# 18.S190 PSET 1

IAP 2023

Due 1/13/2022

**1.** Consider the following map: $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ where

$$d(x,y) = \begin{cases} \|x-y\|_{\mathbb{R}^2} & x, y, 0 \text{ collinear} \\ \|x\|_{\mathbb{R}^2} + \|y\|_{\mathbb{R}^2} & \text{otherwise.} \end{cases}$$

Here, I use $\|\cdot\|_{\mathbb{R}^2}$ to denote the Euclidean norm/magnitude of a vector in $\mathbb{R}^2$. Show that this map is a metric on $\mathbb{R}^2$. This is called the British Railway metric. (Try to figure out why!)

Hint: Try drawing a picture, and use the fact that $\|\cdot\|_{\mathbb{R}^2}$ is a metric.

**2.** Is $d : C^1([0,1]) \times C^1([0,1]) \to [0, \infty)$ defined by

$$d(f,g) = \sup_{x \in [0,1]} |f'(x) - g'(x)|$$

a metric on $C^1([0,1])$? If so, prove it. If not, show what properties of a metric $d$ satisfies, and explain which properties of a metric $d$ fails.

**3.** Show that $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ where

$$d(x,y) = \frac{|x-y|}{1 + |x-y|}$$

is a metric on $\mathbb{R}$.

**4.** Define a *semi-metric* on $X$ as a metric that satisfies symmetry, the triangle inequality, and $d(x,y) \geq 0$ for all $x, y \in X$, but doesn't necessarily satisfy $d(x,y) = 0 \iff x = y$. Specifically, $x = y \implies d(x,y) = 0$ but the opposite implication need not be true. Show that the sum of a metric and a semi-metric on $X$ is a metric on $X$. In other words, if $d$ is a metric on $X$, and $d'$ is a semi-metric on $X$, then $d + d'$ is a metric on $X$.

**5.** Show that $I_t : C^0([a,b]) \to C^1([a,b])$ is a continuous map where

$$I_t(f) = \int_a^t f(x)\,\mathrm{d}x$$

for some $t \in [a,b]$.

Hint: This proof is semi-similar to an example done in class, though you will need to mess with $\epsilon$s and $\delta$s.

**6.** (Optional) In this problem, you will show that the $\ell^p$-metric is in fact a metric.

**(a)** (Hölder's Inequality) Suppose that $n \in \mathbb{N}$, and let $a_k, b_k \in \mathbb{R}$, $1 \leq k \leq n$. Prove that if $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^n |a_k b_k| \leq \left( \sum_{k=1}^n |a_k|^p \right)^{1/p} \left( \sum_{k=1}^n |b_k|^q \right)^{1/q}.$$

Hint: Prove that if $A, B > 0$ and $t \in (0,1)$, then $A^t B^{1-t} \le tA + (1-t)B$ by showing the function

$$f(x) = tx + (1-t)B - x^t B^{1-t}, \quad x > 0,$$

has a minimum at $x = B$.

**(b)** (Minkowski's inequality) Suppose that $n \in \mathbb{N}$ and let $a_k, b_k \in \mathbb{R}$, $1 \le k \le n$. Prove that if $1 \le p < \infty$, then

$$\left( \sum_{k=1}^{n} |a_k + b_k|^p \le \sum_{k=1}^{n} |a_k + b_k|^p \right)^{1/p} \le \left( \sum_{k=1}^{n} |a_k|^p \right)^{1/p} + \left( \sum_{k=1}^{n} |b_k|^p \right)^{1/p}.$$

Hint: by the triangle inequality,

$$\sum_{k=1}^{n} |a_k + b_k|^p \le \sum_{k=1}^{n} |a_k||a_k + b_k|^{p-1} + \sum_{k=1}^{n} |b_k||a_k + b_k|^{p-1}.$$

Now apply Hölder's inequality.

**7.** (Optional) We denote the space of infinitely differentiable functions on an interval $[a, b]$ as $C^\infty([a, b])$. Denote

$$\sup_{x \in [a,b]} |f^{(n)}(x) - g^{(n)}(x)| = d_n(f, g).$$

Problem 2 shows that $d_n$ is a semi-metric on $C^\infty([a, b])$ for all $n \in \mathbb{N}$, and $d_0$ is a metric as we showed in class. Show that

$$d(f, g) := \sum_{n=0}^{\infty} 2^{-n} \frac{d_n(f, g)}{1 + d_n(f, g)}$$

is a metric on $C^\infty([a, b])$.

**Remark 1.** *This concept is related to what is called a Fréchet space, named after Maurice Fréchet who first wrote about metric spaces!*