18.S097: Introduction to Metric Spaces

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Introduction

At MIT, our real analysis class is split into four sections: 18.100A, 18.100B, 18.100P, and 18.100Q. The key differences can be summarized using the following diagram:

	$\mathbb{R}^n$	Abstract
Non-Communication Intensive	18.100A	18.100B
Communication Intensive	18.100P	18.100Q

The goal of this class is to illuminate the key differences between studying the Euclidean space $\mathbb{R}^n$ and studying more abstract spaces. This is particularly done through the use of **metric spaces**.

1 January 4, 2022

Motivation, Intuition, and Examples

In today's lecture, I will give the definition of a metric space, give many many examples, and then relate this new concept back to vocabulary we use throughout 18.100A. Let's start with a key example that we use throughout 18.100A: the Euclidean distance.

Example 1 (Euclidean Distance)

We define the **Euclidean distance** (or **metric**) between two points $x, y \in \mathbb{R}^n$ as

$$\left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}.$$

Conceptually, this is the magnitude of the shortest line segment connecting x and y. We most commonly study this in $\mathbb{R}$ (where $n = 1$). Then, the metric is defined as $|x-y|$ (as there is only one coordinate, and $(|x-y|^2)^{1/2} = |x-y|$). What are the most important features of the absolute value bars? Given $x, y, z \in \mathbb{R}$ we have the following properties:

1. Positive definite: $|x-y| \geq 0$, and $|x-y| = 0 \iff x = y$.

2. Symmetric: $|x-y| = |y-x|$.

3. Triangle Inequality: $|x-z| \leq |x-y| + |y-z|$.

To some extent, these properties may feel inherent at this point as this concept of absolute values is taught to us in elementary school. We are taught how to define a distance between two natural numbers, often times before we are taught what negative numbers are. But notice, we use absolute values to define nearly every term we use in 18.100A. Our goal is to define our "real analysis" vocabulary (which we will get to later) more abstractly. To do so, we define a metric space.

Definition 2 (Metric Space)
A **metric space** is a set X with a metric $d : X \times X \to [0, \infty)$ such that $\forall x, y, z \in X$, d satisfies the following properties:

1. Positive definite: $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$.
2. Symmetric: $d(x, y) = d(y, x)$.
3. Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

We notice that these properties are exactly the same with how the absolute value bars, which to some extent should make sense. We want to understand the distance between things (whether that be vectors or functions or weirder objects), and this idea of a distance comes with these properties. We don't talk about distances being negative, and it makes sense for the distance between me and you to be the same distance between you and I. Yet nonetheless, this new concept of metric spaces allows us to study more abstract concepts (which comes down to the idea that our set X can be weirder).

Here is the outline for the rest of the lecture:

- We will define other metrics on $\mathbb{R}^n$.
- We will *redefine* the terminology we use in 18.100A.
- Then, we will discuss metrics on weirder spaces!

Example 3 (Supremum Metric)
Consider the following function: $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$,

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

This metric is often called the **supremum metric** or supremum norm. We check that this is in fact a metric.

Proof:

1. Positive definite: It is clear that $d_\infty(x, y) \geq 0$ as $|x - y| \geq 0$ for all $x, y \in \mathbb{R}$. If $x = y$, then $x_i - y_i = 0$ for all i, and thus $d(x, y) = 0$. If $d(x, y) = 0$, then we want to show that $x = y$. Assume for the sake of contradiction that $x \neq y$. Then, there exists an i such that $x_i \neq y_i$. Hence, $|x_i - y_i| > 0$. Therefore,

$$0 < |x_i - y_i| \leq d_\infty(x, y) \implies 0 < d_\infty(x, y)$$

which is a contradiction.

2. Symmetric: $d_\infty(x, y) = \max_i |x_i - y_i| = \max_i |y_i - x_i| = d_\infty(y, x)$. This uses the fact that absolute values are symmetric.

3. Triangle inequality: Often times, the triangle inequality is the hardest properties to prove. One common thing to do, however, is to consider *one* term (sorry this advice is very general). Let $x, y, z \in \mathbb{R}^n$. Then, consider an arbitrary $1 \leq i \leq n$. We know that

$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$$

as absolute values satisfy the triangle inequality. Then, taking the maximum of both sides, we get that

$$d_\infty(x, z) = \max_i |x_i - z_i| \leq \max_i |x_i - y_i| + \max_i |y_i - z_i| = d_\infty(x, y) + d_\infty(y, z).$$

□

Example 4 (ℓ^1 metric)
Define $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ such that

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

This is called the **ℓ^1 metric**. We again check that this is a metric.

Proof:

1. Positive definite: it is clear that $d_1(x, y) \geq 0$ for all $x, y \in \mathbb{R}^n$. Further, if $x = y$ then $d_1(x, y) = 0$ as $x_i = y_i \forall i$. If $d_1(x, y) = 0$ then $x_i = y_i \forall i$, and thus $x = y$.
2. Symmetric: $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| \sum_{i=1}^{n} |y_i - x_i| = d_1(y, x)$.
3. Triangle Inequality: This follows immediately from the triangle inequality for absolute values:

$$d_1(x, z) = \sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + |y_i - z_i| = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = d_1(x, y) + d_1(y, z).$$

□

Remark 5. *Notice that the ℓ^1 metric and the Euclidean metric take on the same form. In PSET 1, you will prove that $d_p : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ given by*

$$d_p(x, y) = (|x_i - y_i|^p)^{1/p}$$

is a metric.

Now that we have went through three examples of metrics on a set, let's relate this concept back to real analysis (after all, that is why we are studying this). What are the key definitions we used in 18.100A? We had convergent sequences, Cauchy sequences, and continuity. Let's write down these definitions in terms of 18.100A:

- A sequence $\{a_n\}$ in $\mathbb{R}$ **converges** to $a \in \mathbb{R}$ if and only if $\forall \epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|a_n - a| \leq \epsilon.$$

- A sequence $\{a_n\}$ in $\mathbb{R}$ is a **Cauchy sequence** if and only if $\forall \epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\forall n, m \geq N$,

$$|a_n - a_m| \leq \epsilon.$$

- A set $A \subset \mathbb{R}^n$ is **open** if and only if for all $x \in A$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset A$.

- A function $f : \mathbb{R} \supset A \to \mathbb{R}$ is **continuous** if and only if given $x \in A$, $\forall \epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - y| \leq \delta \implies |f(x) - f(y)| \leq \epsilon.$$

We thus have these (almost immediate) definitions for Metric Spaces:

Definition 6 (Convergent sequence)
A sequence $\{x_n\}$ in a metric space (X, d) **converges** to $x \in X$ if and only if $\forall \epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$d(x_n, x) \leq \epsilon.$$

Definition 7 (Cauchy sequence)
A sequence $\{x_n\}$ in (X, d) is a **Cauchy sequence** if and only if $\forall \epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\forall n, m \geq N$,

$$d(x_n, x_m) \leq \epsilon.$$

Definition 8 (Open Set)
A set in $A \subseteq X$ is **open** if and only if $\forall x \in A$, there exists an $\epsilon > 0$ such that

$$B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\} \subset A.$$

We say that $B(x, \epsilon)$ is a ball of radius epsilon centered at x.

Continuous functions however are a bit different. Normally, continuous functions map to $\mathbb{R}$, but here we can let them map to any other metric space, getting the following definition

Definition 9 (Continuous functions)
Let X and Y be metric spaces with metrics d_X, d_Y respectively. Then, a function $f : X \supset A \to Y$ is **continuous** if and only if given $x \in A$, $\forall \epsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(x, y) \leq \delta \implies d_Y(f(x), f(y)) \leq \epsilon.$$

The reason I bring these up now as opposed to later, is to point out how one-to-one the definitions between 18.100A and 18.100B are. All we are doing is making our definition a bit more general so we can study things other than $\mathbb{R}^n$. Speaking of which, lets consider some more metric spaces on sets that *aren't* $\mathbb{R}^n$. As we do so, I will bring up useful examples of the above definitions that are great to picture/use as intuition.

Example 10 (Metric on Continuous Functions)
We define $C^0([a, b])$ to be the set of functions (that map to the real numbers) that are continuous on the interval $[a, b]$. Show that $d : C^0([0, 1]) \times C^0([0, 1]) \to [0, \infty)$ defined by

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

is a metric.

Proof: I leave you to prove positive definiteness and symmetry, but as usual the hard part comes down to proving the triangle inequality. Let $f, g, h \in C^0([0,1])$. First, lets evaluate what $d(f,h)$ is in terms of this metric.

$$d(f,h) = \sup_{x\in[0,1]} |f(x) - h(x)|.$$

Let x_0 be a point in the interval $[0,1]$ such that

$$d(f,h) = \sup_{x\in[0,1]} |f(x) - h(x)| = |f(x_0) - h(x_0)|.$$

Remark 11. *Why does such a point exist?*

Furthermore, note that for all $x \in [0,1]$,

$$|f(x) - g(x)| \leq \sup_{x\in[0,1]} |f(x) - g(x)| \quad \text{and} \quad |g(x) - h(x)| \leq \sup_{x\in[0,1]} |g(x) - h(x)|.$$

Thus,

$$\begin{aligned} d(f,h) &= |f(x_0) - h(x_0)| \\ &\leq |f(x_0) - g(x_0)| + |g(x_0) - h(x_0)| \\ &\leq \sup_{x\in[0,1]} |f(x) - g(x)| + \sup_{x\in[0,1]} |g(x) - h(x)| \\ &= d(f,g) + d(g,h). \end{aligned}$$

Thus,

$$d(f,h) \leq d(f,g) + d(g,h).$$

□

Question 12. *Pick $f \in C^0([a,b])$. What does $B(f,\epsilon)$ look like in $C^0([a,b])$ ($\epsilon > 0$)?*

Example 13

We can take this one step further. Define $C^1([0,1])$ as the space of continuously differentiable functions. In other words, functions that are continuous, and whose first derivative is continuous. Consider $d : C^1([0,1]) \times C^1([0,1]) \to [0,\infty)$ where

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| + \sup_{x\in[0,1]} |f'(x) - g'(x)|.$$

Show d is a metric on the space.

Proof: I leave you again to check positive definiteness and symmetry, but to prove the triangle inequality we can use a nice trick. Firstly, note that if $f, g, h \in C^1([a,b])$ then they are all continuous (i.e. they are all in $C^0([a,b])$. Therefore,

$$\sup_{x\in[0,1]} |f(x) - h(x)| \leq \sup_{x\in[0,1]} |f(x) - g(x)| + \sup_{x\in[0,1]} |g(x) - h(x)|$$

as we proved in the previous example. Similarly, f', g', h' are continuous by assumption. Hence,

$$\sup_{x\in[0,1]} |f'(x) - h'(x)| \leq \sup_{x\in[0,1]} |f'(x) - g'(x)| + \sup_{x\in[0,1]} |g'(x) - h'(x)|.$$

Adding these two inequalities together gives the desired inequality. $\square$

Question 14. *Consider again $C^1([0,1])$. Is $d(f,g) = \sup_{x\in[0,1]} |f'(x) - g'(x)|$ a metric on $C^1([0,1])$? You will answer this on PSET 1.*

While we will stop here with regards to continuous functions on a bounded interval, notice that we can easily continue this argument, and we can in fact define a metric on functions that are infinitely differentiable. However, we would have to be careful as we done want to take an infinite sum of things if the sum doesn't converge. There will be an optional problem discussing this on PSET 1.

Example 15

Show the map $\frac{\mathrm{d}}{\mathrm{d}x} : C^1([a,b]) \to C^0([a,b])$ is continuous.

Proof: Let $f, g \in C^1([a,b])$. We want to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_{C^1}(f,g) < \delta$, then $d_{C^0}\left(\frac{\mathrm{d}}{\mathrm{d}x}f, \frac{\mathrm{d}}{\mathrm{d}x}g\right) < \epsilon$. To see this, calculate both equations:

$$d_{C^1}(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)| + \sup_{x\in[a,b]} |f'(x) - g'(x)|.$$

$$d_{C^0}\left(\frac{\mathrm{d}}{\mathrm{d}x}f, \frac{\mathrm{d}}{\mathrm{d}x}g\right) = \sup_{x\in[a,b]} |f'(x) - g'(x)|.$$

Hence, notice that $d_{C^0}\left(\frac{\mathrm{d}}{\mathrm{d}x}f, \frac{\mathrm{d}}{\mathrm{d}x}g\right) \leq d_{C^1}(f,g)$. Thus, let $\delta = \epsilon$. $\square$

Remark 16. *On PSET 1, you will show that integration is continuous.*

So far, we have been studying metrics on vector spaces. (I.e. the sum of two vectors in $\mathbb{R}^n$ is in $\mathbb{R}^n$, and the sum of two continuous functions is continuous, etc etc.) We will discuss the notion of a vector space more in Lecture 3. However, it is important to note that we don't need the set to be a vector space in order to define a metric on it. Consider the following two examples:

Example 17 (Geodesic)

Consider the unit ball in $\mathbb{R}^3$. We can define two metrics on it. The first, immediately follows from the Euclidean metric on the ball. However, notice, that we can define a metric on the ball defined as the shortest "line segment" between two points that lie on the sphere. This concept is loosely defined as a "geodesic".

Example 18 (Trivial Metric)

We define the trivial metric. Pick a set X. Then, define

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Show d is a metric.

Proof: It is clear that d is positive definite and symmetric. We prove the triangle inequality. Consider $d(x,z)$. We split this into four cases. If $x = z$, then $d(x,z) = 0$. If $y = x = z$, then $d(x,z) = d(x,y) + d(y,z) = 0$. If $y \neq x$ and thus $y \neq z$, then $0 = d(x,z) \leq d(x,y) + d(y,z) = 2$. If $x \neq z$, then $d(x,z) = 1$. If $x = y$, then $1 = d(x,z) = d(x,y) + d(y,z) = 1$. If $y \neq x$ and $y \neq z$, then $1 = d(x,z) \leq d(x,y) + d(y,z) = 2$. $\square$

Remark 19. *On PSET 1, you will prove the British Railway metric is a metric (this is a similar example to the trivial metric problem).*

We finish with one last example for the day.

Example 20

Once again consider the space $C^0([0,1])$. Define the function $I_1 : C^0([0,1] \times C^0([0,1]) \to [0,\infty)$ where

$$I_1(f,g) = \int_0^1 |f(x) - g(x)|\, \mathrm{d}x.$$

Show that I is a metric.

Proof: I leave you to prove positive definiteness and symmetry (note for positive definiteness, you need to use continuity of $f-g$). However, the loose proof of the triangle inequality isn't terribly bad this time! We simply notice that for all x, $|f(x)-h(x)| \le |f(x)-g(x)|+|g(x)-h(x)|$. Hence, using properties of integration, we get (given $f, g, h \in C^0([0,1])$)

$$\begin{aligned} I_1(f,h) &= \int_0^1 |f(x) - h(x)|\, \mathrm{d}x \\ &\le \int_0^1 |f(x) - g(x)| + |g(x) - h(x)|\, \mathrm{d}x \\ &= \int_0^1 |f(x) - g(x)|\, \mathrm{d}x + \int_0^1 |g(x) - h(x)|\, \mathrm{d}x \\ &= I_1(f,g) + I_1(g,h). \end{aligned}$$

□

We call this the L^1 metric.

Remark 21. *One can (similar to the proof of the ℓ^p metrics) prove that for $1 \le p < \infty$,*

$$I_p(f,g) = \left(\int_0^1 |f(x) - g(x)|^p\, \mathrm{d}x \right)^{1/p}$$

defines a metric on $C^0([0,1])$.

2 January 6, 2022

General Theory

We now go into some of the general theory regarding metric spaces. Metric spaces are not only intuitively related to our understanding of $\mathbb{R}^n$, but they also behave similarly. For our purposes, we want to study metric spaces as they act nicely, and we now show some ways in which they are "nice".

Remark 22. *Not every space is as nice as a metric space! The fancy way to say this is "Not every space is metrizable." 18.901 explores weirder spaces like this, but we will not do so in this class.*

Let's start with convergent sequences, just like we did when we first started studying the real numbers.

Proposition 23

Let (X, d) be a metric space and let x_n be a convergent sequence in X such that $x_n \to x$. This limit is unique.

Proof: Suppose there exists a y such that $x_n \to y$. We want to show that if this is the case, then $x = y$. What property about metric spaces tells us when points are equal? In the real line, $x = y \iff |x - y| = 0$, which is how we proved this property in 18.100x. Here, we similarly have $x = y \iff d(x, y) = 0$. Hence, we use that to our advantage:

$$0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y).$$

Given that $x_n \to x$ and $x_n \to y$, we can make the right hand side arbitrarily small. More formally, let $\epsilon > 0$. Then, there exists an N such that for all $n \geq N$,

$$0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y) < \epsilon.$$

This is true for all $\epsilon > 0$, and thus $d(x, y) = 0 \implies x = y$. $\square$

Proposition 24

Let $x_n \to x$. Then, $\forall y \in X$, $d(x_n, y) \to d(x, y)$.

In other words, when you have a convergent sequence in a metric space, the distance also behaves how one would expect. A similar way to think about this: fix $y \in X$. Then $a_n = d(x_n, y)$ is a convergent sequence in the real numbers.

Proof: Let $y \in X$. Firstly, note that given $x_n \to x$, $d(x_n, x) \to 0$. Hence, let $\epsilon > 0$. Then, there exists an N such that for all $n \geq N$,

$$d(x_n, y) \leq d(x, x_n) + d(x, y) < \epsilon + d(x, y).$$

We now want a similar lower bound, which we obtain by the triangle inequality again:

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \implies d(x, y) - d(x_n, x) \leq d(x_n, y).$$

For $n \geq N$, we have $d(x, y) - \epsilon < d(x_n, y)$. Therefore, given $\epsilon > 0$, there exists an N such that for all $n \geq N$,

$$d(x, y) - \epsilon < d(x_n, y) < d(x, y) + \epsilon \implies |d(x_n, y) - d(x, y)| < \epsilon.$$

Therefore, $d(x_n, y) \to d(x, y)$. $\square$

Proposition 25

We can take this concept one step further, studying two convergent sequences at once. Let $x_n \to x$ and $y_n \to y$. Then, $d(x_n, y_n) \to d(x, y)$.

This problem will be on your PSET!

Definition 26 (Bounded)
A sequence $\{x_n\}$ in (X, d) is bounded if and only if there exists a $p \in X$ and a $B \in \mathbb{R}$ such that

$$d(x_n, p) \le B \quad \forall n \in \mathbb{N}.$$

Similarly, a subset $A \subseteq X$ is bounded if and only if there exists a $p \in X$ and a $B \in \mathbb{R}$ such that

$$d(x, p) \le B \quad \forall x \in A.$$

Proposition 27
Every convergent sequence in a metric space is bounded.

Proof: Let $x_n \to x$ and let $\epsilon = 1 > 0$. Then, there exists an N such that for all $n \ge N$, $d(x_n, x) < 1$. Now this is almost exactly our definition of bounded with $p = x$ and $B = 1$, but the issue is that this isn't true for all n, only for all $n \ge N$ (which is still infinitely many!). We thus use a common and useful technique: Let

$$B = \max\{d(x_n, x), 1 \mid 1 \le n < N\}.$$

Is B finite? Yes; B is the maximum of finitely many finite elements and thus finite. Furthermore, we now have that for all $n \ge N$, $d(x_n, x) < 1 \le B$, and for all $n < N$, $d(x_n, x) \le B$. Hence, $\{x_n\}$ is bounded. □

We will prove two more theorems about convergent sequences, and then we will shift our focus to open and closed sets.

Proposition 28
Every convergent sequence is a Cauchy sequence.

Proof: Let $x_n \to x$, and let $\epsilon > 0$. Then, there exists an N such that for all $n \ge N$, $d(x_n, x) < \frac{\epsilon}{2}$. Hence, for all $n, m \ge N$,

$$d(x_n, x_m) \le d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

We showed this before for the real numbers! In fact, we showed that Cauchy sequences are convergent for the real line. However, this isn't always true. A space in which every Cauchy sequence is convergent, is called **Cauchy complete**.

Remark 29. *You will show on PSET 2 that $C^0([0,1])$ is Cauchy complete.*

Proposition 30
Every subsequence of a convergent sequence is convergent.

Proof: This proof will help give an example of why Cauchy sequences are useful. Let $x_n \to x$, and consider the subsequence x_{n_k}. We want to show that x_{n_k} is convergent, and to do so we will show that $x_{n_k} \to x$. Firstly notice that

$$d(x_{n_k}, x) \le d(x_{n_k}, x_n) + d(x_n, x).$$

We know that $x_n \to x$, and thus for $\epsilon > 0$ there exists an N_1 such that for all $n \geq N_1$, $d(x_n, x) < \frac{\epsilon}{2}$. In other words, we can make $d(x_n, x)$ arbitrarily small; but what can we do about $d(x_{n_k}, x_n)$? Well we note that $\{x_n\}$ is a Cauchy sequence. Thus, there exists an N_2 such that for all $n, n_k \geq N$, $d(x_{n_k}, x_n) < \frac{\epsilon}{2}$. Hence, for all $n \geq \max\{N_1, N_2\}$,

$$d(x_{n_k}, x) \leq d(x_{n_k}, x_n) + d(x_n, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

You may be wondering "Why don't we have as many theorems for convergent sequences like we used to?" Well notice, that metric spaces are *much* more general than $\mathbb{R}$. For instance, we can't show sums of convergent sequences converge, because we don't *always* have a notion of addition. Similarly, we don't have a direct analog of the squeeze theorem, as we don't *always* have a notion of "ordering" (i.e. what it means for one element to be bigger than another). Thus we have to study new tools, like open sets.

Recall (Open Set)
A set in $A \subset X$ is **open** if and only if $\forall x \in A$, there exists an $\epsilon > 0$ such that

$$B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\} \subset A.$$

We say that $B(x, \epsilon)$ is a ball of radius epsilon centered at x.

While it may seem out of left field, open sets prove very useful in understanding concepts of convergence and continuity. We will show this connection today, but let's start with some useful and powerful propositions.

Theorem 31 (Topological Properties of Open Sets)
Let X be a metric space, and let $\{A_i\}_{i \in \Lambda}$ be open sets in X. Then,

1. $\emptyset$ and X are open sets in X.
2. $\bigcup_{i \in I} A_i$ is open in X. (I.e., the arbitrary union of open sets is open.)
3. $\bigcap_{i=1}^{n} A_i$ is open in X. (I.e., the finite intersection of open sets is open.)

Proof: All we can use so far is the definition given to us.

1. Consider $\emptyset$. It is vacuously true that $\forall x \in \emptyset$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset \emptyset$, as there are no elements in the empty set. Now consider X. Recall the definition of $B(x, \epsilon)$:

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}.$$

By definition, $\forall x \in X$ and in fact for all $\epsilon > 0$ (though we only need one), $B(x, \epsilon) \subset X$. Thus, X is an open set.

2. Consider some $x \in \bigcup_{i \in I} A_i$. Then, by assumption, there exists a $\lambda \in \Lambda$ such that $x \in A_\lambda$. Furthermore, A_λ is an open set, and thus there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset A_\lambda$. Notice though, that $A_\lambda \subset \bigcup_{i \in I} A_i$, and thus $B(x, \epsilon) \subset \bigcup_{i \in I} A_i$. Hence, the arbitrary union of open sets is open.

3. The proof for the intersection will act similarly, but let's see why we can only consider a finite intersection. Let $x \in \bigcap_{i=1}^{n} A_i$. Then, for each $1 \leq i \leq n$, $x \in A_i$. Therefore, there exists an ϵ_i such that $B(x, \epsilon_i) \subset A_i$. The issue though, is A_i is not automatically a subset of the intersection. However, we can take $\epsilon = \min\{\epsilon_i\} > 0$. Thus, $B(x, \epsilon) \subseteq B(x, \epsilon_i) \subset A_i$ for every i. Hence, $B(x, \epsilon) \subset \bigcap_{i=1}^{n} A_i$. □

Remark 32. *These three properties can help us understand why open sets are so useful (at least conceptually). As we will see, open sets are directly related to convergence and continuity, and are yet so much more general. In point-set topology (18.901), you actually* **start** *with defining open sets abstractly using these three properties, and go from there. It's very interesting, and leads to very interesting examples! We will discuss this more in Lecture 6.*

Theorem 33

An open subset U in a metric space (X, d) can be written as a union of open balls in X. This is an optional problem on PSET 2.

Definition 34 (Closed Set)

Let $A \subset X$. We say that A is **closed** if $X \setminus A := A^c$ is open in X. We call A^c the **complement** of A.

Theorem 35

Let X be a metric space, and let $\{A_i\}_{i\in\Lambda}$ be closed sets in X. Then,

1. $\emptyset$ and X are closed sets in X.
2. $\bigcap_{i\in I} A_i$ is closed in X. (I.e., the arbitrary intersection of closed sets is closed.)
3. $\bigcup_{i=1}^{n} A_i$ is closed in X. (I.e., the finite union of closed sets is closed.)

To prove this, we use DeMorgan's Law in set theory (which is proven in Lebl's Theorem 0.3.5).

Proposition 36 (DeMorgan's Law)

Consider the sets $\{U_i\}_{i\in\Lambda}$. Then,

$$\left(\bigcup_{i\in\Lambda} U_i\right)^c = \bigcap_{i\in\Lambda} U_i^c \quad \text{and} \quad \left(\bigcap_{i\in\Lambda} U_i\right)^c = \bigcup_{i\in\Lambda} U_i^c.$$

To put this into words, the complement of a union is the intersection of the complements, and the complement of an intersection is the union of the complements.

Proof:

1. Well firstly, notice $\emptyset^c = X$ and $X^c = \emptyset$ in X. Hence, given $\emptyset$ and X are open sets, $\emptyset$ and X are closed sets.
2. Given A_i are closed, A_i^c is open. Hence, using DeMorgan's Law,

$$\left(\bigcap_{i\in\Lambda} A_i\right)^c = \bigcup_{i\in\Lambda} A_i^c,$$

   and the arbitrary union of open sets is open. Hence, $\bigcap_{i\in\Lambda} A_i$ is closed.
3. We use DeMorgan's Law in exactly the same way to prove that the finite union of closed sets is closed.

□

Lets look at a useful examples:

Example 37

Let (X, d) be a metric space, and let $x \in X$. Then, for any $\epsilon > 0$, $B(x, \epsilon)$ is open in X. In fact, this ball is sometimes referred to as an *open ball*.

Proof: Let $y \in B(x, \epsilon)$. If $x = y$ then this is automatically true, just take $\epsilon' = \frac{\epsilon}{2}$. Suppose that $y \neq 0$, and let $r = \epsilon - d(x, y) > 0$. We want to show that $B(y, r) \subset B(x, \epsilon)$. Let $z \in B(y, r)$. Then $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r = \epsilon$. Therefore, $B(y, r) \subset B(x, \epsilon)$, and thus $B(x, \epsilon)$ is an open set. □

Example 38

Let (X, d) be a metric space, and $x \in X$. Then, $\{x\}$ is a closed set in X.

Proof: We want to show that for all $y \in X \setminus \{x\}$, there is an open ball around y such that x is not in the ball. Fix $y \in X \setminus \{x\}$; then, $y \neq x$ and thus $d(x, y) > 0$. Let $r = \frac{d(x,y)}{2}$. Hence, consider $B(y, r)$. We know that $x \notin B(y, r)$ as if this were the case, then $d(x, y) < r < d(x, y)$ which is a contradiction. Hence, $B(y, r) \subset X \setminus \{x\}$. Therefore, $X \setminus \{x\}$ is an open set, and thus $\{x\}$ is a closed set in X. □

Remark 39. *One can similarly prove that any finite set in a metric space is closed.*

Let's now see again how open sets relate to convergence and continuity. To do so, we first observe a fact about convergent sequences in $\mathbb{R}$.

Proposition 40

Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then, $\{x_n\}$ is convergent (and converges to x) if and only if $\forall \epsilon > 0$, all but finitely many terms in $\{x_n\}$ are in $(x - \epsilon, x + \epsilon)$.

Proof: Given $x_n \to x$, given $\epsilon > 0$ there exists an N such that for all $n \geq N$, $|x_n - x| < \epsilon$. Therefore, for all $n \geq N$, $x_n \in (x - \epsilon, x + \epsilon)$. For the other direction, fix arbitrary $\epsilon > 0$ and consider $(x - \epsilon, x + \epsilon)$. Given that all but finitely many terms in $\{x_n\}$ are in $(x - \epsilon, x + \epsilon)$, there exists an M such that for all $n \geq M$, $x_n \in (x - \epsilon, x + \epsilon) = B_\epsilon(x)$. Therefore, x_n is convergent. □

The same can be generally said for metric spaces.

Definition 41 (Neighborhood)

Suppose that $x \in U$ and U is open in X. Then we can U a **neighborhood** of x.

Theorem 42

Let $\{x_n\}$ be a sequence in the metric space (X, d). Then, x_n is convergent and converges to x if and only if for every neighborhood of x, all but finitely many terms in $\{x_n\}$ are not in the neighborhood of x.

Proof: The proof is exactly the same as the proof of $X = \mathbb{R}$, only changing to metric notation. □

Remark 43. *Every closed set has the property that every convergent sequence converges in the set. This will be shown on PSET 2, and gives yet another connection between open/closed sets and convergence.*

We now shift our focus to continuous functions.

Recall (Continuous functions)

Let (X, d_X) and (Y, d_Y) be metric spaces. Then, a function $f : X \supset A \to Y$ is **continuous** if and only if given $x \in A$, $\forall \epsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(x, y) \leq \delta \implies d_Y(f(x), f(y)) \leq \epsilon.$$

We will first show how continuity is related to convergence, and then how continuity is related to open sets.

Theorem 44

Let (X, d_X) and (Y, d_Y) be metric spaces. Then, $f : X \to Y$ is continuous at $c \in X$ if and only if for every sequence $\{x_n\}$ in X converging to c, $f(x_n) \to f(c)$.

Proof: Suppose that f is continuous at c. Let $\{x_n\}$ be a sequence in X converging to c. Given $\epsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, c) < \delta \implies d_Y(f(x), f(c)) < \epsilon$. Given $x_n \to c$, there exists an N such that for all $n \geq N$, $d_X(x_n, c) < \delta$. Therefore, $d_Y(f(x_n), f(c)) < \epsilon$. Thus, $f(x_n) \to f(c)$.

Suppose that f is not continuous at c. Let $\epsilon > 0$. Then, for all $n \in \mathbb{N}$, there exists an x_n such that $d(x_n, c) < \frac{1}{n}$ but $d_Y(f(x_n), f(c)) \geq \epsilon$. Then, $x_n \to c\}$ but $f(x_n)$ does not converge to $f(c)$. □

Lemma 45

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous at $c \in X$ if and only if for every open neighborhood U of $f(c)$ in Y, the set $f^{-1}(U)$ contains an open neighborhood of c in X.

Proof: Suppose that f is continuous at c. Let U be an open neighborhood of $f(c)$ in Y. Then, $B_Y(f(c), \epsilon) \subset U$ for some $\epsilon > 0$. By the continuity of f, there exists a $\delta > 0$ such that $d_X(x, c) \implies d_Y(f(x), f(c)) < \epsilon$. Hence,

$$B_X(c, \delta) \subset f^{-1}(B_Y(f(c), \epsilon)) \subset f^{-1}(U)$$

and $B_X(c, \delta)$ is an open neighborhood of c.

For the other direction, let $\epsilon > 0$. If $f^{-1}(B_Y(f(c), \epsilon))$ contains an open neighborhood V of c, then it contains a ball $B_X(c, \delta)$ such that

$$B_X(c, \delta) \subset W \subset f^{-1}(B_Y(f(c), \epsilon)).$$

Therefore, if $d_X(x, c) < \delta \implies d_Y(f(x), f(c)) < \epsilon$. Hence, f is continuous at c. □

Remark 46. *In fact, one can show that a function $f : X \to Y$ is continuous if and only if given $U \subset Y$ open, $f^{-1}(U)$ is open in X. This is an optional problem on PSET 2. This idea is once again integral to 18.901.*

3 January 11, 2022

Compact Sets in $\mathbb{R}^n$

Today, we will first discuss another useful concept tangentially related to metrics (norms), which will then motivate an important concept: compact sets.

Have you heard of a norm in other contexts before? A classic place to first hear of a "norm" is in 18.02 with the **Euclidean norm**, which defines the length of a vector in $\mathbb{R}^n$. How does this definition fundamentally work? One way to understand the idea of a Euclidean norm, is to visualize it as the distance between a point in $\mathbb{R}^n$ and the origin. This gives a direct relationship between this word "norm" in this context, to a metric. Given this, we define a norm in a more general context.

We first define a vector space.

Definition 47 (Vector Space)
A **vector space** V over a field k is a set of vectors which come with addition $(+ : V \times V \to V)$ and scalar multiplication $(\cdot : k \times V \to V)$ along with some classic axioms: commutativity, associativity, identity, and inverse of addition, identity of multiplication, and distributivity.

For our purposes in this class, we will only study vector spaces over the field $\mathbb{R}$. In essence, when we add two elements in the vector space, we stay in the vector space, and you can multiply an element in the space by a constant and stay in the space. The three key examples of a vector space, for our purposes, are $\mathbb{R}^n$, $\mathbb{C}^n$, and $C^0([a, b])$ (or more generally, $C^n([a, b])$). We can now define a norm:

Definition 48 (Norm)
A **norm** on a vector space V over the real numbers is a function $\|\cdot\| : V \to [0, \infty)$ satisfying the following three properties:

1. Positive Definite: $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$.
2. Homogeneity: $\|\lambda v\| = |\lambda|\|v\|$ for all $v \in V$ and $\lambda \in \mathbb{R}$.
3. Triangle Inequality: $\|x + y\| \leq \|x\| + \|y\|$.

A vector space with a norm on it is defined as a **normed space**.

Remark 49. *In a vector space V, 0 is always in V (why?). In PSET 2, you will directly show how the norm can relate to metrics.*

We can thus view some of the metrics we have defined thus far in the class to be analogous to norms.

Example 50 (Norm on Continuous Functions)
Show that $\|\cdot\| : C^0([0, 1]) \to [0, \infty)$ defined by

$$\|f\| = \sup_{x \in [0,1]} |f(x)|$$

is a norm.

Proof: Most of this proof will follow directly from the proof given by Example 10 in Lecture 1, but I will write the proof fully nonetheless.

1. It is clear that $\|f\| \geq 0$ for all $f \in C^0([0, 1])$ as absolute values are always non-negative, and $\|f\| = 0$ if and only if $\forall x \in [0, 1]$, $f(x) = 0$.

2. Let $\lambda \in \mathbb{R}$. Then,

$$\|\lambda f\| = \sup_{x\in[0,1]} |\lambda f(x)| = \sup_{x\in[0,1]} |\lambda||f(x)| = |\lambda| \sup_{x\in[0,1]} |f(x)| = |\lambda|\|f\|.$$

3. Let $f, g \in C^0([0,1])$. Then,

$$\|f+g\| = \sup_{x\in[0,1]} |f(x)+g(x)| \le \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |g(x)| = \|f\| + \|g\|$$

by the triangle inequality we proved for the metric on the space of continuous functions.

□

Example 51

Show that $\|\cdot\| : C^0([0,1]) \to [0,\infty)$ defined by

$$\|f\| = \int_0^1 |f(x)|\,\mathrm{d}x$$

is a norm.

Proof: This example is related to Example 20 in Lecture 1.

1. It is clear that $\|f\| \ge 0$ for all $f \in C^0([0,1])$ as absolute values are always non-negative. Additionally, notice that $|f\| = l_1(f,0)$ where l_1 which we discussed in that example. Hence, since d is positive definite, $l_1(f,0) = \|f\| = 0$ if and only if $f = 0$, which implies the norm is positive definite.

2. Let $\lambda \in \mathbb{R}$. Then,

$$\|\lambda f\| = \int_0^1 |\lambda f(x)|\,\mathrm{d}x = \int_0^1 |\lambda||f(x)|\,\mathrm{d}x = |\lambda| \int_0^1 |f(x)|\,\mathrm{d}x = |\lambda|\|f\|$$

using the linearity of the integral.

3. The triangle inequality we proved for the metric l_1 proves the triangle inequality here.

□

Given that the concept of a norm is very clearly analogous to metrics in some respects, you may wonder why we study norms in particular. A few key remarks about this: firstly, 18.102 explores this concept much further. In essence, norms help us understand vector spaces better, and 18.102 studies infinite dimensional vector spaces. (Conceptually: infinite dimensional linear algebra.) Secondly, proving a given function is a norm is a similar process to proving a given function is a metric, which is a useful skill.

Finally, norms give us an intuition behind *magnitude*. In $\mathbb{R}$, the magnitude is again related to absolute values, the very thing we used to motive metrics. In our last example, we could consider a function f to be large if $\|f\|$ is large (this is not official terminology, just conceptual). What, then, does $\|f'\|$ convey? This would measure "how large" or "how much change" f goes through over the interval $[0,1]$. One could ask the question: How does $\|f'\|$ relate to $\|f\|$? This is a very interesting question, and becomes even more interesting in higher dimensions, but I digress.

Question 52. *Why have we been studying metrics/norms on the space of continuous functions over intervals,* $[a,b]$ *or* $[0,1]$*, and not over* $\mathbb{R}$*?*

Notice, that we want both norms and metrics to be finite. However, scattered throughout our proofs, we have been using the fact that continuous functions on bounded intervals are themselves bounded (the Extreme Value Theorem).

What condition would we need to impose on the space of continuous functions to get the metrics and norms to be finite?

Let $f \in C^0(\mathbb{R})$. When will $\int_{-\infty}^{\infty} |f(x)|\, dx$ be finite? It will be finite if outside of some bounded interval, $f = 0$. This space of functions is very useful to study, and even has its own name:

Definition 53 (Compact Support)
A function $f \in C^0(\mathbb{R})$ has **compact support** if $f = 0$ outside of some interval $[-n, n]$ for a finite n.

Remark 54. *The* **support** *of a function* $f \in C^0(\mathbb{R})$ *is the closure of the set*

$$\{x \in \mathbb{R} \mid f(x) \neq 0\}.$$

A more general definition states that a function is compactly supported if it is zero outside of a **compact set**. Before we study compact sets, I want to quickly bring up three small lemmas to serve as a starting point. If you asked an analyst what intuition there is behind "compactness", many would say that compactness is a generalization of finiteness. Compact sets are to continuous functions as finite sets are to functions in general. Hence, recall the following three lemmas regarding finite sets.

Recall 55
Let A be a finite set of a metric space (X, d). Then,

- Every sequence in A has a convergent subsequence.
- A is closed and bounded.
- Given any function $f : A \to \mathbb{R}$, f achieves a maximum and minimum on A, and f is bounded.

Proof:

1. Let $\{x_n\}$ be a sequence in A. Then, there are only finitely many values x_i can take on, as A is finite. However, given that a sequence is infinitely long, there must exist some element $x \in A$ that is in the sequence $\{x_n\}$ infinitely many times. If this wasn't the case, the sequence $\{x_n\}$ wouldn't be infinitely long. Thus, take $x_{n_k} = x_i$ for $i \in I = \{n \in \mathbb{N} \mid x_n = x\}$. Then, $x_{n_k} \to x$ as $x_{n_k} = x$ for all k by construction.
2. Well firstly, we know that A is closed in X by the previous lecture. Furthermore, we know that A is bounded, as we can simply fix an $x \in A$, and let $B = \max_i\{d(x_i, x)\}$ for $x_i \in A$.
3. To find the maximum and minimum, simply look at the image of A under f. There are only finitely many elements in A, and then we can simply let

$$B_1 = \max_i\{f(x_i) \mid x_i \in A\} \quad \text{and} \quad B_2 = \min_i\{f(x_i) \mid x_i \in A\}.$$

We know B_1 and B_2 are achieved as there are only finitely many terms in A. It is then immediate to see then that every function is bounded.

□

These are extremely nice properties! We will shortly see analogs of these lemmas with regards to compact sets, but first:

Definition 56 (Covers)
Let $A \subset X$ where X is a metric space. Then, $\{U_i\}_{i \in I}$ is an **open cover** of A if $A = \bigcup_{i \in I} U_i$ and U_i is open for each i. A **subcover** of an open cover is a subcollection of the sets U_i that still cover A. A **finite subcover** of an open cover is a finite subcollection of the sets U_i that still cover A.

Definition 57 (Compactness)
Let (X, d) be a metric space. A set $A \subset X$ is **sequentially compact** if and only if every sequence in A has a convergent subsequence in A. A set $A \subset X$ is **compact** or *topologically compact* if every open cover of A has a finite subcover.

Remark 58. *Notice that the definition of sequential continuity is the same as the first lemma regarding finite sets we talked about a second ago.*

Conceptually, this idea can be kind of confusing, but let's look at some examples.

Example 59
$\mathbb{R}$ is not a compact subset of $\mathbb{R}$.

To see this, consider the open sets $U_j = (-j, j)$ for $j \in \mathbb{N}$. It is clear that the union of all the U_j will cover $\mathbb{R}$. However, is there a finite subcover? Assume for the sake of contradiction that there was a finite subcover. Then,

$$\mathbb{R} = \bigcup_{k=1}^{n} U_{j_k}. = (-j_k, j_k).$$

However, notice $j_k \in \mathbb{R}$ but $j_k \notin (-j_k, j_k)$. Hence, we have found an open cover of $\mathbb{R}$ that does not have a finite subcover.

Example 60
$(0,1]$ is not compact or sequentially compact in $\mathbb{R}$.

Similarly, consider $U_j = (1/j, 2)$ for $j \in \mathbb{N}$. To see why sequential compactness fails, consider the subsequences of the sequence $\{\frac{1}{n}\}$.

Example 61
$[0,1]$ is a compact subset of $\mathbb{R}$.

Proof: We will prove this directly, though it will take some work. Ultimately, we will develop more theorems about compact sets that will make similar examples like this easier. Take an open cover of $[0,1]$

$$[0,1] \subseteq \bigcup_{i \in I} U_i.$$

Then, for every $x \in [0,1]$, we have that

$$[0,x] \subseteq \bigcup_{i \in I} U_i.$$

Hence, let

$$c = \sup\{x \in [0,1] \mid [0,x] \text{ is covered by finitely many elements in the open cover}\}.$$

Clearly, $0 \leq c < 1$, as the closed interval $[0,0] = \{0\}$ must be contained in one U_i. Hence, we want to show that $c = 1$, in order to show that $[0,1]$ has a finite subcover. Assume for the sake of contradiction that $c < 1$]. Then, it follows that c is contained in some open set, and thus contained in some open interval U_i. This implies that there is an element c' such that $c' > c$ and $c' \in U_i$. Thus, $[0, c']$ is covered by finitely many open sets from the cover, which is a contradiction. Therefore, $c = 1$. □

Remark 62. *Notice, that a similar proof will show that $[a,b]$ is compact in $\mathbb{R}$. Also note, that a similar proof can show that $[a,b] \times [c,d]$ is compact in $\mathbb{R}^2$, and so on and so forth. This will be useful for an optional problem on PSET 2.*

We now want to prove some more general theorems regarding compact sets. Today, we will focus on compact sets in $\mathbb{R}^n$, and next time we will discuss compact subsets of general metric spaces.

By the previous few examples, we have some insight as to what compact sets in $\mathbb{R}$ might look like.

Theorem 63

Compacts sets in $\mathbb{R}$ are closed and bounded.

Proof: Assume that $A \Subset \mathbb{R}$. We want to take an open cover of A that shows its bounded. Pick an arbitrary $p \in A$. Then,

$$A \subset \bigcup_{i=1}^{\infty} B(p,i) = \mathbb{R}.$$

Given that A is compact, and the right hand side is an open cover, there exists a finite subcover. Hence,

$$A \subset \bigcup_{k=1}^{n} B(p, i_k) = B(p, i_n).$$

Therefore, A is bounded, as given any $x \in A$, $d(x,p) \leq i_n < \infty$.

We now prove closure. To do so, we want to show that $X \setminus A$ is open. Let $p \in X \setminus A$. For arbitrary $q \in A$, define

$$V_q = B\left(p, \frac{d(p,q)}{2}\right) \quad \text{and} \quad W_q = B\left(q, \frac{d(p,q)}{2}\right).$$

Notice that $V_q \cap W_q = \emptyset$ for all $q \in A$. Furthermore, $A \subset \bigcup_{q \in A} W_q$. Therefore, there exists a finite subcover of A, given by $A \subset \bigcup_{k=1}^{n} W_{q_k}$.

Thus, consider the sets $V_{q_1}, \ldots, V_{q_k}$. Given that there are finitely many open sets, the intersection of them all is open. Furthermore, by construction, for all $q_k, W_{q_k} \not\subseteq V_{q_k}$. Therefore, $\bigcap_{k=1}^{n} V_{q_k}$ is a neighborhood of p, and $\bigcap_{k=1}^{n} V_{q_k} \cap K = \emptyset$. We know this last intersection is the emptyset, as if it weren't, then there would exist an element in A in the intersection of the V_qs, and thus an element in a W_{q_j} such that $W_{q_j} \cap V_{q_j} \neq \emptyset$, which is a contradiction.

Therefore, there exists a neighborhood of p contained in $X \setminus A$. Thus, $X \setminus K$ is open, and hence A is closed. □

Notice that this proof does not rely on the fact that we are looking at $\mathbb{R}$. In fact,

Lemma 64

A compact set in a metric space (X, d) is closed and bounded.

Is the converse true? To see why it is true **in the case of** $\mathbb{R}$, we show a quick lemma.

Lemma 65

Let K be a compact set in a metric space (X, d), and let F be a closed subset of K. Then, F is a compact set.

Proof: Given that F is closed, F^c is open. Hence, let $\{U_i\}_{i \in I}$ be an open cover of F. Then,

$$F \subset K \subset F^c \cup \bigcup_{i \in I} U_i.$$

Therefore, given K is compact, there exists a finite subcover of K. Hence, there is a finite open subcover of F. ☐

Theorem 66 (Heine-Borel)

Let K be a subset of $\mathbb{R}$. Then, K is compact if and only if K is closed and bounded.

Proof: We know that compact implies closed and bounded, and we thus need to prove the other direction! Let K be a closed and bounded subset of $\mathbb{R}$. Then, given K is bounded, K is contained in some closed interval $[a, b]$, which we have shown to be compact. Hence, K is a closed subset of a compact set, and thus K is compact. ☐

Remark 67. *The Heine-Borel theorem does not carry over to an arbitrary metric space. Here, we used the fact that* $[a, b]$ *is compact in* $\mathbb{R}$. *A metric space is said to have the Heine-Borel property if every closed and bounded set in* X *is compact.*

At this point, you may be wondering why we mention the idea of sequential compactness, and how this actually relates to the idea of topological compactness. Firstly, recall the following theorem:

Theorem 68 (Bolzano-Weierstrass)

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Lemma 69

Consider $A \subset \mathbb{R}^n$ such that A is closed and bounded. Then, A is sequentially compact.

Proof: Let $\{x_n\}$ be a sequence in A. Then $\{x_n\}$ is bounded as A is bounded, and thus by Bolzano-Weierstrass, there exists a convergent subsequence of $\{x_n\}$. How do we know that $\{x_n\}$ converges in A? This uses the fact that A is closed. Therefore, every sequence in A has a convergent subsequence in A. ☐

Is the converse true? Yes!

Theorem 70 (Bolzano-Weierstrass)

Let K be s subset of $\mathbb{R}$. Show that K is sequentially compact if and only if K is closed and bounded.

Proof: We have shown the backwards direction, and we now show the forward direction. Let $K \subset \mathbb{R}$ be sequentialy compact. Let $\{x_n\}$ be a sequence in K that converges to arbitrary $x \in \mathbb{R}$. Then, every subsequence of $\{x_n\}$ converges to x. Therefore, $x \in K$. Hence, K contains all of its limit points, and is thus closed.

Suppose for the sake of contradiction that K is unbounded. Then, there is a sequence $\{x_n\}$ in K such that $|x_n| \to \infty$ as $n \to \infty$. Therefore, every subsequence of $\{x_n\}$ is unbounded and diverges, and thus $\{x_n\}$ has no convergent subsequence. This is a contradiction as K is sequentially compact. ☐

Remark 71. *You can generalize this proof to* $\mathbb{R}^n$*; try to do so!*

Corollary 72

Given $A \subset \mathbb{R}$, A is sequentially compact if and only if A is topologically compact.

In our next lecture, we will show this is true for all metric spaces! However, the proof will need to be different, as a closed and bounded set is not necessarily compact in a general metric space.

4 January 13, 2022

Compact Metric Spaces

Last time, we showed that a set in $\mathbb{R}^n$ is sequentially compact if and only if it is topologically compact, by showing

$$\text{sequentially compact} \iff \text{closed and bounded} \overset{\text{Heine-Borel}}{\iff} \text{topologically compact}.$$

However, by the previous remark, we don't have Heine-Borel for arbitrary metric spaces. Which begs the question: is sequentially compact the same as topologically compact in metric spaces? The answer is yes. To prove this, we first show a handful of preliminary results.

Lemma 73 (Lebesgue Number Lemma)

Let (X, d) be a sequentially compact metric space and $\{U_i\}_{i\in I}$ be an open cover of X. Then, there exists an $r > 0$ such that for each $x \in X$, $B_r(x) \subseteq U_i$ for some $i \in I$.

Proof: Before proving this, try to visualize the result!

We prove this lemma through contradiction. Assume that for some $r > 0$ there exists an $x \in X$ (possibly depending on r) such that for each $i \in I$, $B_r(x) \not\subseteq U_i$. Consider the sequence $\{x_n\}_n$ in X such that $B_{1/n}(x_n) \not\subseteq U_i$ for all $i \in I$.

Given that X is sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}_k$. Let $x_{n_k} \to x \in X$. Given that $\{U_i\}$ is an open cover of X, there exists a U_{i_0} such that $x \in U_{i_0}$. Given U_{i_0} is open, it also follows that there exists an r_0 such that $B_{r_0}(x) \subseteq U_{i_0}$. Hence, choose N large enough such that $d(x, x_N) < \frac{r_0}{2}$ and $\frac{1}{N} < \frac{r_0}{2}$. Then, if $y \in B_{1/N}(x_N)$, then

$$d(x, y) \leq d(x, x_N) + d(x_N, y) < r_0.$$

Therefore, $y \in B_{r_0}(x) \subseteq U_{i_0}$. Hence,

$$B_{1/N}(x_N) \subseteq B_{r_0}(x) \subseteq U_{i_0}$$

which is a contradiction. □

We call this r the **Lebesgue number** of the open cover of X, which is useful in other applications.

Definition 74

A metric space X it **totally bounded** if, for every $\epsilon > 0$, there exists $x_1, x_2, \ldots, x_k \in X$ with k finite such that $\{B_\epsilon(x_i) \mid 1 \leq i \leq k\}$ is an open cover of X.

Lemma 75

A metric space X is sequentially compact implies that X is totally bounded.

Proof: Assume that X is sequentially compact and not totally bounded. Therefore, there exists an $\epsilon > 0$ such that X cannot be covered by a collection of open sets of only finitely many ϵ-balls. Hence, let $x_1 \in X$, $x_2 \in X \setminus B_\epsilon(x_1)$, then $x_3 \in X \setminus B_\epsilon(x_1) \setminus B_\epsilon(x_2)$ and so on. We know that there exists such x_i by the previous statement. Hence, for all $i \neq j$, $d(x_i, x_j) \geq \epsilon$. Therefore, $\{x_n\}_n$ has no convergent subsequence as if there was a convergent subsequence it would be Cauchy, and the previous line shows that no subsequence of $\{x_n\}$ will be Cauchy. This is a contradiction to X being sequentially compact. □

Theorem 76

A metric space X is (topologically) compact if and only if X is sequentially compact.

Proof: We first show that topologically compact implies sequentially compact. Assume for the sake of contradiction there there exists a sequence $\{x_n\}_n$ in X with no convergent subsequence. Notice that no term in the sequence can appear infinitely many times, as otherwise there would be a trivial subsequence of $\{x_n\}$. Hence, we assume without loss of generality that $x_i \neq x_j$ if $i \neq j$. Furthermore, notice then that for every n there exists an $\epsilon_n > 0$ such that $B_{\epsilon_n}(x_n)$ contains no other terms in the sequence. If this wasn't the case, then there would again be a convergent subsequence of $\{x_n\}_n$. Therefore, for each i, there exists an open ball U_i centered at x_i such that $x_j \notin U_i$ for all $i \neq j$.

Additionally, consider $U_0 = X \setminus \{x_n \mid n \in \mathbb{N}\}$. U_0 is open, as $U_0^c = \{x_n \mid n \in \mathbb{N}\}$ is closed (it contains all of it's limit points). Hence,

$$U_0 \cup \{U_n \mid n \in \mathbb{N}\}$$

is an open cover of X. However, this open cover has no finite subcover as any finite collection of the cover will fail to include infinitely many terms from the sequence $\{x_n\}_n$. This is a contradiction, and thus topologically compact implies sequentially compact.

We now prove the other direction. Let X be sequentially compact and let $\{U_i\}_{i \in I}$ be an open cover of X. By the Lebesgue number lemma, there exists an $r > 0$ such that for each $x \in X$, $B_r(x) \subset U_i$ for some $i \in I$. Furthermore, by Lemma 5, X is totally bounded. Hence, there exists $y_1, \ldots, y_k \in X$ such that

$$X \subset B_r(y_1) \cup \cdots \cup B_r(y_k).$$

However, for each $i \in I$, we have $B_r(y_i) \subset U_{j(i)}$ for some $j(i) \in I$. (This notation just means for each i, there exists a $j \in I$ which depends on i such that $B_r(y_i) \subseteq U_j$). Thus, $\{U_{j(1)}, \ldots, U_{j(k)}\}$ is a finite subcover for X. Therefore, every open cover of X has a finite subcover, and thus sequentially compact implies topologically compact. □

Remark 77. *Notice that we technically could've used this proof in the previous lecture, but the Heine-Borel Theorem is so vastly important that I decided to do that proof before today's lecture.*

We will now start to look at some illuminating applications of compact sets to reach an even more powerful theorem.

Recall 78

Let X, Y be metric spaces and $f : X \to Y$ be a continuous function. Then, for all U open in Y, $f^{-1}(U)$ is open in X.

Theorem 79

Let X, Y be metric spaces and $f : X \to Y$ be continuous. Given $K \Subset X$, $f(K) \subset Y$ is compact.

Proof: Let $\{U_i\}_{i\in I}$ be an open cover of $f(K)$. Then, define $V_i = \{f^{-1}(U_i)\}_{i\in I}$, which is open as f is continuous. Therefore, $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of K. Hence, there exists a finite subcover $\{V_{i_1}, \dots V_{i_k}\}$ of K as K is compact. Thus, $\{U_{i_1}, \dots U_{i_k}\} = \{f(V_{i_1}), \dots, f(V_{i_k})\}$ is a finite subcover of $f(K)$. Therefore, $f(K)$ is compact. □

Corollary 80

Let X be a metric space and $K \Subset X$. Then, given a continuous function $f : X \to \mathbb{R}$, f obtains a maximum and minimum finite value on K.

Proof: The proof follows from the previous theorem, and Problem 5.(a) on PSET 2. □

Corollary 81

Sometimes in particular we want to study bounded continuous functions, and the previous corollary gives us a nice property. Given a compact metric space X, every continuous function on f is bounded.

Proof: Follows immediately. □

Theorem 82 (Cantor's Intersection Theorem)

If $K_1 \supset K_2 \supset K_3 \supset \dots$ is a decreasing sequence of nonempty sequentially compact subsets of $\mathbb{R}^n$, then $\cap_{i\geq 1} K_i$ is non-empty.

Proof: Choose a sequence $\{a_n\}_n$ such that $a_n \in K_n$ for each n. We know that there exists such an a_n as each K_n is nonempty. Then, $\{a_n\}_n$ is a sequence in K_1, and thus there exists a convergent subsequence $\{a_{n_k}\}_k$ such that $a_{n_k} \to a \in K_1$. Furthermore, $\{a_n\}_{n=2}^{\infty}$ is a sequence in K_2, and thus contains a a convergent subsequence. Therefore, $a \in K_2$. Continuing this process, we get that $a \in K_i$ for all i. Thus, $a \in \cap_{i\geq 1} K_i$. □

Definition 83 (Finite Intersection Property)

A collection of closed sets $\{C_i\}_i$ has the **finite intersection property** if every finite subcollection has a nonempty intersection.

Given Lemma 5 and the Cantor Intersection Theorem, it is clear that there are some relations between compact sets, nonempty intersections of sets, and totally bounded sets. We hence show the following theorem.

Theorem 84

Given a metric space (X, d), the following are equivalent.

(1) X is compact.

(2) X is sequentially compact.

(3) X is Cauchy complete and totally bounded.

(4) Every collection of closed subsets of X with the finite intersection property has a non-empty intersection.

We have shown (1) $\iff$ (2), and thus we show (1) $\iff$ (4) and (2) $\iff$ (3) to finish the proof.

Proof: (1) $\implies$ (4): Assume for the sake of contradiction that there exists a collection of closed subsets $\{C_i\}_{i\in I}$ with the finite intersection property such that $\cap_{i\in I} C_i =$. Given C_i is closed in X for all i, $U_i = C_i^c$ is open in X for each i. Then,

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} C_i^c = \left(\bigcap_{i\in I} C_i\right)^c = \emptyset^c = X.$$

Hence, the U_i cover X. Given X is compact, there exists a finite subcover $\{U_{i_1}, \dots, U_{i_k}\}$ of X. Thus,

$$X = \bigcup_{n=1}^{k} U_{i_n} = \left(\bigcap_{n=1}^{k} U_{i_n}^c\right)^c = \left(\bigcap_{n=1}^{k} C_{i_n}\right)^c.$$

Therefore, $\bigcap_{n=1}^{k} C_{i_n} = \emptyset$ which is a contradiction with the finite intersection property.

(4) $\implies$ (1): Suppose that $\{U_i\}_{i\in I}$ is an open cover of X, and let $C_i = U_i^c$ for each $i \in I$. Assume for the sake of contradiction that no finite subset of the U_i covers X. We show that C_i has the finite intersection property. Assume for the sake of contradiction that $\{C_{n_1}, \dots, C_{n_k}\}$ satisfies $C_{n_1} \cap \cdots \cap C_{n_k} = \emptyset$. Then,

$$\bigcup_{i=1}^{k} U_{n_i} = \left(\bigcap_{i=1}^{k} U_{n_i}^c\right)^c = \left(\bigcap_{i=1}^{k} C_{i_k}\right)^c = \emptyset^c = X.$$

This is a contradiction with the assumption that no subset of the U_i covers X. Hence, $\{C_i\}_{i\in I}$ satisfies the finite intersection property. Therefore, $\{C_i\}_{i\in I}$ has non-empty intersection; i.e. $\bigcap_{i\in I} C_i \neq \emptyset$. Then, $\bigcup_{i\in I} U_i \neq X$, which is a contradiction to the U_i being an open cover for X. Thus, every open cover of X has a finite open subcover.

(2) $\implies$ (3): We have already shown that X being sequentially compact implies totally bounded, and hence we only need show that sequentially compact implies Cauchy complete. Let $\{x_n\}$ be a Cauchy sequence in X. Given $\{x_n\}$ is a sequence in X, there exists a convergent subsequence $\{x_{n_k}\}$ in X such that $x_{n_k} \to x \in X$. Let $\epsilon > 0$, and choose N such that $d(x_i, x_j) < \epsilon/2$ whenever $i, j \geq N$. Next, choose $n_k > N$ such that $d(x_{n_k}, x) < \epsilon/2$. Then,

$$d(x, x_N) \leq d(x, x_{n_k}) + d(x_{n_k}, x_N) < \epsilon.$$

Thus, $x_n \to x \in X$ as $n \to \infty$. Therefore, every Cauchy sequence in X converges to a point in X. Hence, X is Cauchy complete.

(3) $\implies$ (2): This part of the proof is quite difficult. Consider a sequence $\{x_n\}_n$ in X. Given X is totally bounded, for every $n \in \mathbb{N}$, there exists a finite set of points $\{y_1^{(n)}, \dots, y_{r(n)}^{(n)}\}$ such that $X \subset B_{\frac{1}{n}}(y_1^{(n)}) \cup \cdots \cup B_{\frac{1}{n}}(y_{r(n)}^{(n)})$. Define

$$S_n = \{y_1^{(n)}, \dots, y_{r(n)}^{(n)}\}.$$

We want to find a convergent subsequence of $\{x_n\}_n$. We do so by construction. Given S_1 is finite, there exists a $y_{n(1)}^{(1)} \in S_1$ such that $B_1(y_{n(1)}^{(1)})$ contains infinitely many points from $\{x_n\}_n$. Choose z_1 from this ball. Then, given S_2 is finite, there is a $y_{n(2)}^{(2)}$ such that infinitely many points from $\{x_n\}_n$ are in $B_1(y_{n(1)}^{(1)}) \cap B_{1/2}(y_{n(2)}^{(2)})$. Choose z_2 from this set. Continue this procedure for each $k > 1$, selecting a z_k from $\bigcap_{i=1}^{k} B_{\frac{1}{k}}(y_{n(k)}^{(k)})$. Then, we show $\{z_n\}_n$ is Cauchy. Let $\epsilon > 0$. Then, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Hence, for all $n, m \geq N$,

$$d(z_n, z_m) < \frac{1}{N} < \epsilon.$$

Therefore, by the Cauchy completeness of X, $\{z_n\}$ converges to a point in X. □

Remark 85. *Where do we use the fact that each ball has infinitely many points? We do in fact use this property in the proof. Try to figure out how!*

5 January 18, 2022

The Fixed Point Theorem

In this section of the notes, we focus on examples and theorems that are useful to know with very useful applications. Some of the most insightful examples, involve "Lipschitz" functions.

Definition 86 (Lipschitz)
Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called **Lipschitz** or K-Lipschitz if there exists a $K \in \mathbb{R}$ such that

$$d_Y(f(x), f(y)) \leq K d_X(x, y) \quad \text{for all } x, y \in X.$$

These functions are sometimes called *Lipschitz continuous functions*. Why? Well, consider a K-Lipschitz function for some $K > 0$, and let $\epsilon > 0$. Then, choose $\delta = \frac{\epsilon}{K}$. Hence, when $d_X(p, q) < \delta$, we have that

$$d_Y(f(p), f(q)) \leq K d_X(p, q) < \epsilon.$$

Therefore, f is continuous. The same is immediately true when $K \leq 0$, simply choose $\delta = 1$ and use positive definiteness of d_Y.

Lipschitz functions are a key motivator for *uniformly continuous functions*.

Definition 87 (Uniform continuity)
Let (X, d_X) and (Y, d_Y) be metric spaces. Then, $f : X \to Y$ is **uniformly continuous** if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \epsilon$.

Remark 88. *You may be wondering what the difference is between uniform continuity and regular continuity. Well notice that in the definition of uniform continuity, δ only depends on ϵ and f. I.e., δ does not depend on x. We say a function is continuous if it is continuous at every $x \in X$, and thus δ depends on x. This is the difference between uniform continuity and regular continuity.*

Notice that a uniformly continuous function is continuous, but the other direction is not necessarily true.

Theorem 89
Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $f : X \to Y$ is continuous and X is compact. Then, f is uniformly continuous.

Proof: Let $\epsilon > 0$. For each $c \in X$, choose δ_c such that

$$d_X(x, c) < \delta_c \implies d_Y(f(x), f(c)) < \epsilon/2.$$

We know that such a δ_c exists as f is continuous. Furthermore, the balls $B(c, \delta_c)$ cover X and the space X is compact. Then, by the Lebesgue Number Lemma, there exists a $\delta > 0$ such that for all $x \in X$, there is a $c \in X$ such that $B(x, \delta) \subset B(c, \delta_c)$.

If $x, y \in X$ and $d_X(x, y) < \delta$, choose a $c \in X$ such that $B(x, \delta) \subset B(c, \delta_c)$. Then, $y \in B(c, \delta_c)$ by assumption. Therefore, by the triangle inequality,

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(c)) + d_Y(f(c), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

We discuss one more application of uniform continuity, and then we will move onto another useful application of Lipschitz functions.

Proposition 90

If $f : [a, b] \times [c, d] \to \mathbb{R}$ is a continuous function, then $g : [c, d] \to \mathbb{R}$ defined by

$$g(y) = \int_a^b f(x, y)\, dx$$

is continuous.

Proof: Let $\epsilon > 0$. Fix $y \in [c, d]$ and let $\{y_n\}$ be a sequence in $[c, d]$ such that $y_n \to y$. As we have shown in Lecture 2, g is continuous if and only if $g(y_n) \to g(y)$. This is what we will show. Firstly, note that as f is continuous on $[a, b] \times [c, d]$ which is compact, f is uniformly continuous. I.e., there exists a $\delta > 0$ such that given $y' \in [c, d]$ and $|y' - y| < \delta$, then $|f(x, y') - f(x, y)| < \epsilon$ for all $x \in [a, b]$.

Let $h_n(x) = f(x, y_n)$ and $h(x) = f(x, y)$. We have thus shown that $h_n \to h$ uniformly as $n \to \infty$. Uniform convergence implies we can swap limits and integrals, obtaining

$$\lim_{n\to\infty} g(y_n) == \lim_{n\to\infty} \int_a^b f(x, y_n)\, dx = \int_a^b \lim_{n\to\infty} f(x, y_n)\, dx = \int_a^b f(x, y)\, dx = g(y).$$

Therefore, g is continuous. □

We now return back to the usefulness of Lipschitz functions.

Definition 91 (Contraction)

Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $f : X \to Y$ is said to be a **contraction** if it is a $k-$Lipschitz map for some $0 \leq k < 1$. In other words, there exists a $k < 1$ such that

$$d_Y(f(x), f(y)) \leq k d_X(x, y) \quad \text{for all } x, y \in X.$$

Definition 92 (Fixed point)

If $f : X \to X$ is a map, $x \in X$ is called a **fixed point** if $f(x) = x$.

We thus have a useful theorem that follows from these simple definitions.

Theorem 93

Banach Fixed Point Theorem Let (X, d) be a nonempty complete metric space, and $f : X \to X$ be a contraction. Then, f has a unique fixed point.

Note: This is sometimes called the contraction mapping principle.

Proof: Try to picture this!

We want to show that there exists an $x \in X$ such that $f(x) = x$, and then we want to show x is unique. How can we find such an x though?

Pick some random $x_0 \in X$, and define a sequence $\{x_n\}$ such that $f(x_n) = x_{n+1}$. Then, by definition, we have that

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq kd(x_n, x_{n-1}) \leq \cdots \leq k^n d(x_1, x_0).$$

We will show that $\{x_n\}$ is a Cauchy sequence.

Question 94. *What good does this do? What property of the theorem will we use here?*

Suppose $m \geq n$. Then,

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \\ &\leq \sum_{i=n}^{m-1} k^i d(x_1, x_0) \\ &= k^n d(x_1, x_0) \sum_{i=0}^{m-n-1} k^i \\ &\leq k^n d(x_1, x_0) \sum_{i=0}^{\infty} k^i \\ &= \frac{k^n}{1-k} d(x_1, x_0). \end{aligned}$$

Given $0 \leq k < 1$, as $n \to \infty$, $d(x_m, x_n) \to 0$. Therefore, $\{x_n\}$ is a Cauchy sequence, and thus there exists an x such that $x_n \to x$. We claim that x is a fixed point:

$$x = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

We also claim that x is unique. Suppose that y is also a fixed point of f. Then,

$$d(x, y) = d(f(x), f(y)) \leq kd(x, y) \implies (1-k)d(x, y) \leq 0.$$

Given $0 \leq k < 1$, it follows that $d(x, y) = 0 \implies x = y$. □

As stated in Lebl's book: "The proof is constructive. Not only do we know a unique fixed point exists. We also know how to find it" (7.6.1 page 268). We use this fact to consider an interesting application of the fixed point theorem: differential equations.

Often, we wonder when a differential equation has a solution. In 18.03, we tried to produce formulas to precisely solve differential equations. But as we approach more and more complex differential equations (complex in the sense of difficulty, not inherently complex valued), we need a different approach. Analysis and metric spaces, and especially the Banach fixed point theorem can be a very useful tool for such questions. Especially since, as we have shown, $C^0([a, b])$ is a complete metric space under the uniform metric/norm.

Remark 95. *Using the contraction mapping principle to solve differential equation is a central topic in 18.152.*

Consider the simple ordinary differential equation

$$\begin{cases} \frac{df}{dx} = F(x, f(x)) \\ f(x_0) = y_0. \end{cases}.$$

We want to solve this initial value problem (IVP), finding a function $f(x)$ such that $f'(x) = F(x, f(x))$ where F is a general function. For instance, consider the IVP:

$$\begin{cases} \frac{dy}{dx} = y' = y \\ y(0) = 1. \end{cases}.$$

We can solve this IVP with the solution $y = e^x$ as $(e^x)' = e^x$ and $e^0 = 1$. A more complicated example to consider is $y' = -2xy$, $y(0) = 1$. You can check that $y(x) = e^{x^2}$ is a solution.

One can ask how long a solution exists for. For instance, consider $y' = y^2$, $y(0) = 1$. This has solution $y(x) = \frac{1}{1-x}$. While y^2 is a nice function (i.e. existing for all x and y), the solution blows up at $x = 1$. So how can we use the contraction mapping theorem to approach this problem?

Consider the following equation:

$$f(x) = y_0 + \int_0^x F(t, f(t))\,dt.$$

Notice that $f(0) = y_0$, and $f'(x) = F(x, f(x))$ by the Fundamental Theorem of Calculus. Using this equation as a motivator, we can prove the following theorem:

Theorem 96 (Picard's Theorem)
Let $I, J \subset \mathbb{R}$ be closed and bounded intervals, let I°, J° be their interiors, and let $(x_0, y_0) \in I^\circ \times J^\circ$. Suppose $F : I \times J \to \mathbb{R}$ is continuous and Lipschitz in the second variable. I.e., there exists an $L \in \mathbb{R}$ such that

$$|F(x, y) - F(x, z)| \leq L|y - z|$$

for all $x \in I$ and $y, z \in J$. Then, there exists an $h > 0$ and a unique differentiable function $f : [x_0 - h, x_0 + h] \to J \subset \mathbb{R}$ such that

$$f'(x) = F(x, f(x)) \quad \text{and} \quad f(x_0) = y_0.$$

By "interiors", I mean that if $I = [0, 1]$, then $I^\circ = (0, 1)$. There is a more general definition of the interior of a set, but we move on for now. Also note that we may assume without loss of generality that $x_0 = 0$.

Proof: We will prove this by constructing the convergent sequence used in the Banach fixed point theorem, and then I will outline another approach that creates a contraction that satisfies the properties we want.

The first method is called Picard iteration. To solve $f'(t) = F(x, t)$ with $f(0) = 0$, we first start with a guess. Consider the simple function $f_0(t) = y_0$. Then, it is clear that $f_0(0) = y_0$, but clearly this only solves the equation if $F(x, f_0(t)) = 0$. We thus need to keep improving our guesses. Consider a function f_1 such that

$$f_1'(t) = F(t, f_0(t)), \quad f_1(0) = y_0.$$

We can solve this ODE using an integral, obtaining

$$f_1(x) - f_1(0) = \int_0^x F(t, f_0(t))\,dt \implies f_1(x) = y_0 + \int_0^x F(t, f_0(t))\,dt.$$

Now *this* is a function we can keep on reiterating. Consider

$$f_{n+1}(x) = y_0 + \int_0^x F(t, f_n(t))\,dt.$$

We thus want to show that this sequence of functions converges as $k \to \infty$ and that the limit

$$f(x) = \lim_{k\to\infty} f_k(x)$$

is a solution to the ODE.

We first check that f_k is well-defined for all k. Pick $\alpha > 0$ such that $[-\alpha, \alpha] \subset I$ and $[y_0 - \alpha, y_0 + \alpha] \subset J$. Given $F(x, y)$ is continuous over the compact set $I \times J$, there exists an M such that $|F(x, y)| \leq M$ for all $(x, y) \in I \times J$. Hence, define

$$h = \min\left\{\alpha, \frac{\alpha}{M + L\alpha}\right\}.$$

Notice that $[-h, h] \subset I$. We prove that f_k is well-defined inductively. Assuming that $f_{k-1}([-h, h]) \subset [y_0 - \alpha, y_0 + \alpha]$, it follows that $F(t, f_{k-1}(t))$ is well defined for all $t \in [-h, h]$. Therefore, $f_k(x) = y_0 + \int_0^x F(t, f_{k-1}(t))\,\mathrm{d}t$ is well defined for all $x \in [-h, h]$. We thus need to show that $f_k([-h, h]) \subset [y_0 - \alpha, y_0 + \alpha]$. Given $x \in [-h, h]$, we have

$$|f_k(x) - y_0| = \left|\int_0^x F(t, f_{k-1}(t))\,\mathrm{d}t\right| \leq M|x| \leq Mh \leq \frac{M\alpha}{M + L\alpha} \leq \alpha.$$

Therefore, f_k is well-defined for all k on the interval $[-h, h] \subset I$. Now we want to show that f_k converge to some function f. We can do this by showing $\{f_k\}$ is a Cauchy sequence (just like we did for the proof of the Banach fixed point theorem!):

$$\begin{aligned}
|f_m(x) - f_x(x)| &= \left|\int_0^x F(t, f_{m-1}(t)) - F(t, f_{n-1}(t))\,\mathrm{d}x\right| \\
&\leq \int_0^x |F(t, f_{m-1}(t)) - F(t, f_{n-1}(t))|\,\mathrm{d}t \\
&\leq L\int_0^t |f_{m-1}(t) - f_{n-1}(t)| \\
&\leq L\|f_{m-1} - f_{n-1}\||x| \\
&\leq \frac{L\alpha}{M + L\alpha}\|f_{m-1} - f_{n-1}\|.
\end{aligned}$$

Let $C = \frac{L\alpha}{M+L\alpha} \leq 1$. Therefore, taking the supremum of the left-hand side, we get

$$\|f_m - f_n\| \leq C\|f_{m-1} - f_{n-1}\|.$$

By induction, through a similar proof used in the Banach fixed point theorem, it follows that $\{f_n\}$ is a Cauchy sequence, and thus $f_n \to f \in C^0([-h, h])$.

We want to show that f satisfies the ODE. Note that $f([-h, h]) \subset [y_0 - \alpha, y_0 + \alpha]$. Firstly, notice that

$$|F(t, f_n(t)) - F(t, f(t))| \leq L|f_n(t) - f(t)| \leq L\|f_n - f\|.$$

Therefore, given $f_n \to f$ uniformly, $F(t, f_n(t)) \to F(t, f(t))$ uniformly for $t \in [-h, h]$. Thus,

$$\begin{aligned}
y_0 + \int_0^x F(t, f(t))\,\mathrm{d}t &= y_0 + \int_0^x F(t, \lim_{n\to\infty} f_n(t))\,\mathrm{d}t \\
&= y_0 + \int_0^x \lim_{n\to\infty} F(t, f_n(t))\,\mathrm{d}t \\
&= \lim_{n\to\infty} y_0 + \int_0^x F(t, f_n(t)) \\
&= \lim_{n\to\infty} f_{n+1}(x) \\
&= f(x).
\end{aligned}$$

By the FTC, it is then clear that f is differentiable, $f'(x) = F(x, f(x))$, and $f(0) = y_0$. □

To prove this by proving the premises of the Banach fixed point theorem, you can consider the space

$$Y = \{f \in C([-h, h]) \mid f([-h, h]) \subset J\},$$

and show the following:

1. Y is a closed subset of continuous functions.
2. A closed subset of a complete metric space is a complete metric space.
3. Consider $T : Y \to C([-h, h])$ given by

   $$T(f)(x) = y_0 + \int_0^x F(t, f(t))\,\mathrm{d}t,$$

   and show that T is a contraction from $Y \to Y$.
4. Then T has a unique fixed point by the fixed point theorem, which solves the ODE.

□

Remark 97. *This will be an optional problem on PSET 4.*

We can consider one more (harder to motivate) example of the Banach fixed point theorem.

Example 98

Let $\lambda \in \mathbb{R}$, $f, g \in C^0([a, b])$, and $k \in C^0([a, b] \times [a, b])$. Then, consider the operator $T : C^0([a, b]) \to C^0([a, b])$

$$T(f)(x) = g(x) + \lambda \int_a^b k(x, y) f(y)\,\mathrm{d}y.$$

For which λ is T a contraction?

By the Proposition 5, we know that $T(f)$ is continuous. Given that k is continuous on a compact set, k is bounded. Thus, there exists a c such that

$$|k(x, y)| \leq c \;\forall x, y \in [a, b].$$

Then, we have

$$\begin{aligned} d(T(f_1), T(f_2)) &= \sup_{x\in[a,b]} |T(f_1)(x) - T(f_2)(x)| \\ &= |\lambda| \sup_{x\in[a,b]} \left| \int_a^b k(x,y)(f_1(y) - f_2(y))\,\mathrm{d}y \right| \\ &\le |\lambda| \sup_{x\in[a,b]} \int_a^b |k(x,y)||f_1(y) - f_2(y)|\,\mathrm{d}y \\ &\le |\lambda| \sup_{x\in[a,b]} |f_1(x) - f_2(x)| \sup_{x\in[a,b]} \int_a^b |k(x,y)|\,\mathrm{d}y \\ &\le c|\lambda|(b-a)d(f_1, f_2). \end{aligned}$$

Therefore, if $|\lambda| < \frac{1}{c(b-a)}$, it follows that T is a contraction on a complete metric space. Therefore, by the Banach fixed point theorem, there exists a unique $f \in C^0([a,b])$ such that

$$T(f)(x) = f(x) = g(x) + \lambda \int_a^b k(x,y)f(y)\,\mathrm{d}y.$$

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Where We Go From Here

In this section of the notes (the final one for this class!), we will discuss a bit of the history of metric spaces, and give a preview of how concepts learned here apply to future classes (i.e. 18.901, 18.102, etc.).

History: In the early 1900s, the usual approach to mathematics was far less abstract and axiomatic. Hence, at the time, *various spaces* that mathematicians studied (such as function spaces as we have studied a bit of in this class) had different notions of convergence. Each space has it's own notion of the word, which was studied in its own respect. There were some similarities between these notions, but there was no general understanding of the term.

Then, in 1906, Fréchet introduced the idea of metric spaces in his *Ph.D. Dissertation*.

Remark 99. *Fréchet, however, did not coin the term "Metric Space"; the term was coined by Felix Hausdorff.*

This allowed him (and many other mathematicians) to prove a result for a metric space, and have it be applicable to all other specific examples. This was the highlight of §2 of our class– the General Theory of metric spaces.

In this class, we discussed *normed spaces*, and saw how such spaces were in fact metric spaces under the metric induced by the norm. In this way, metric spaces are a generalization of normed spaces.

Question 100. *Is there a generalization of metric spaces?*

Yes, there is: Topological spaces.

18.901: Introduction to Topology: Topological spaces were first defined by Hausdorff in 1914 in his book "Principles of Set Theory". His book increased the popularity of metric spaces as a mathematical tool.

Definition 101 (Topology)

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**.

Question 102. *Where has we seem properties like this before?*

We saw these properties when we defined open sets! In fact,

Definition 103 (Open Set)

Let $(X, \mathcal{T})$ be a topological space. Then, $U \subset X$ is called an **open set** if $U \in \mathcal{T}$. In other words, we define the set in $\mathcal{T}$ as open sets. Similarly, $V \subset X$ is a **closed set** if $X \setminus V \in \mathcal{T}$.

Thus, inherently, open sets in the topological sense automatically follow the topological properties of open sets in metric spaces (as discussed in Lecture 2). In fact, given a metric space (X, d), there is a *topology induced by the metric*. Intuitively, the topology is defined as the collection of unions of ϵ-balls for all $\epsilon > 0$. This intuitive definition follows from what we have discussed about metric spaces: an open set in a metric space is the union of (arbitrarily many) open balls.

Remark 104. *To rigorously define the topology induced by the metric: ϵ-balls form a basis for the topology on X.*

At first, this definition can feel too general to particularly seem useful, but then again the definition of metric spaces can feel the same way at first. Given a topological space X, we can define notions of neighborhoods, convergence, and continuity that align with what we have proven for metric spaces.

Definition 105

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then,

1. A **neighborhood** U of a point $x \in X$ is an open set (i.e. $U \in \mathcal{T}_X$) such that $x \in U$.
2. A sequence $\{x_n\}$ in X **converges** to $x \in X$ if for every neighborhood U of x, there is an N such that $x_n \in U$ for all $n \geq N$.
3. A function $f : X \to Y$ is said to be **continuous** if for each open set $V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X$.

This definition may feel very abstract, but as we have shown throughout 18.S097, these definitions are related to our understanding of convergence and continuity in metric space.

One may ask, given that topological spaces are a generalization of metric spaces, why we study metric spaces. To this, I say: why do we study calculus before we study real analysis? In theory, we could prove calculus using real analysis without having actually taken 18.01 or 18.02. And yet, taking these pre-requisites give us an intuition for *why* certain theorems should be true, and in some cases give us an intuition of how to approach proofs. Even more generally, calculus gives us an intuition for how derivatives of functions "should" look for nice functions. Even if we study weirder functions (or analogously weirder spaces that aren't metric spaces), having the intuition allows us to play with abstract objects.

This is why throughout this class I have tried to draw diagrams when possible to describe what a theorem is actually saying, or to visualize an example. This technique of drawing an entire "space" as a blob on a chalkboard or on paper is **very** useful when trying to approach a problem, which you can see more of if you decide to take a class like 18.901.

That being said, there is a *lot* more we can know about metric spaces than we can about topological spaces, just in the same way we can know more about normed spaces than metric spaces. We have already seen this to a certain extent. One can show that given a normed space $(X, \|\cdot\|)$, and two convergent sequences $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$. We cannot say the same about a general metric space, as we don't inherently have always have a notion of addition in a metric space. So metric spaces are still an interesting area to study even *after* someone learns about topological spaces. In fact, metric spaces are currently an active area of research, even though most research has ended for topological spaces.

So so far, we have talked about topological spaces which are more general than metric spaces. Is there a space that is more specific than a metric space that is useful to study? Yes, in fact: normed spaces.

18.102: Introduction to Functional Analysis: In Lecture 3 for this class, as a way to motivate the usefulness of compact sets, we defined normed spaces.

Recall 106

A vector space with a norm on it is defined as a **normed space**.

And as you showed in PSET 2, a normed space is in fact a metric space, by defining $d(x, y) = \|x - y\|$: the metric induced by the norm. In fact, using our concept of convergence, Cauchy sequences, and open sets for metric spaces, we can define these ideas once again for normed spaces.

Definition 107

Let $(X, \|\cdot\|)$ be a normed space and $\{x_n\}$ be a sequence in X. Let d be the metric induced by the norm. Then,

1. x_n converges to x if and only if for all $\epsilon > 0$, there exists an N such that for all $n \geq N$,

$$d(x_n, x) = \|x_n - x\| < \epsilon.$$

2. $\{x_n\}$ is a Cauchy sequence if and only if for all $\epsilon > 0$, there exists an N such that for all $n, m \geq N$,

$$d(x_n, x_m) = \|x_n - x_m\| < \epsilon.$$

3. A set A is open in X if for all $x \in A$, there exists an $\epsilon > 0$ such that

$$B_\epsilon(x) = \{y \in X \mid d(x, y) = \|x - y\| < \epsilon\} \subset A.$$

We similarly have a definition of Cauchy completeness in a normed space (i.e. a space is Cauchy complete if every Cauchy sequence converges in the space). To be honest, when I studied this in 18.100B, Cauchy sequences did not seem all that useful. So, I found it somewhat shocking that Cauchy sequences were extremely important in functional analysis.

Definition 108 (Banach Space)

A **Banach space** is a normed space that is Cauchy complete with respect to the norm.

Banach spaces are named after Stefan Banach who studied the spaces in 1920-1922.

Remark 109. *Fun fact: The term "Banach space" was coined by Fréchet, and the term "Fréchet space" (which we did not cover in this class) was coined by Banach.*

Example 110

As we have shown (either in 18.100A/P or in this class), $\mathbb{R}^n$, $\mathbb{C}^n$, and $C^0([a,b])$ are Banach spaces. In fact, one can show that the space

$$C_\infty(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous and bounded}\}$$

is a metric space with respect to the uniform norm on metric spaces.

In our study of Cauchy sequences in metric spaces, there was one fact that we use over and over again to help finish proofs: the fact that $\mathbb{R}$ is Cauchy complete. This fact is so important, we obtain a new definition:

Definition 111 (Functional)

Let $(V, \|\cdot\|)$ be a normed space. A **functional** is a bounded linear map from $f : V \to \mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ depending on the context.

This term is directly related to why 18.102 is called functional analysis! As it turns out, studying normed spaces is heavily related to studying functionals on those spaces. If you have taken linear algebra, this idea is similar to how studying a vector space is heavily related to studying the dual space of that vector space, but I digress.

This class is also heavily related to quantum mechanics, so if you are someone interested in physics/this concept, a class like 18.102 may be interesting to take.

18.152: Introduction to Partial Differential Equations: I can't say too much about this class as I haven't taken it myself, but just to bring it up again, the Banach fixed point theorem is very useful for proofs in this class regarding differential equations as we have seen.

On that note, I want to discuss one specific problem in partial differential equations that actually motivated the development of material in 18.100x classes: the Dirichlet problem. Consider some subset $\Omega \subset \mathbb{R}^2$, and picture this set as a metal plate. Suppose I heated the plate with a blowtorch for a certain amount of time. After a long period of time, the plate will reach thermal equilibrium. This is represented by the steady-state heat equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where $u(x, y)$ is the temperature at point (x, y). This operator is so important, we abbreviate it with Δ, referred to as the Laplace operator or **Laplacian**. Suppose that I also knew the temperature on the boundary of this plate (denoted $\partial\Omega$). Let $f = u$ on $\partial\Omega$ (i.e. f is the temperature on the boundary of the plate).

Question 112. *Can we find the temperature distribution on Ω? Is this temperature distribution unique? This question is called the Dirichlet problem (in $\mathbb{R}^2$).*

We want to solve the differential equation

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u\big|_{\partial\Omega} = f \end{cases}$$

for some $u \in C^2(\mathbb{R}^2)$ and $f \in C^0(\mathbb{R}^2)$.

Early on, mathematicians studied this problem by studying the "Dirichlet energy" of u, given by the function

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dA.$$

Consider all of the C^2 functions on Ω with the given boundary condition. Let E_{inf} be the infimal energy of functions in this set. Dirichlet and others showed that if E_{inf} is the energy of a function $u \in C^2$ with the given boundary condition, then $\Delta u = 0$. Thus, u would be a solution to the Dirichlet problem! However, this raises the question:

Question 113 (Question 1)**.** *Does there exist a C^2 function u with the given boundary condition, such that $E(u) = E_{inf}$?*

If the answer is yes, then we have solved the problem! How might we approach finding such a function u? We could take a sequence of functions u_n such that $E(u_n) \to E(u)$! However, this raises yet another question:

Question 114 (Question 2)**.** *Does u_n converge in C^2 to a limit function u?*

If this is true, then we can easily show that $E(u) = E_{\text{inf}}$ and then we solve the problem! In fact, if u_n converges in C^1 to u, then $E(u) = E_{\text{inf}}$. This raises the following question again:

Question 115 (Question 3)**.** *Consider the set of functions $u \in C^2$ with the given boundary data, with $E(u)$ at most $E_0 > 0$. Is this set of functions compact in C^2 or C^1?*

If *this* is true, then we have sequential compactness, which means we can find a subsequence u_{n_k} converging to a limit u with $E(u) = E_{\text{inf}}$.

The answer to question 3, is no. This set of functions it *not* compact in either C^1 or C^2. This ruins this entire, arguably very intuitive approach to the problem. Even though this idea didn't solve the problem though, it did highlight some key issues at play. In fact, the ideas of convergence and compactness in a metric space were developed partly to see if this approach to the Dirichlet problem works or not. It turns out, this approach doesn't work.

The answer to question 2 is also no. We can in fact build a sequence of C^2 functions u_n with the given boundary condition such that $E(u_n) \to E_{\text{inf}}$, and yet u_n fails to converge in C^1 (let alone C^2). One can show that a function with zero boundary data and very small Dirichlet energy can *still* have a large C^1 norm.

However, the answer to question 1 is *yes*. Eventually, mathematicians were able to solve the Dirichlet problem, using techniques beyond the scope of this class. While we won't discuss the solution, this rich history highlights how mathematical concepts were/are developed to solve problems like this. This topic is sometimes talked about more in 18.102, and in some graduate level classes.

Remark 116. *An optional problem on PSET 4 asks you to solve the Dirichlet problem on an interval of $\mathbb{R}$.*

Thank you for taking this class with me this IAP. It has been a fun time developing the material and teaching the class. Please feel free to send me any feedback by emailing me or talking to me after class. Have a great spring semester!