18.S097: Introduction to Metric Spaces

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Compact Metric Spaces

Last time, we showed that a set in $\mathbb{R}^n$ is sequentially compact if and only if it is topologically compact, by showing

$$\text{sequentially compact} \iff \text{closed and bounded} \overset{\text{Heine-Borel}}{\iff} \text{topologically compact.}$$

However, by the previous remark, we don't have Heine-Borel for arbitrary metric spaces. Which begs the question: is sequentially compact the same as topologically compact in metric spaces? The answer is yes. To prove this, we first show a handful of preliminary results.

Lemma 1 (Lebesgue Number Lemma)
Let (X, d) be a sequentially compact metric space and $\{U_i\}_{i\in I}$ be an open cover of X. Then, there exists an $r > 0$ such that for each $x \in X$, $B_r(x) \subseteq U_i$ for some $i \in I$.

Proof: Before proving this, try to visualize the result!

We prove this lemma through contradiction. Assume that for some $r > 0$ there exists an $x \in X$ (possibly depending on r) such that for each $i \in I$, $B_r(x) \not\subseteq U_i$. Consider the sequence $\{x_n\}_n$ in X such that $B_{1/n}(x_n) \not\subseteq U_i$ for all $i \in I$.

Given that X is sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}_k$. Let $x_{n_k} \to x \in X$. Given that $\{U_i\}$ is an open cover of X, there exists a U_{i_0} such that $x \in U_{i_0}$. Given U_{i_0} is open, it also follows that there exists an r_0 such that $B_{r_0}(x) \subseteq U_{i_0}$. Hence, choose N large enough such that $d(x, x_N) < \frac{r_0}{2}$ and $\frac{1}{N} < \frac{r_0}{2}$. Then, if $y \in B_{1/N}(x_N)$, then

$$d(x, y) \leq d(x, x_N) + d(x_N, y) < r_0.$$

Therefore, $y \in B_{r_0}(x) \subseteq U_{i_0}$. Hence,

$$B_{1/N}(x_N) \subseteq B_{r_0}(x) \subseteq U_{i_0}$$

which is a contradiction. □

We call this r the **Lebesgue number** of the open cover of X, which is useful in other applications.

Definition 2
A metric space X it **totally bounded** if, for every $\epsilon > 0$, there exists $x_1, x_2, \ldots, x_k \in X$ with k finite such that $\{B_\epsilon(x_i) \mid 1 \leq i \leq k\}$ is an open cover of X.

Lemma 3

A metric space X is sequentially compact implies that X is totally bounded.

Proof: Assume that X is sequentially compact and not totally bounded. Therefore, there exists an $\epsilon > 0$ such that X cannot be covered by a collection of open sets of only finitely many ϵ-balls. Hence, let $x_1 \in X$, $x_2 \in X \setminus B_\epsilon(x_1)$, then $x_3 \in X \setminus B_\epsilon(x_1) \setminus B_\epsilon(x_2)$ and so on. We know that there exists such x_i by the previous statement. Hence, for all $i \neq j$, $d(x_i, x_j) \geq \epsilon$. Therefore, $\{x_n\}_n$ has no convergent subsequence as if there was a convergent subsequence it would be Cauchy, and the previous line shows that no subsequence of $\{x_n\}$ will be Cauchy. This is a contradiction to X being sequentially compact. □

Theorem 4

A metric space X is (topologically) compact if and only if X is sequentially compact.

Proof: We first show that topologically compact implies sequentially compact. Assume for the sake of contradiction there there exists a sequence $\{x_n\}_n$ in X with no convergent subsequence. Notice that no term in the sequence can appear infinitely many times, as otherwise there would be a trivial subsequence of $\{x_n\}$. Hence, we assume without loss of generality that $x_i \neq x_j$ if $i \neq j$. Furthermore, notice then that for every n there exists an $\epsilon_n > 0$ such that $B_{\epsilon_n}(x_n)$ contains no other terms in the sequence. If this wasn't the case, then there would again be a convergent subsequence of $\{x_n\}_n$. Therefore, for each i, there exists an open ball U_i centered at x_i such that $x_j \notin U_i$ for all $i \neq j$.

Additionally, consider $U_0 = X \setminus \{x_n \mid n \in \mathbb{N}\}$. U_0 is open, as $U_0^c = \{x_n \mid n \in \mathbb{N}\}$ is closed (it contains all of it's limit points). Hence,

$$U_0 \cup \{U_n \mid n \in \mathbb{N}\}$$

is an open cover of X. However, this open cover has no finite subcover as any finite collection of the cover will fail to include infinitely many terms from the sequence $\{x_n\}_n$. This is a contradiction, and thus topologically compact implies sequentially compact.

We now prove the other direction. Let X be sequentially compact and let $\{U_i\}_{i \in I}$ be an open cover of X. By the Lebesgue number lemma, there exists an $r > 0$ such that for each $x \in X$, $B_r(x) \subset U_i$ for some $i \in I$. Furthermore, by Lemma 5, X is totally bounded. Hence, there exists $y_1, \ldots, y_k \in X$ such that

$$X \subset B_r(y_1) \cup \cdots \cup B_r(y_k).$$

However, for each $i \in I$, we have $B_r(y_i) \subset U_{j(i)}$ for some $j(i) \in I$. (This notation just means for each i, there exists a $j \in I$ which depends on i such that $B_r(y_i) \subseteq U_j$). Thus, $\{U_{j(1)}, \ldots, U_{j(k)}\}$ is a finite subcover for X. Therefore, every open cover of X has a finite subcover, and thus sequentially compact implies topologically compact. □

Remark 5. *Notice that we technically could've used this proof in the previous lecture, but the Heine-Borel Theorem is so vastly important that I decided to do that proof before today's lecture.*

We will now start to look at some illuminating applications of compact sets to reach an even more powerful theorem.

Recall 6

Let X, Y be metric spaces and $f : X \to Y$ be a continuous function. Then, for all U open in Y, $f^{-1}(U)$ is open in X.

Theorem 7

Let X, Y be metric spaces and $f : X \to Y$ be continuous. Given $K \Subset X$, $f(K) \subset Y$ is compact.

Proof: Let $\{U_i\}_{i\in I}$ be an open cover of $f(K)$. Then, define $V_i = \{f^{-1}(U_i)\}_{i\in I}$, which is open as f is continuous. Therefore, $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of K. Hence, there exists a finite subcover $\{V_{i_1}, \dots V_{i_k}\}$ of K as K is compact. Thus, $\{U_{i_1}, \dots U_{i_k}\} = \{f(V_{i_1}), \dots, f(V_{i_k})\}$ is a finite subcover of $f(K)$. Therefore, $f(K)$ is compact. $\square$

Corollary 8

Let X be a metric space and $K \Subset X$. Then, given a continuous function $f : X \to \mathbb{R}$, f obtains a maximum and minimum finite value on K.

Proof: The proof follows from the previous theorem, and Problem 5.(a) on PSET 2. $\square$

Corollary 9

Sometimes in particular we want to study bounded continuous functions, and the previous corollary gives us a nice property. Given a compact metric space X, every continuous function on f is bounded.

Proof: Follows immediately. $\square$

Theorem 10 (Cantor's Intersection Theorem)

If $K_1 \supset K_2 \supset K_3 \supset \dots$ is a decreasing sequence of nonempty sequentially compact subsets of $\mathbb{R}^n$, then $\cap_{i\geq 1} K_i$ is non-empty.

Proof: Choose a sequence $\{a_n\}_n$ such that $a_n \in K_n$ for each n. We know that there exists such an a_n as each K_n is nonempty. Then, $\{a_n\}_n$ is a sequence in K_1, and thus there exists a convergent subsequence $\{a_{n_k}\}_k$ such that $a_{n_k} \to a \in K_1$. Furthermore, $\{a_n\}_{n=2}^{\infty}$ is a sequence in K_2, and thus contains a a convergent subsequence. Therefore, $a \in K_2$. Continuing this process, we get that $a \in K_i$ for all i. Thus, $a \in \cap_{i\geq 1} K_i$. $\square$

Definition 11 (Finite Intersection Property)

A collection of closed sets $\{C_i\}_i$ has the **finite intersection property** if every finite subcollection has a nonempty intersection.

Given Lemma 5 and the Cantor Intersection Theorem, it is clear that there are some relations between compact sets, nonempty intersections of sets, and totally bounded sets. We hence show the following theorem.

Theorem 12

Given a metric space (X, d), the following are equivalent.

(1) X is compact.

(2) X is sequentially compact.

(3) X is Cauchy complete and totally bounded.

(4) Every collection of closed subsets of X with the finite intersection property has a non-empty intersection.

We have shown (1) $\iff$ (2), and thus we show (1) $\iff$ (4) and (2) $\iff$ (3) to finish the proof.

Proof: (1) $\implies$ (4): Assume for the sake of contradiction that there exists a collection of closed subsets $\{C_i\}_{i\in I}$ with the finite intersection property such that $\cap_{i\in I} C_i =$. Given C_i is closed in X for all i, $U_i = C_i^c$ is open in X for each i. Then,

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} C_i^c = \left(\bigcap_{i\in I} C_i\right)^c = \emptyset^c = X.$$

Hence, the U_i cover X. Given X is compact, there exists a finite subcover $\{U_{i_1}, \ldots, U_{i_k}\}$ of X. Thus,

$$X = \bigcup_{n=1}^{k} U_{i_n} = \left(\bigcap_{n=1}^{k} U_{i_n}^c\right)^c = \left(\bigcap_{n=1}^{k} C_{i_n}\right)^c.$$

Therefore, $\bigcap_{n=1}^{k} C_{i_n} = \emptyset$ which is a contradiction with the finite intersection property.

(4) $\implies$ (1): Suppose that $\{U_i\}_{i\in I}$ is an open cover of X, and let $C_i = U_i^c$ for each $i \in I$. Assume for the sake of contradiction that no finite subset of the U_i covers X. We show that C_i has the finite intersection property. Assume for the sake of contradiction that $\{C_{n_1}, \ldots, C_{n_k}\}$ satisfies $C_{n_1} \cap \cdots \cap C_{n_k} = \emptyset$. Then,

$$\bigcup_{i=1}^{k} U_{n_i} = \left(\bigcap_{i=1}^{k} U_{n_i}^c\right)^c = \left(\bigcap_{i=1}^{k} C_{i_k}\right)^c = \emptyset^c = X.$$

This is a contradiction with the assumption that no subset of the U_i covers X. Hence, $\{C_i\}_{i\in I}$ satisfies the finite intersection property. Therefore, $\{C_i\}_{i\in I}$ has non-empty intersection; i.e. $\bigcap_{i\in I} C_i \neq \emptyset$. Then, $\bigcup_{i\in I} U_i \neq X$, which is a contradiction to the U_i being an open cover for X. Thus, every open cover of X has a finite open subcover.

(2) $\implies$ (3): We have already shown that X being sequentially compact implies totally bounded, and hence we only need show that sequentially compact implies Cauchy complete. Let $\{x_n\}$ be a Cauchy sequence in X. Given $\{x_n\}$ is a sequence in X, there exists a convergent subsequence $\{x_{n_k}\}$ in X such that $x_{n_k} \to x \in X$. Let $\epsilon > 0$, and choose N such that $d(x_i, x_j) < \epsilon/2$ whenever $i, j \geq N$. Next, choose $n_k > N$ such that $d(x_{n_k}, x) < \epsilon/2$. Then,

$$d(x, x_N) \leq d(x, x_{n_k}) + d(x_{n_k}, x_N) < \epsilon.$$

Thus, $x_n \to x \in X$ as $n \to \infty$. Therefore, every Cauchy sequence in X converges to a point in X. Hence, X is Cauchy complete.

(3) $\implies$ (2): This part of the proof is quite difficult. Consider a sequence $\{x_n\}_n$ in X. Given X is totally bounded, for every $n \in \mathbb{N}$, there exists a finite set of points $\{y_1^{(n)}, \ldots, y_{r(n)}^{(n)}\}$ such that $X \subset B_{\frac{1}{n}}(y_1^{(n)}) \cup \cdots \cup B_{\frac{1}{n}}(y_{r(n)}^{(n)})$. Define

$$S_n = \{y_1^{(n)}, \ldots, y_{r(n)}^{(n)}\}.$$

We want to find a convergent subsequence of $\{x_n\}_n$. We do so by construction. Given S_1 is finite, there exists a $y_{n(1)}^{(1)} \in S_1$ such that $B_1(y_{n(1)}^{(1)})$ contains infinitely many points from $\{x_n\}_n$. Choose z_1 from this ball. Then, given S_2 is finite, there is a $y_{n(2)}^{(2)}$ such that infinitely many points from $\{x_n\}_n$ are in $B_1(y_{n(1)}^{(1)}) \cap B_{1/2}(y_{n(2)}^{(2)})$. Choose z_2 from this set. Continue this procedure for each $k > 1$, selecting a z_k from $\bigcap_{i=1}^{k} B_{\frac{1}{k}}(y_{n(k)}^{(k)})$. Then, we show $\{z_n\}_n$ is Cauchy. Let $\epsilon > 0$. Then, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Hence, for all $n, m \geq N$,

$$d(z_n, z_m) < \frac{1}{N} < \epsilon.$$

Therefore, by the Cauchy completeness of X, $\{z_n\}$ converges to a point in X. □

Remark 13. *Where do we use the fact that each ball has infinitely many points? We do in fact use this property in* *the proof. Try to figure out how!*