18.S097 PSET 4

IAP 2022

Due 1/26/2022

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = kx + b$ for $0 < k < 1$ and $b \in \mathbb{R}$. Show that f is a contraction, find the fixed point of f, and directly show the fixed point is unique.

2. (Optional, 5pts) Consider the initial value problem

$$x'(t) = \sqrt{x} + x^3, \quad x(1) = 2.$$

Take $x_0(t) = 2$, and use Picard iteration to find what x_1 and x_2 are. Your solution should have x_1 defined without an integral, but you can leave x_2 as an integral (whose final form does not depend on x_1!).

Remark 1. *It should be clear from this exercise that Picard iteration results in worse and worse integrals, even though this method is extremely useful as we have seen.*

3. (Optional, 5pts) Let $\Delta = \partial_x^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, and let $\Omega = (a, b) \subset \mathbb{R}$. Then, solve the differential equation below:

$$\begin{cases} \Delta u(x) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \text{ i.e. } x \in \{a, b\} \end{cases}$$

where $g \in C^0([a, b])$ (arbitrary). In other words, find a function $u \in C^2([a, b])$ such that $\Delta u(x) = 0$ and $u(a) = g(a)$, $u(b) = g(b)$. Your final answer should only depend on a, b, and $g(x)$.

Remark 2. *Here, $\partial\Omega$ can be understood to be the "boundary" of Ω. While we didn't study the definition of boundaries of sets (or in fact, closure and interiors of sets), you can find more information about this topic in Lebl 7.2.3.*

4. (Optional, 10pts) Prove that a closed subset of a complete metric space (X, d) is complete.

5. (Optional, 10pts) Prove Picard's theorem using the approach not used in class (Lecture 5 notes). If you do Problem 3 above, you do not need to show this again in this problem.