18.100A: Typed Lecture Notes

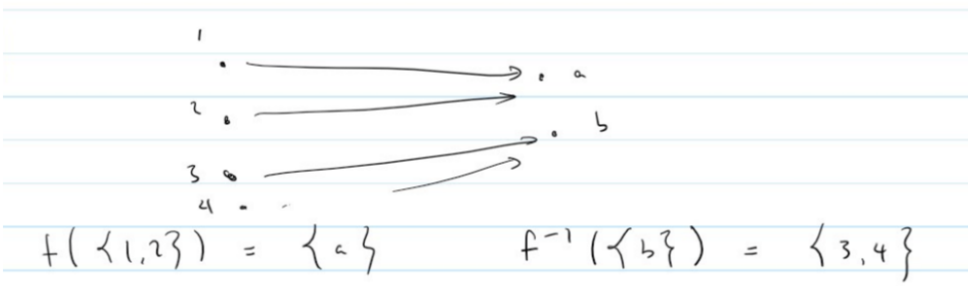
Lecture 2:
Cantor's Theory of Cardinality (Size)

Functions

If A and B are sets, a **function** $f : A \to B$ is a mapping that assigns each $x \in A$ to a unique element in B denoted $f(x)$. Let $f : A \to B$. Then

1. If $C \subset A$, we define $f(C) := \{y \in B \mid y \in f(x) \text{ for some } x \in C\}$.
2. If $D \subset B$, we define $f^{-1}(D) := \{x \in A \mid f(x) \in D\}$.

As an example, consider the following mapping $f : \{1, 2, 3, 4\} \to \{a, b\}$:



We can categorize functions in 3 important ways. Let $f : A \to B$.

1. f is injective or one-to-one (1-1) if $f(x_1) = f(x_2) \implies x_1 = x_2$.
2. f is surjective or onto if $f(A) = B$.
3. f is bijective if it is 1-1 and onto.

If a function $f : A \to B$ is bijective, then $f^{-1} : B \to A$ is the function which assigns each $y \in B$ to the unique $x \in A$ such that $f(x) = y$. Note that $f(f^{-1}(x)) = x$.

Cardinality

Question 1. *When do two sets have the same* ***size****?*

Cantor answered this question in the 1800s, stating that two sets have the same size when you can pair each element in one set with a unique element in the other.

> **Definition 2** (Cardinality)
> We state that two sets A and B have the same **cardinality** if there exists a bijection $f : A \to B$.

With this new concept comes some new notation:

1. $|A| = |B|$ if A and B have the same cardinality.
2. $|A| = n$ if $|A| = |\{1, \ldots, n\}|$. If this is the case we say A is **finite**.

3. $|A| \leq |B|$ if there exists an injection $f : A \to B$.

4. $|A| < |B|$ if $|A| \leq |B|$ but $|A| \neq |B|$.

Theorem 3 (Cantor-Schröder-Bernstein)
If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

If $|A| = |\mathbb{N}|$, then A is countably infinite. If A is finite or countably infinite, we say A is **countable**. Otherwise, we say A is **uncountable.**

Example 4
There are a few key theorems that we can prove with this new concept:

1. $|\{2n \mid n \in \mathbb{N}\}| = |\mathbb{N}|$.

2. $|\{2n - 1 \mid n \in \mathbb{N}\}| = |\mathbb{N}|$.

3. $|\{x \in \mathbb{Q} \mid x > 0\}| = |\mathbb{N}|$.

The first two statements can be summarized by Feynman: "There are twice as many numbers as numbers."
Proof:

1. Define the function $f : \mathbb{N} \to \{2n \mid n \in \mathbb{N}\}$ as $f(n) = 2n$. Then, f is 1-1– if $f(n) = f(m)$ then $2n = 2m \implies n = m$. Furthermore, f is also onto, as if $m \in \{2n \mid n \in \mathbb{N}\}$ then $\exists n \in \mathbb{N}$ such that $m = 2n = f(n)$.

2. The second statement can be proven similarly.

3. This is left as an exercise to the reader in Assignment 1.

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