}$ is bounded.

Proof: If $x_n \geq 0$ for all $n \in \mathbb{N}$ then

$$s_{m+1} = \sum_{n=1}^{m+1} x_n = \sum_{n=1}^{m} x_n + x_{m+1} = s_m + x_{m+1} \geq s_m$$

Thus, $\{s_m\}$ is a monotone increasing sequence. Therefore, $\{s_m\}$ converges if and only if $\{s_m\}$ is bounded. $\square$

Definition 7

$\sum x_n$ **converges absolutely** if $\sum |x_n|$ converges.

Theorem 8

If $\sum x_n$ converges absolutely then $\sum x_n$ converges.

Proof: Suppose $\sum |x_n|$ converges. We will then show that $\sum x_n$ is Cauchy.

Claim: $\forall m \geq 2$, $|\sum_{n=1}^{m} x_n| \leq \sum_{n=1}^{m} |x_n|$. We prove this claim by induction. For $m = 2$, this states that $|x_1 + x_2| \leq |x_1| + |x_2|$, which follows by the Triangle Inequality. Suppose for all $\left|\sum_{n=1}^{\ell} x_n\right| \leq \sum_{n=1}^{\ell} |x_n|$. Then,

$$\left|\sum_{n=1}^{\ell+1} x_n\right| \leq \left|\sum_{n=1}^{\ell} x_n\right| + |x_{\ell+1}| \leq \sum_{n=1}^{\ell} |x_n| + |x_{\ell+1}| = \sum_{n=1}^{\ell+1} |x_n|.$$

We now prove that $\sum x_n$ is Cauchy. Let $\epsilon > 0$. Since $\sum |x_n|$ converges, $\sum |x_n|$ is Cauchy. Therefore, there exists an $M_0 \in \mathbb{N}$ such that for all $\ell > m \geq M_0$,

$$\sum_{n=m+1}^{\ell} |x_n| < \epsilon.$$

Choose $M = M_0$. Then, for all $\ell > m \geq M$,

$$\left|\sum_{n=m+1}^{\ell} x_n\right| \leq \sum_{n=m+1}^{\ell} |x_n| < \epsilon.$$

Hence, $\sum x_n$ is Cauchy, and thus converges. $\square$

Remark 9. *We will see that* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *is convergent but not absolutely convergent.*

Notice that it is immediately clear that this series is not absolutely convergent as $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ (the harmonic series), which doesn't converge.

Convergence tests

Theorem 10 (Comparison Test)
Suppose for all $n \in \mathbb{N}$ $0 \leq x_n \leq y_n$. Then,

1. if $\sum y_n$ converges, then $\sum x_n$ converges.
2. if $\sum x_n$ diverges, then $\sum y_n$ diverges.

Proof:

1. If $\sum y_n$ converges, then $\{\sum_{n=1}^m y_n\}_{m=1}^\infty$ is bounded. In other words, there exists a $B \geq 0$ such that for all $m \in \mathbb{N}$,
$$\sum_{n=1}^m y_n \leq B.$$
Thus, for all $m \in \mathbb{N}$, $\sum_{n=1}^m x_n \leq \sum_{n=1}^m y_n \leq B$. Therefore, the partial sums of $\{x_n\}$ are bounded, which implies $\sum x_n$ converges.
2. If $\sum x_n$ diverges, then $\{\sum_{n=1}^m x_n\}_{m=1}^\infty$ is unbounded. We now prove that
$$\left\{\sum_{n=1}^m y_n\right\}_{m=1}^\infty$$
is also unbounded. Let $B \geq 0$. Then, $\exists m \in \mathbb{N}$ such that
$$\sum_{n=1}^m x_n \geq B.$$
Therefore, $\sum_{n=1}^m y_n \geq \sum_{n=1}^m x_n \geq B$. Thus, $\{\sum_{n=1}^m y_n\}_{m=1}^\infty$ is unbounded, which implies $\sum y_n$ diverges.

$\square$

Remark 11. *We will see that geometric series and the Comparison Test imply everything!*

Theorem 12
For $p \in \mathbb{R}$, the series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if $p > 1$.

Proof: ($\implies$) We prove this direction through contradiction. Suppose $\sum_{n=1}^\infty \frac{1}{n^p}$ converges and $p \leq 1$. Then, $\frac{1}{n^p} \geq \frac{1}{n}$, and $\sum \frac{1}{n}$ diverges. Therefore, by the Comparison Test, $\sum \frac{1}{n^p}$ also diverges. Hence, if $\sum \frac{1}{n^p}$ converges, then $p > 1$.

($\impliedby$) Suppose $p > 1$. We first prove that a subsequence of the partial series is bounded.

Claim 1: $\forall k \in \mathbb{N}$, $s_{2^k} \leq 1 + \frac{1}{1-2^{-(p-1)}}$. Proof:

$$\begin{aligned}
s_{2^k} &= 1 + \sum_{\ell=1}^{k} \sum_{n=2^{\ell-1}+1}^{2^\ell} \frac{1}{n^p} \\
&\leq 1 + \sum_{\ell=1}^{k} \sum_{n=2^{\ell-1}+1}^{2^\ell} \frac{1}{(2^{\ell-1}+1)^p} \\
&\leq 1 + \sum_{\ell=1}^{k} 2^{-p(\ell-1)}(2^\ell - (2^{\ell-1}+1)+1) \\
&= 1 + \sum_{\ell=1}^{k} 2^{-(p-1)(\ell-1)} \\
&= 1 + \sum_{\ell=0}^{k-1} 2^{-(p-1)\ell} \\
&\leq 1 + \sum_{\ell=0}^{\infty} 2^{-(p-1)\ell} \\
&= 1 + \frac{1}{1-2^{-(p-1)}}
\end{aligned}$$

using the fact that $p - 1 > 0$, and using properties of geometric series. Thus, Claim 1 is proven.

Claim 2: $\{s_m = \sum_{n=1}^{m} \frac{1}{n^p}\}$ is bounded. Proof: Let $m \in \mathbb{N}$. Since $2^m > m$, we have that

$$s_m = \sum_{n=1}^{m} \frac{1}{n^p} \leq \sum_{n=1}^{2^m} n^{-p} \leq 1 + \frac{1}{1-2^{-(p-1)}}.$$

Hence, the partial sums are bounded, which implies $\{s_m\}$ converges. $\square$