18.S097 PSET 1

IAP 2022

Due 1/09/2022

1. Consider the following map: $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ where

$$d(x,y) = \begin{cases} \|x-y\|_{\mathbb{R}^2} & x, y, 0 \text{ collinear} \\ \|x\|_{\mathbb{R}^2} + \|y\|_{\mathbb{R}^2} & \text{otherwise.} \end{cases}$$

Here, I use $\|\cdot\|_{\mathbb{R}^2}$ to denote the Euclidean norm/magnitude of a vector in $\mathbb{R}^2$. Show that this map is a metric on $\mathbb{R}^2$. This is called the British Railway metric. (Try to figure out why!)
Hint: Try drawing a picture.
2. Is $d : C^1([0,1]) \times C^1([0,1]) \to [0, \infty)$ defined by

$$d(f,g) = \sup_{x \in [0,1]} |f'(x) - g'(x)|$$

a metric on $C^1([0,1])$? If so, prove it. If not, show what properties of a metric d satisfies, and explain which properties of a metric d fails.
3. Show that $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ where

$$d(x,y) = \frac{|x-y|}{1+|x-y|}$$

is a metric on $\mathbb{R}$.
4. Define a *semi-metric* on X as a metric that satisfies symmetry, the triangle inequality, and $d(x, y) \geq 0$ for all $x, y \in X$, but doesn't necessarily satisfy $d(x, y) = 0 \iff x = y$. Specifically, $x = y \implies d(x, y) = 0$ but the opposite implication need not be true. Show that the sum of a metric and a semi-metric on X is a metric on X. In other words, if d is a metric on X, and d' is a semi-metric on X, then $d + d'$ is a metric on X.
5. Show that $I_t : C^0([a, b]) \to C^1([a, b])$ is a continuous map where

$$I_t(f) = \int_a^t f(x)\,\mathrm{d}x$$

for some $t \in [a, b]$.
Hint: This proof is semi-similar to an example done in class, though you will need to mess with ϵs and δs.
6. (Optional) In this problem, you will show that the ℓ^p-metric is in fact a metric.

(a) (Hölder's Inequality) Suppose that $n \in \mathbb{N}$, and let $a_k, b_k \in \mathbb{R}$, $1 \leq k \leq n$. Prove that if $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q \right)^{1/q}.$$

Hint: Prove that if $A, B > 0$ and $t \in (0,1)$, then $A^t B^{1-t} \le tA + (1-t)B$ by showing the function

$$f(x) = tx + (1-t)B - x^t B^{1-t}, \quad x > 0,$$

has a minimum at $x = B$.

(b) (Minkowski's inequality) Suppose that $n \in \mathbb{N}$ and let $a_k, b_k \in \mathbb{R}$, $1 \le k \le n$. Prove that if $1 \le p < \infty$, then

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p \le \sum_{k=1}^{n} |a_k + b_k|^p \right)^{1/p} \le \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{1/p}.$$

Hint: by the triangle inequality,

$$\sum_{k=1}^{n} |a_k + b_k|^p \le \sum_{k=1}^{n} |a_k||a_k + b_k|^{p-1} + \sum_{k=1}^{n} |b_k||a_k + b_k|^{p-1}.$$

Now apply Hölder's inequality.

7. (Optional) We denote the space of infinitely differentiable functions on an interval $[a,b]$ as $C^\infty([a,b])$. Denote

$$\sup_{x \in [a,b]} |f^{(n)}(x) - g^{(n)}(x)| = d_n(f,g).$$

Problem 2 shows that d_n is a semi-metric on $C^\infty([a,b])$ for all $n \in \mathbb{N}$, and d_0 is a metric as we showed in class. Show that

$$d(f,g) := \sum_{n=0}^{\infty} 2^{-n} \frac{d_n(f,g)}{1 + d_n(f,g)}$$

is a metric on $C^\infty([a,b])$.

Remark 1. *This concept is related to what is called a Fréchet space, named after Maurice Fréchet who first wrote about metric spaces!*

18.S097 PSET 2

IAP 2022

Due 1/16/2022

Review / helpful information:

- The uniform distance on $C^0([a,b])$ is defined as

$$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|.$$

- Given a vector space V over the real numbers (i.e. a space where addition of vectors and multiplication by real numbers is well-defined), we define a norm to be a function $\|\cdot\| : V \to [0,\infty)$ satisfying the following properties:
  - Positive Definite: $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$.
  - Homogeneity: $\|\lambda v\| = |\lambda|\|v\|$ for all $v \in V$ and $\lambda \in \mathbb{R}$.
  - Triangle Inequality: $\|x + y\| \leq \|x\| + \|y\|$.
- We denote "K is a subset of a metric space X" by $K \Subset X$.
- Let A be a set of real numbers and $a = \sup A < \infty$. Then, for all $n \in \mathbb{N}$ there exists an $x_n \in A$ such that

$$a - \frac{1}{n} < x_n \leq a.$$

Throughout this problem set, let (X, d) be a metric space.

1. Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in X. Show that $d(x_n, y_n)$ converges.
Hint: show that in a metric space, $|d(a,b) + d(a',b')| \leq d(a,a') + d(b,b')$.

Remark 1. *You may **not** assume x_n and y_n converges. This is only true in a Cauchy complete space.*

2. In class, we have defined a set $A \subset X$ to be closed if its complement is an open set in X. There is another useful definition of a closed set however. Show that $A \subset X$ is closed if and only if every convergent sequence in A converges in A. In other words, if $\{x_n\}$ is a convergent sequence in A such that $x_n \to x$, then $x \in A$.

3. Here, we will show that $C^0([0,1])$ is Cauchy complete with respect to the uniform distance. Suppose that $f_n \in C^0([0,1])$ is a Cauchy sequence.

(a) Fix an arbitrary $x_0 \in [0,1]$. Show that $\lim_{n\to\infty} f_n(x_0)$ exists.
Hint: $\mathbb{R}$ is Cauchy complete.

(b) Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| \leq \epsilon$$

for all $x \in [0,1]$ and for all $n \geq N$.

(c) Show that $f(x)$ is continuous on $[0,1]$. I.e., $f \in C^0([0,1])$.
Hint: To show $f(x)$ is continuous at x_0, consider

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

(d) Using parts a-c, explain why $\lim_{n\to\infty} f_n = f$ as a sequence in $C^0([0,1])$.

4. Let $\|\cdot\|$ be a norm on a vector space V, and let $d(x,y) = \|x - y\|$ for all $x, y \in V$.
Show the following three properties:

(a) $d(\lambda x, \lambda y) = |\lambda| d(x,y)$ for all $\lambda \in \mathbb{R}$, and for all $x, y \in V$.

(b) Translation invariance: $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in V$.

(c) Prove d is a metric on V. This metric is called the *metric induced by the norm.*

5. The following are important properties of compact sets in $\mathbb{R}$.

(a) Let $K \Subset \mathbb{R}$. Show that there exists a maximum and a minimum value in K.

(b) (Optional) Generalize the Heine-Borel theorem to $\mathbb{R}^n$. (This proof is very similar to that in class.)

6. (Optional) Let U be an open set in the metric space (X,d). Show that U can be written as a union of arbitrarily many open balls.

7. (Optional) Show that a function $f : X \to Y$ is continuous if and only if given $U \subset Y$ where U is open in Y, $f^{-1}(U)$ is open in X.

8. (Optional) We call two norms $\|\cdot\|_1$, $\|\cdot\|_2$ equivalent if there exists constants $C_1 > 0$ and C_2 such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

One can similarly define equivalent metrics. On $\mathbb{R}^n$ we define the supremum norm and ℓ^p norms (for $1 \leq p < \infty$):

$$\|x\|_\infty = \max_i |x_i| \quad \text{and} \quad \|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}.$$

(You can check that these are in fact norms, but do not have to.) Show that the supremum norm, ℓ^1, and ℓ^2 norms are equivalent on $\mathbb{R}^n$ by showing

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty.$$

Briefly explain why this shows the norms are pairwise equivalent.

18.S097 PSET 3

IAP 2022

Due 1/23/2022

Review / helpful information:

- Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$. Then, we define

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

- We define the ℓ^p norm on a sequence $a = \{a_n\}_n$ of real numbers as

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_n|^p \right)^{1/p}$$

where $1 \leq p < \infty$. We define ℓ^p space as the space of (infinite) sequences $x = \{x_n\}_n$ in $\mathbb{R}$ such that $\|x\|_p < \infty$.

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1. Let A be a closed subset of $\mathbb{R}^n$ and let B be a compact subset of $\mathbb{R}^n$. Show that $A + B$ is closed.
Hint: Let $\{a_n\}_n$ be a sequence in $A + B$ such that $a_n \to z \in \mathbb{R}^n$. Show that $z \in A + B$.

Remark 1. *In fact, you can show that the sum of a closed set and a compact subset in a "topological vector space" is a closed set, but that goes beyond the scope of this class.*

2. Let (X, d) be a metric space and $S \subset X$. Then, a point $x \in S$ is an **isolated point** if there exists an $\epsilon > 0$ such that $B_\epsilon(x)$ contains no other points of S. Show that a point $x \in S$ is an isolated point if and only if give a sequence $\{a_n\}$ in S converges to x, it must be the case that there exists an N such that for all $n \geq N$, $a_n = x$.
3. Let X be a metric space. Show that a finite union of compact subsets in X is compact.
4. Let $1 \leq p < \infty$. Consider the set

$$S := \{a = \{a_n\} \in \ell^p \mid \|a\|_p \leq 1\}.$$

Explain why S is closed and bounded in ℓ^p (under the metric induced by the norm in PSET 2), and prove that S is not a compact subset of ℓ^p.
Hint: Let $e_n = \{\delta_{k,n}\}_k \in S$ where

$$\delta_{k,n} = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}.$$

Show that $\{e_n\}_n$ does not have a convergent subsequence in S. Make sure to explain why this shows S is not compact.
5. Consider the set

$$S := \{f \in C([0, 1]) \mid \|f\|_\infty = \sup_{x \in [0,1]} |f(x)| \leq 1\}.$$

Prove that S is not compact.
Hint: consider the sequence $f_n(x) = x^n$.

6. (Optional) Consider the space ℓ^2. Show that the set

$$A = \{a = \{a_k\}_k \in \ell^2 \mid |a_k| < k^{-3}\}$$

is a compact subset of ℓ^2.

7. (Optional) Consider the set of functions of the form

$$\sum_n a_n e^{inx}$$

with $|a_n| < (1 + |n|)^{-2}$.

(a) Show that every function in this set lies in $C^0([0, 2\pi])$.

(b) Show that this set is compact in $C^0([0, 2\pi])$.

18.S097 PSET 4

IAP 2022

Due 1/26/2022

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = kx + b$ for $0 < k < 1$ and $b \in \mathbb{R}$. Show that f is a contraction, find the fixed point of f, and directly show the fixed point is unique.

2. (Optional, 5pts) Consider the initial value problem

$$x'(t) = \sqrt{x} + x^3, \quad x(1) = 2.$$

Take $x_0(t) = 2$, and use Picard iteration to find what x_1 and x_2 are. Your solution should have x_1 defined without an integral, but you can leave x_2 as an integral (whose final form does not depend on x_1!).

Remark 1. *It should be clear from this exercise that Picard iteration results in worse and worse integrals, even though this method is extremely useful as we have seen.*

3. (Optional, 5pts) Let $\Delta = \partial_x^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, and let $\Omega = (a, b) \subset \mathbb{R}$. Then, solve the differential equation below:

$$\begin{cases} \Delta u(x) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \text{ i.e. } x \in \{a, b\} \end{cases}$$

where $g \in C^0([a, b])$ (arbitrary). In other words, find a function $u \in C^2([a, b])$ such that $\Delta u(x) = 0$ and $u(a) = g(a)$, $u(b) = g(b)$. Your final answer should only depend on a, b, and $g(x)$.

Remark 2. *Here, $\partial\Omega$ can be understood to be the "boundary" of Ω. While we didn't study the definition of boundaries of sets (or in fact, closure and interiors of sets), you can find more information about this topic in Lebl 7.2.3.*

4. (Optional, 10pts) Prove that a closed subset of a complete metric space (X, d) is complete.

5. (Optional, 10pts) Prove Picard's theorem using the approach not used in class (Lecture 5 notes). If you do Problem 3 above, you do not need to show this again in this problem.