18.100A: Typed Lecture Notes

Lecture 15:
The Continuity of Sine and Cosine
and the Many Discontinuities of Dirichlet's Function

Theorem 1

Let $S \subset \mathbb{R}$, $c \in S$, and $f : S \to \mathbb{R}$. Then,

1. if c is not a cluster point of f, then f is continuous at c.
2. if c is a cluster point of S, then f is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$.
3. f is continuous at c if and only if for every sequence $\{x_n\}$ of elements of S such that $x_n \to c$, we have $f(x_n) \to f(c)$.

Proof:

1. Let $\epsilon > 0$. Since c is not a cluster point of S, $\exists \delta_0 > 0$ such that $(c - \delta_0, c + \delta_0) \cap S = \{c\}$. Choose $\delta = \delta_0$. If $x \in S$ and $|x - c| < \delta \implies x = c \implies |f(x) - f(c)| = 0 < \epsilon$. Therefore, f is continuous at c.
2. This part of the theorem is left as an exercise (or read the short proof in the book).
3. ($\implies$) Suppose f is continuous at c. Let $\{x_n\}$ be a sequence such that $x_n \to c$. Let $\epsilon > 0$. Since f is continuous at c, $\exists \delta > 0$ such that if $x \in S$ and $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$. Since $x_n \to c$, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$, $|x_n - c| < \delta$. Choose $M = M_0$. Then, $\forall n \geq M$,

$$|x_n - c| < \delta \implies |f(x_n) - f(c)| < \epsilon.$$

   Thus, $f(x_n) \to f(c)$.

   ($\impliedby$) Suppose that for every sequence $\{x_n\}$ of elements of S such that $x_n \to c$, we have that $f(x_n) \to f(c)$. We will work towards a contradiction. Suppose $f(x)$ is not continuous at c. Then, $\exists \epsilon_0$ such that $\forall \delta > 0$ $\exists x \in S$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon_0$.

   Thus, $\forall n \in \mathbb{N}$, $\exists x_n \in S$ such that $|x_n - c| < \frac{1}{n}$ and

$$|f(x_n) - f(c)| \geq \epsilon_0.$$

   Thus, by the Squeeze Theorem, $|x_n - c| \to 0 \implies x_n \to c$. Therefore,

$$0 = \lim_{n \to \infty} |f(x_n) - f(c)| \geq \epsilon_0$$

   which is a contradiction.

$\square$

Theorem 2

The functions $f(x) = \sin x$ and $g(x) = \cos x$ are continuous functions on $\mathbb{R}$.

Proof: From their definitions in terms of the unit circle, we have that $\sin^2(x) + \cos^2(x) = 1$. Also note the following:

1. $\forall x \in \mathbb{R}$, $|\sin x| \leq 1$ and $|\cos x| \leq 1$
2. $\forall x \in \mathbb{R}$, $|\sin x| \leq |x|$
3. The angle formulae:

$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b) \quad \text{and} \quad \sin(a) - \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right).$$

We now show that $\sin x$ is continuous on $\mathbb{R}$. Let $\epsilon > 0$. Choose $\delta = \epsilon$. Then, if $|x-c| < \delta$, then

$$|\sin x - \sin c| = 2\left|\sin\frac{x-c}{2}\cos\frac{x+c}{2}\right| \leq 2\left|\sin\frac{x-c}{2}\right| \leq 2\frac{|x-c|}{2} = |x-c| < \delta = \epsilon.$$

Therefore, $\sin x$ is continuous on $\mathbb{R}$. We now show that $\cos x$ is continuous. Recall that $\forall x \in \mathbb{R}$, $\cos x = \sin(x+\pi/2)$. Let $c \in \mathbb{R}$ and let $\{x_n\}$ be a sequence such that $x_n \to c$. Then, $x_n + \pi/2 \to c + \pi/2$. Since $\sin x$ is continuous on $\mathbb{R}$,

$$\lim_{n\to\infty} \cos x_n = \lim_{n\to\infty} \sin\left(x_n + \frac{\pi}{2}\right) = \sin\left(c + \frac{\pi}{2}\right) = \cos c.$$

Therefore, $\cos x$ is continuous on $\mathbb{R}$. $\square$

Theorem 3

Let f be a polynomial, in other words let f be of the form

$$f(x) = a_d x^d + \cdots + a_1 x + a_0.$$

Then, f is continuous on all of $\mathbb{R}$.

Proof: Let $c \in \mathbb{R}$ and let $\{x_n\}$ be a sequence such that $x_n \to c$. Then, by the limit theorem for sequences,

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} (a_d x^d + \cdots + a_1 x + a_0) \\
&= a_d(\lim_{n\to\infty} x_n)^d + \cdots + a_1(\lim_{n\to\infty} x_n) + a_0 \\
&= a_d c^d + \cdots + a_1 c + a_0 \\
&= f(c).
\end{aligned}$$

Thus, f is continuous at c for all $c \in \mathbb{R}$. $\square$

Theorem 4

If $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ are continuous at $c \in S$, then

1. $f + g$ is continuous at c,
2. $f \cdot g$ is continuous at c,
3. and if $\forall x \in S$ $g(x) \neq 0$, then $\frac{f}{g}$ is continuous at c.

Proof: These proofs are left to the reader. $\square$

Theorem 5

Let $A, B \subset \mathbb{R}$, $f : B \to \mathbb{R}$, $g : A \to B$. Then, if g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c.

Proof: Suppose $x_n \to c$. Then, $g(x_n) \to g(c)$, and thus

$$f(g(x_n)) \to f(g(c)).$$

$\square$

Example 6

These theorems allow us to say that some functions are continuous without a huge $\epsilon - \delta$ proof:

i) $\frac{1}{x^2}$ is continuous on $(0, \infty)$. This follows as $g(x) = x^2$ is continuous on $(0, \infty)$ and thus $\frac{1}{g(x)} = 1/x^2$ is continuous on $(0, \infty)$.

ii) $(\cos \frac{1}{x^2})^2$ is continuous on $(0, \infty)$. This follows as $\cos x$ is continuous on $\mathbb{R}$, and thus $g(x) = \cos(1/x^2)$ is continuous on $(0, \infty)$. Furthermore, since $f(x) = x^2$ is continuous on $\mathbb{R}$, $(f \circ g)(x) = (\cos 1/x^2)^2$ is continuous on $(0, \infty)$.

Question 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Does there exists a point $c \in \mathbb{R}$ such that f is continuous at c?*

Theorem 8

The function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not continuous on all of $\mathbb{R}$. This function is called the **Dirichlet function**.

Proof: We have two cases: $c \in \mathbb{Q}$ or $c \notin \mathbb{Q}$.

1. $c \in \mathbb{Q}$. For each $n \in \mathbb{N}$, $\exists x_n \notin \mathbb{Q}$ such that $c < x_n < c + 1/n$, and thus $x_n \to c$ but $f(x_n) = 1$ for all n so

$$1 = \lim_{n \to \infty} f(x_n) \neq f(c) = 0.$$

2. $c \notin \mathbb{Q}$. Similarly, for each $n \in \mathbb{N}$, $\exists x_n \in \mathbb{Q}$ such that $c < x_n < c + 1/n$, and thus $x_n \to c$ but $f(x_n) = 0$ for all n so

$$0 = \lim_{n \to \infty} f(x_n) \neq f(c) = 1.$$

$\square$