18.100A: Typed Lecture Notes

Lecture 5:
The Archimedian Property, Density of the Rationals, and Absolute Value

For all $x, y \in \mathbb{R}$ and $x < y$, there exists an $r \in \mathbb{R}$ such that $x < r < y$ (take $r = \frac{x+y}{2}$).

Question 1. *Can we find $r \in \mathbb{Q}$ such that $x < r < y$?*

Theorem 2

The answer is yes!

i) (Archimedian Property) If $x, y \in \mathbb{R}$ and $x > 0$, then $\exists n \in \mathbb{N}$ such that $nx > y$.

ii) (Density of $\mathbb{Q}$) If $x, y \in \mathbb{R}$ and $x < y$ then $\exists r \in \mathbb{Q}$ such that $x < r < y$.

Proof:

i) Suppose that $x, y \in \mathbb{R}$ and $x > 0$. Then we wish to show that $\exists n \in \mathbb{N}$ such that $n > \frac{y}{x}$. Suppose this is not the case. Then, $\forall n \in \mathbb{N}$, $n \leq \frac{y}{x}$. In other words, $\mathbb{N}$ is bounded above by $\frac{y}{x}$. Hence, $\exists a = \sup \mathbb{N} \in \mathbb{R}$. Since a is the least upper bound for $\mathbb{N}$, $a - 1$ cannot be an upper bound for $\mathbb{N}$. Hence, $\exists m \in \mathbb{N}$ such that

$$a - 1 < m \implies a < m + 1 \in \mathbb{N}.$$

However, this is a contradiction, because then a is not an upper bound for $\mathbb{N}$. Therefore, $\exists n \in \mathbb{N}$ such that $n \geq \frac{y}{x}$.

ii) Suppose $x, y \in \mathbb{R}$ and $x < y$. Then, there are three cases:

- $0 \leq x < y$,
- $x < 0 < y$, and
- $x < y \leq 0$.

For the second case, take $r = 0 \in \mathbb{Q}$. So, assume that $0 \leq x < y$. Then, by the Archimedian Property, $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$. Again by the Archimedean property, $\exists l \in \mathbb{N}$ such that $l > nx$. Thus, consider the set

$$S = \{k \in \mathbb{N} \mid k > nx\}.$$

By the well-ordering property of $\mathbb{N}$, S has a least element, $m \in S \implies nx < m \implies x < \frac{m}{n} \in \mathbb{Q}$.

Since $m - 1 \notin S$, $m - 1 \leq nx \implies m \leq nx + 1 < ny$. Hence, $\frac{m}{n} < y$. Therefore,

$$x < \frac{m}{n} < y.$$

If instead we have $x < y \leq 0$, then $0 \leq -y < -x \implies \exists \tilde{r} \in \mathbb{Q}$ such that

$$-y < \tilde{r} < x \implies x < -\tilde{r} < y$$

by the previous case.

$\square$

Theorem 3
$1 = \sup\left\{1 - \frac{1}{n} \mid n \in \mathbb{N}\right\}$.

Proof: If $n \in \mathbb{N}$, then $1 - \frac{1}{n} < 1 \implies 1$ is an upper bound of this set. Suppose that x is an upper bound for the set $\{1 - 1/n \mid n \in \mathbb{N}\}$. We now prove that $x \geq 1$. For the sake of contradiction, assume that $x < 1$. By the Archimedean property, there exists an $n \in \mathbb{N}$ such that $1 < n(1 - x)$. Therefore, $\exists n \in \mathbb{N}$ such that $x < 1 - 1/n$. Hence, x is not an upper bound for the set $\{1 - 1/n \mid n \in \mathbb{N}\}$ if $x < 1$. Thus, if x is an upper bound, $x \geq 1$. Therefore,

$$\sup\left\{1 - \frac{1}{n} \mid n \in \mathbb{N}\right\} = 1.$$

$\square$

We now begin proving some theorems about supremums and infinimums which will make them easier to use.

Theorem 4
Suppose that $S \subset \mathbb{R}$ is nonempty and bounded above. Then, $x = \sup S$ if and only if

1. x is an upper bound for S.
2. for all $\epsilon > 0$, $\exists y \in S$ such that $x - \epsilon < y \leq x$.

Proof: This is left as an exercise in Assignment 3. $\square$

Notation 5
For $x \in \mathbb{R}$ and $A \subset \mathbb{R}$, define

$$x + A := \{x + a \mid a \in A\}$$
$$xA := \{xa \mid a \in A\}.$$

Theorem 6
Using this new notation, we have the following theorems:

1. If $x \in \mathbb{R}$ and A is bounded above, then $x + A$ is bounded above and
$$\sup(x + A) = x + \sup A.$$
2. If $x > 0$ and A is bounded above then xA is bounded above and
$$\sup(xA) = x \sup A.$$

Proof:

1. Suppose that $x \in \mathbb{R}$ and A is bounded above. Therefore, $\sup A \in \mathbb{R}$ by the least upper bound property of $\mathbb{R}$. Then, $\forall a \in A$, $a \leq \sup A$. Hence,
$$\forall a \in A, \quad x + a \leq x + \sup A.$$
Hence, $x + \sup A$ is an upper bound for $x + A$. Let $\epsilon > 0$. Then, $\exists y \in A$ such that
$$\sup A - \epsilon < y \leq \sup A \implies (x + \sup A) - \epsilon < y + x \leq x + \sup A.$$

Therefore, by our previous theorem, $x + \sup A = \sup(x + A)$.

2. Suppose that $x > 0$ and A is bounded above. Thus, $\sup A \in \mathbb{R}$. Then, $\forall a \in A$, $a \leq \sup A$ and thus $xa \leq x \sup A$. Hence, $x \sup A$ is an upper bound of xA. Let $\epsilon > 0$. Then $\exists y \in A$ such that

$$\sup A - \frac{\epsilon}{x} < y \leq \sup A \implies x \sup A - \epsilon < xy \leq x \sup A.$$

Therefore, by the previous theorem, $\sup(xA) = x \sup A$.

$\square$

Theorem 7

Let $A, B \subset \mathbb{R}$ such that $\forall x \in A, \forall y \in B$, $x \leq y$. Then, $\sup A \leq \inf B$.

Proof: The proof of this is left to the reader. $\square$

Absolute Value

Definition 8

If $x \in \mathbb{R}$ we define

$$|x| := \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}.$$

Theorem 9

We can prove a bunch of theorems about the absolute value function that we usually take for granted:

1) $|x| \geq 0$ and $|x| = 0 \iff x = 0$.

2) $\forall x \in \mathbb{R}$, $|-x| = |x|$.

3) $\forall x, y \in \mathbb{R}$, $|xy| = |x||y|$.

4) $|x^2| = x^2 = |x|^2$.

5) If $x, y \in \mathbb{R}$, then $|x| \leq y \iff -y \leq x \leq y$.

6) $\forall x \in \mathbb{R}$, $x \leq |x|$.

Proof:

1) If $x \geq 0$ then $|x| = x \geq 0$. If $x \leq 0$, then $-x \geq 0 \implies |x| = -x \geq 0$. Thus, $|x| \geq 0$. Now suppose $x = 0$. Then, $|x| = x = 0$. For the other direction, suppose $|x| = 0$. Then, if $x \geq 0 \implies x = |x| = 0$. If $x \leq 0$, then $-x = |x| = 0$. Therefore, $x = 0 \iff |x| = 0$.

2) If $x \geq 0$ then $-x \leq 0$. Thus, $|x| = x = -(-x) = |-x|$. If $x \leq 0$ then $-x \geq 0$ and thus $|-x| = |-(-x)| = |x|$.

3) If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$ and $|xy| = xy = |x||y|$. If $x \leq 0$ and $y \leq 0$, then

$$xy \leq 0 \implies |xy| = -xy = (-x)y = |x||y|.$$

4) Take $x = y$ in 3). Then, $|x^2| = |x|^2$. Since $x^2 \geq 0$, it follows that $|x^2| = x^2$.

5) Suppose $|x| \leq y$. If $x \geq 0$, then $-y \leq 0 \leq x = |x| \leq y$. Therefore, $-y \leq x \leq y$. If $x \leq 0$, then $-x \geq 0$ and $|-x| \leq y$. Hence, $-y \leq -x \leq y \implies -y \leq x \leq y$.

6) Take $y = |x|$ in 5).

$\square$

On its own, these properties of the absolute values may not seem all that useful, but in the next lecture we will prove the *extremely* important Triangle Inequality.