18.S190 PSET 2

IAP 2023

Due 1/18/2022

Review / helpful information:

- The uniform distance on $C^0([a,b])$ is defined as

$$d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|.$$

- Given a vector space V over the real numbers (i.e. a space where addition of vectors and multiplication by real numbers is well-defined), we define a norm to be a function $\|\cdot\| : V \to [0,\infty)$ satisfying the following properties:
  - Positive Definite: $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$.
  - Homogeneity: $\|\lambda v\| = |\lambda|\|v\|$ for all $v \in V$ and $\lambda \in \mathbb{R}$.
  - Triangle Inequality: $\|x + y\| \leq \|x\| + \|y\|$.
- We denote "K is a compact subset of a metric space X" by $K \Subset X$.
- Let A be a set of real numbers and $a = \sup A < \infty$. Then, for all $n \in \mathbb{N}$ there exists an $x_n \in A$ such that

$$a - \frac{1}{n} < x_n \leq a.$$

Throughout this problem set, let (X,d) be a metric space.

1.

1. Let $x_n \to x$ and $y_n \to y$ be two convergent sequences in (X,d). Then, $d(x_n, y_n) \to d(x,y)$.
2. Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in X. Show that $d(x_n, y_n)$ converges.
   Hint: show that in a metric space, $|d(a,b) + d(a',b')| \leq d(a,a') + d(b,b')$.

Remark 1. *You may* ***not*** *assume x_n and y_n converges in the second part of this problem. This is only true in a Cauchy complete space.*

2. In class, we have defined a set $A \subset X$ to be closed if its complement is an open set in X. There is another useful definition of a closed set however. Show that $A \subset X$ is closed if and only if every convergent sequence in A converges in A. In other words, if $\{x_n\}$ is a convergent sequence in A such that $x_n \to x$, then $x \in A$.

3. Here, we will show that $C^0([0,1])$ is Cauchy complete with respect to the uniform distance. Suppose that $f_n \in C^0([0,1])$ is a Cauchy sequence.

(a) Fix an arbitrary $x_0 \in [0,1]$. Show that $\lim_{n\to\infty} f_n(x_0)$ exists.
Hint: $\mathbb{R}$ is Cauchy complete.

(b) Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| \leq \epsilon$$

for all $x \in [0, 1]$ and for all $n \geq N$.

(c) Show that $f(x)$ is continuous on $[0, 1]$. I.e., $f \in C^0([0, 1])$.
Hint: To show $f(x)$ is continuous at x_0, consider

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

(d) Using parts a-c, explain why $\lim_{n\to\infty} f_n = f$ as a sequence in $C^0([0, 1])$.

4. Let $\|\cdot\|$ be a norm on a vector space V, and let $d(x, y) = \|x - y\|$ for all $x, y \in V$.
Show the following three properties:

(a) $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $\lambda \in \mathbb{R}$, and for all $x, y \in V$.

(b) Translation invariance: $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in V$.

(c) Prove d is a metric on V. This metric is called the *metric induced by the norm.*

5. Let U be an open set in the metric space (X, d). Show that U can be written as a union of arbitrarily many open balls.

6. (Optional) The following are important properties of compact sets in $\mathbb{R}$.

(a) Let $K \Subset \mathbb{R}$. Show that there exists a maximum and a minimum value in K.

(b) Generalize the Heine-Borel theorem on $\mathbb{R}$ to $\mathbb{R}^n$. (This proof is very similar to that in class.)

7. (Optional) Show that a function $f : X \to Y$ is continuous if and only if given $U \subset Y$ where U is open in Y, $f^{-1}(U)$ is open in X.

8. (Optional) We call two norms $\|\cdot\|_1$, $\|\cdot\|_2$ equivalent if there exists constants $C_1 > 0$ and C_2 such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1.$$

One can similarly define equivalent metrics. On $\mathbb{R}^n$ we define the supremum norm and ℓ^p norms (for $1 \leq p < \infty$):

$$\|x\|_\infty = \max_i |x_i| \quad \text{and} \quad \|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}.$$

(You can check that these are in fact norms, but do not have to.) Show that the supremum norm, ℓ^1, and ℓ^2 norms are equivalent on $\mathbb{R}^n$ by showing

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty.$$

Briefly explain why this shows the norms are pairwise equivalent.

9. (Optional) Let $f : X \to Y$ be a map. Then, given $A \subset B$, show that $f(A) \subset f(B)$.

Remark 2. *There are a number of other properties of maps you can show, as outlined on this StackExchange post.*