18.100A: Typed Lecture Notes

Lecture 8:
The Squeeze Theorem and Operations Involving Convergent Sequences

Facts About Limits

Theorem 1 (Squeeze Theorem)
Let $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ be sequences such that $\forall n \in \mathbb{N}$,

$$a_n \leq x_n \leq b_k.$$

Suppose that $\{a_n\}$ and $\{b_n\}$ converge and

$$\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n.$$

Therefore, $\{x\}$ converges and $\lim_{n\to\infty} x_n = x$.

Remark 2. *We sometimes abbreviate the Squeeze Theorem to ST.*

Proof: Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = x$, there exists an $M_0 \in \mathbb{N}$ such that for all $n \geq M_0$,

$$|a_n - x| < \epsilon \implies x - \epsilon < a_n.$$

Since $\lim_{n\to\infty} b_n = x$, $\exists M_1 \in \mathbb{N}$ such that $\forall n \geq M_1$,

$$|b_n - x| < \epsilon \implies b_n < x + \epsilon.$$

Choose $M = \max\{M_0, M_1\}$. Then, if $n \geq M$, then

$$x - \epsilon < a_n \leq x_n \leq b_n < x - \epsilon \implies |x_n - x| < \epsilon.$$

Therefore, $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = x$. $\square$

Theorem 3
Another way to check that a sequence $x_n \to x$, is stated below:

$$\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} |x_n - x| = 0.$$

Hence, we can consider a sequence like the following:

Example 4
Show that

$$\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1.$$

Proof: We have
$$\left|\frac{n^2}{n^2+n+1}-1\right| = \left|\frac{-n-1}{n^2+n+1}\right| = \frac{n+1}{n^2+n+1} \leq \frac{n+1}{n^2+n} = \frac{1}{n}.$$
Thus,
$$0 \leq \left|\frac{n^2}{n^2+n+1}-1\right| \leq \frac{1}{n} \to 0 \implies \lim_{n\to\infty}\left|\frac{n^2}{n^2+n+1}-1\right| = 0$$
by the Squeeze Theorem. ■

Question 5. *How do limits interact with ordering?*

Theorem 6

Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers. Then,

1. if $\{x_n\}$ and $\{y_n\}$ are convergent sequences and $\forall n \in \mathbb{N}\ x_n \leq y_n$, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.
2. if $\{x_n\}$ is a convergent sequence and $\forall n \in \mathbb{N}\ x \leq x_n \leq b_n$ then $a \leq \lim_{n\to\infty} x_n \leq b$.

Proof:

1. Let $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Suppose for the sake of contradiction that $y < x$. Then, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$
$$|y_n - y| < \frac{x-y}{2}$$
And $\exists M_1 \in \mathbb{N}$ such that for all $n \geq M_1$,
$$|x_n - x| < \frac{x-y}{2}.$$
Then, if $M = M_0 + M_1 \geq \max\{M_0, M_1\}$,
$$y_M < \frac{x-y}{2} + y = \frac{x+y}{2} = x - \frac{x-y}{2} + x < x_M.$$
However, this would imply that $y_M < x_M$ which contradicts $\forall n \in \mathbb{N} x_n \leq y_n$.
2. Apply part 1 to proof part 2, by considering $y_n = a \leq x_n \leq b = z_n$ for all $n \in \mathbb{N}$.

□

Question 7. *How do limits interact with algebraic operations?*

Theorem 8

Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then,

1. $\{x_y + y_n\}_n$ is convergent and $\lim_{n\to\infty}(x_n + y_n) = x + y$.
2. $\forall c \in \mathbb{R}$, $\{cx_n\}_n$ is convergent and $\lim_{n\to\infty} cx_n = cx$.
3. $\{x_n \cdot y_n\}$ is convergent and $\lim_{n\to\infty} x_n y_n = xy$.
4. If $\forall n \in \mathbb{N}$, $y_n \neq 0$ and $y \neq 0$, then $\{x_n/y_n\}_n$ is convergent and
$$\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{x}{y}.$$

Proof:

1. Let $\epsilon > 0$. Then, since $x_n \to x$, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$, $|x_n - x| < \frac{\epsilon}{2}$. Since $y_n \to y$, $\exists M_1 \in \mathbb{N}$ such that $\forall n \geq M_1$, $|y_n - y| < \frac{\epsilon}{2}$. Hence, letting $M = \max\{M_0, M_1\}$, we get for all $n \geq M$,

$$|x_n + y_n - (x + y)| \leq |x_n - x| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

2. Let $\epsilon > 0$. Since $x_n \to x$, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$, $|x_n - x| < \frac{\epsilon}{|c|+1}$. Let $M = M_0$. Then, $\forall n \geq M$,

$$|cx_n - cx| = |c||x_n - x| \leq \frac{|c|}{|c| + 1} \cdot \epsilon < \epsilon$$

since $\frac{|c|}{|c|+1} < 1$.

3. Since $y_n \to y$, $\{y_n\}$ is bounded. In other words, $\exists B \geq 0$ such that $\forall n \in \mathbb{N}$, $|y_n| \leq B$. Then,

$$\begin{aligned} |x_n y_n - xy| &= |(x_n - x)y_n + (y_n - y)x| \\ &\leq |x_n - x||y_n + |x||y_n - y| \\ &\leq B|x_n - x| + |x||y_n - y|. \end{aligned}$$

Therefore, $0 \leq |x_n y_n - xy| \leq B|x_n - x| + |x||y_n - y|$. Since $B|x_n - x| + |x||y_n - y| \to 0$, by the Squeeze Theorem $\lim_{n\to\infty} |x_n y_n - xy| = 0$.

4. We prove $\frac{1}{y_n} \to \frac{1}{y}$. We first prove $\exists b > 0$ such that $\forall n \in \mathbb{N}$, $|y_n| \geq b$. Since $y_n \to y$ and $y \neq 0$, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$,

$$|y_n - y| < \frac{|y|}{2}.$$

By the Triangle Inequality, $\forall n \geq M_0$,

$$|y| \leq |y_n - y| + |y_n| \leq \frac{|y|}{2} + |y_n| \implies |y_n| \geq \frac{|y|}{2}.$$

Let $b = \min\left\{|y_1|, \ldots, |y_{M_0-1}|, \frac{|y|}{2}\right\}$. Then, $\forall n \in \mathbb{N}$, $|y_n| \geq b$. Therefore,

$$0 \leq \left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y_n - y|}{|y_n||y|} \leq \frac{1}{b|y|}|y_n - y|.$$

By the Squeeze Theorem, $\lim_{n\to\infty} \left|\frac{1}{y_n} - \frac{1}{y}\right| = 0$. Therefore, $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$. Furthermore, by the proof before this (3.), it follows that $\lim_{n\to\infty} \left(x_n \cdot \frac{1}{y_n}\right) = \frac{x}{y}$.

□

Remark 9. *By induction, one can prove that*

$$\lim_{n\to\infty} (x_n)^k = x^k.$$

Theorem 10

If $\{x_n\}$ is a convergent sequence such that $\forall n \in \mathbb{N}$, $x_n \geq 0$, then $\{\sqrt{x_n}\}$ is convergent and

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}.$$

Proof: Let $x = \lim_{n\to\infty} x_n$.

Case 1: $x = 0$. Let $\epsilon > 0$. Then, since $x_n \to 0$, there exists an $M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$, $x_n = |x_n - 0| < \epsilon^2$. Choose $M = M_0$. Then, $\forall n \geq M$,

$$|\sqrt{x_n} - \sqrt{0}| = \sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon.$$

Case 2: $x > 0$. We have $\forall n \in \mathbb{N}$,

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \left| \frac{\sqrt{x_n} - \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} \cdot (\sqrt{x_n} + \sqrt{x}) \right| \\ &= \frac{1}{\sqrt{x_n} + \sqrt{x}} |x_n - x| \\ &\leq \frac{1}{\sqrt{x}} |x_n - x|. \end{aligned}$$

Hence,

$$0 \leq |\sqrt{x_n} - \sqrt{x}| \leq \frac{1}{\sqrt{x}} |x_n - x|$$

$\forall n \in \mathbb{N}$. Hence, by the Squeeze Theorem,

$$\lim_{n \to \infty} |\sqrt{x_n} - \sqrt{x}| = 0.$$

□

Remark 11. *Why must we do casework in the above proof?*

Theorem 12

If $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = x$, then $\{|x_n|\}$ is convergent and $\lim_{n\to\infty} |x_n| = |x|$.

Proof: Firstly, note that $\forall x \in \mathbb{R}$, $\sqrt{x^2} = |x|$. Then,

$$\lim_{n \to \infty} |x_n| = \lim_{n \to \infty} \sqrt{x_n^2} = \sqrt{x^2} = |x|$$

by the previous theorem. □

Theorem 13

If $c \in (0, 1)$, then $\lim_{n\to\infty} c^n = 0$. If $c > 1$, then $\{c^n\}$ is unbounded.

Proof: If $0 < c < 1$, we claim that $\forall n \in \mathbb{N}$, $0 < c^{n+1} < c^n < 1$. We can prove this through induction. Firstly, notice that $0 < c^2 < c < 1$ since $c > 0$ and $c < 1$. Now assume that $0 < c^{m+1} < c^m$. Then, multiply by $c > 0$ to obtain

$$0 < c^{m+1} \cdot c = c^{(m+1)+1} < c^m \cdot c = c^{(m+1)}.$$

By induction, our claim holds. Thus, $\{c^n\}$ is a monotone decreasing sequence and is bounded below. Thus, $\{c^n\}$ is convergent. Let $L = \lim_{n\to\infty} c^n$. We will prove that $L = 0$. Let $\epsilon > 0$. Then, $\exists M \in \mathbb{N}$ such that $\forall n \geq M$, $|c^n - L| < (1-c)\frac{\epsilon}{2}$. Therefore,

$$\begin{aligned} (1-c)|L| = |L - cL| &= |L - c^{M+1} + c^{M+1} - cL| \\ &\leq |L - c^{M+1}| + c|c^M - L| \\ &< (1-c)\frac{\epsilon}{2} + c(1-c)\frac{\epsilon}{2} < (1-c)\epsilon. \end{aligned}$$

Therefore, $\forall \epsilon > 0$, $|L| < \epsilon \implies L = 0$.

Now let $c > 1$. We have to show that $\forall B \geq 0$, $\exists n \in \mathbb{N}$ such that $c^n > B$. Let $B \geq 0$. Choose $n \in \mathbb{N}$ such that $n > \frac{B}{c-1}$. Then,

$$c^n = (1 + (1 - c))^n \geq 1 + n(c - 1) \geq n(c - 1) > B.$$

To see why this center inequality is true, see the last theorem shown in Lecture 1. $\square$