18.S190: Introduction to Metric Spaces

Lecturer: Paige Dote

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General Theory

We now go into some of the general theory regarding metric spaces. Metric spaces are not only intuitively related to our understanding of $\mathbb{R}^n$, but they also behave similarly. For our purposes, we want to study metric spaces as they act nicely, and we now show some ways in which they are "nice".

Remark 1. *Not every space (i.e. topological space) is as nice as a metric space! The fancy way to say this is "Not every topological space is metrizable." 18.901 explores spaces like this, but we will not do so in this class.*

Let's start with convergent sequences, just like we did when we first started studying the real numbers.

Proposition 2

Let (X, d) be a metric space and let x_n be a convergent sequence in X such that $x_n \to x$. This limit is unique.

Proof: Suppose there exists a y such that $x_n \to y$. We want to show that if this is the case, then $x = y$.

Question 3. *What property about metric spaces tells us when points are equal?*

On the real line, $x = y \iff |x - y| = 0$, which is how we proved this property in 18.100x. Here, on metric spaces, we similarly have $x = y \iff d(x, y) = 0$. Hence, we use that to our advantage. Notice that

$$0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y).$$

Given that $x_n \to x$ and $x_n \to y$, we can make the right hand side arbitrarily small. More formally, let $\epsilon > 0$. Then, there exists an N such that for all $n \geq N$,

$$0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y) < \epsilon.$$

This is true for all $\epsilon > 0$, and thus $d(x, y) = 0 \implies x = y$. □

Proposition 4

Let $x_n \to x$. Then, $\forall y \in X$, $d(x_n, y) \to d(x, y)$.

In other words, when you have a convergent sequence in a metric space, the distance also behaves how one would hope. A similar way to think about this: fix $y \in X$. Then $a_n = d(x_n, y)$ is a convergent sequence in the real numbers.

Proof: Let $y \in X$. Firstly, note that given $x_n \to x$, $d(x_n, x) \to 0$. Hence, let $\epsilon > 0$. Then, there exists an N such that for all $n \geq N$,

$$d(x_n, y) \leq d(x, x_n) + d(x, y) < \epsilon + d(x, y).$$

We now want a similar lower bound, which we obtain by the triangle inequality again:

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \implies d(x, y) - d(x_n, x) \leq d(x_n, y).$$

For $n \geq N$, we have $d(x, y) - \epsilon < d(x_n, y)$. Therefore, given $\epsilon > 0$, there exists an N such that for all $n \geq N$,

$$d(x, y) - \epsilon < d(x_n, y) < d(x, y) + \epsilon \implies |d(x_n, y) - d(x, y)| < \epsilon.$$

Therefore, $d(x_n, y) \to d(x, y)$. □

Proposition 5

We can take this concept one step further, studying two convergent sequences at once. Let $x_n \to x$ and $y_n \to y$. Then, $d(x_n, y_n) \to d(x, y)$. Similarly, given $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, you can show that $d(x_n, y_n)$ converges (but you cannot assume the sequences have limit points!).

This problem will be on your PSET!

Definition 6 (Bounded)

A sequence $\{x_n\}$ in (X, d) is bounded if and only if there exists a $p \in X$ and a $B \in \mathbb{R}$ such that

$$d(x_n, p) \leq B \quad \forall n \in \mathbb{N}.$$

Similarly, a subset $A \subseteq X$ is bounded if and only if there exists a $p \in X$ and a $B \in \mathbb{R}$ such that

$$d(x, p) \leq B \quad \forall x \in A.$$

Proposition 7

Every convergent sequence in a metric space is bounded.

Proof: Let $x_n \to x$ and let $\epsilon = 1 > 0$. Then, there exists an N such that for all $n \geq N$, $d(x_n, x) < 1$. Now this is almost exactly our definition of bounded with $p = x$ and $B = 1$, but the issue is that so far this isn't true for all n, only for all $n \geq N$ (which is still infinitely many!). We thus use a common and useful technique: let

$$B = \max\{d(x_n, x), 1 \mid 1 \leq n < N\}.$$

Is B finite? Yes; B is the maximum of finitely many finite elements and is thus finite. Furthermore, we now have that for all $n \geq N$, $d(x_n, x) < 1 \leq B$, and for all $n < N$, $d(x_n, x) \leq B$. Hence, $\{x_n\}$ is bounded. □

We will prove two more theorems about convergent sequences, and then we will shift our focus to open and closed sets.

Proposition 8

Every convergent sequence is a Cauchy sequence.

Proof: Let $x_n \to x$, and let $\epsilon > 0$. Then, there exists an N such that for all $n \geq N$, $d(x_n, x) < \frac{\epsilon}{2}$. Hence, for all $n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

We showed this before for the real numbers! In fact, we showed that Cauchy sequences are convergent for the real line. However, this isn't always true.

Definition 9 (Cauchy complete)

A metric space in which every Cauchy sequence is convergent is called **Cauchy complete**.

Remark 10. *You will show on PSET 2 that $C^0([0, 1])$ is Cauchy complete.*

Proposition 11

Every subsequence of a convergent sequence is convergent.

Proof: This proof will help give an example of why Cauchy sequences are useful. Let $x_n \to x$, and consider the subsequence $\{x_{n_k}\}$. We want to show that $\{x_{n_k}\}$ is convergent, and to do so we will show that $x_{n_k} \to x$. Firstly notice that

$$d(x_{n_k}, x) \leq d(x_{n_k}, x_n) + d(x_n, x).$$

We know that $x_n \to x$, and thus for $\epsilon > 0$ there exists an N_1 such that for all $n \geq N_1$, $d(x_n, x) < \frac{\epsilon}{2}$. In other words, we can make $d(x_n, x)$ arbitrarily small; but what can we do about $d(x_{n_k}, x_n)$? Well we note that $\{x_n\}$ is a Cauchy sequence. Thus, there exists an N_2 such that for all $n, n_k \geq N$, $d(x_{n_k}, x_n) < \frac{\epsilon}{2}$. Hence, for all $n \geq \max\{N_1, N_2\}$,

$$d(x_{n_k}, x) \leq d(x_{n_k}, x_n) + d(x_n, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

You may be wondering "Why don't we have as many theorems for convergent sequences like we used to?" Well, notice that metric spaces are *much* more general than $\mathbb{R}$. For instance, we can't show sums of convergent sequences converge, because we don't *always* have a notion of addition. Similarly, we don't have a direct analog of the squeeze theorem, as we don't *always* have a notion of "ordering" (i.e. what it means for one element to be bigger than another). Thus we have to study new tools, like open sets.

Recall (Open Set)

A set in $A \subset X$ is **open** if and only if $\forall x \in A$, there exists an $\epsilon > 0$ such that

$$B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\} \subset A.$$

We say that $B(x, \epsilon)$ is a ball of radius epsilon centered at x.

While it may seem out of left field, open sets prove very useful in understanding concepts of convergence and continuity. We will show this connection today, but let's start with some useful and powerful propositions.

Theorem 12 (Topological Properties of Open Sets)
Let X be a metric space, and let $\{A_i\}_{i\in\Lambda}$ be open sets in X. Then,

1. $\emptyset$ and X are open sets in X.
2. $\bigcup_{i\in I} A_i$ is open in X. (I.e., the arbitrary union of open sets is open.)
3. $\bigcap_{i=1}^{n} A_i$ is open in X. (I.e., the finite intersection of open sets is open.)

Proof: All we can use so far is the definition given to us.

1. Consider $\emptyset$. It is vacuously true that $\forall x \in \emptyset$, there exists an $\epsilon > 0$ such that $B(x,\epsilon) \subset \emptyset$, as there are no elements in the empty set. Now consider X. Recall the definition of $B(x,\epsilon)$:
$$B(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon\}.$$
By definition, $\forall x \in X$ and in fact for all $\epsilon > 0$ (though we only need one), $B(x,\epsilon) \subset X$. Thus, X is an open set.
2. Consider some $x \in \bigcup_{i\in I} A_i$. Then, by assumption, there exists a $\lambda \in \Lambda$ such that $x \in A_\lambda$. Furthermore, A_λ is an open set, and thus there exists an $\epsilon > 0$ such that $B(x,\epsilon) \subset A_\lambda$. Notice though, that $A_\lambda \subset \bigcup_{i\in I} A_i$, and thus $B(x,\epsilon) \subset \bigcup_{i\in I} A_i$. Hence, the arbitrary union of open sets is open.
3. The proof for the intersection will act similarly, but let's see why we can only consider a finite intersection. Let $x \in \bigcap_{i=1}^{n} A_i$. Then, for each $1 \le i \le n$, $x \in A_i$. Therefore, there exists an ϵ_i such that $B(x,\epsilon_i) \subset A_i$. The issue though, is A_i is not automatically a subset of the intersection. However, we can take $\epsilon = \min\{\epsilon_i\} > 0$. Thus, $B(x,\epsilon) \subseteq B(x,\epsilon_i) \subset A_i$ for every i. Hence, $B(x,\epsilon) \subset \bigcap_{i=1}^{n} A_i$. □

Remark 13. *These three properties can help us understand why open sets are so useful (at least conceptually). As we will see, open sets are directly related to convergence and continuity, and are yet so much more general. In point-set topology (18.901), you actually* **start** *with defining open sets abstractly using these three properties, and go from there. It's very interesting, and leads to very interesting examples! We will discuss this more in Lecture 6.*

Definition 14 (Closed Set)
Let $A \subset X$. We say that A is **closed** if $X \setminus A := A^c$ is open in X. We call A^c the **complement** of A.

I want to note that a set being closed *does not* imply it is not open. Consider for instance, the emptyset $\emptyset$. The complement of the emptyset is X, which is open, and this $\emptyset$ is closed. However, by Theorem 12, $\emptyset$ is open!

This concept is deeply tied to the notion of connectedness.

Definition 15 (Disconnected and Connected)
A metric space X is **disconnected** if there exists two disjoint, non-empty, open sets U_1 and U_2 such that $X = U_1 \cup U_2$.

The space is **connected** if it is not disconnected.

Proposition 16
A metric space X is connected if and only if the only open and closed sets are the emptyset and X itself.

Remark 17. *This proposition will be outlined on your third PSET as an optional problem.*

We now make develop some theory for closed sets.

Theorem 18

Let X be a metric space, and let $\{A_i\}_{i\in\Lambda}$ be closed sets in X. Then,

1. $\emptyset$ and X are closed sets in X.
2. $\bigcap_{i\in I} A_i$ is closed in X. (I.e., the arbitrary intersection of closed sets is closed.)
3. $\bigcup_{i=1}^{n} A_i$ is closed in X. (I.e., the finite union of closed sets is closed.)

To prove this, we use DeMorgan's Law from set theory (which is proven in Lebl's Theorem 0.3.5).

Proposition 19 (DeMorgan's Law)

Consider the sets $\{U_i\}_{i\in\Lambda}$. Then,

$$\left(\bigcup_{i\in\Lambda} U_i\right)^c = \bigcap_{i\in\Lambda} U_i^c \quad \text{and} \quad \left(\bigcap_{i\in\Lambda} U_i\right)^c = \bigcup_{i\in\Lambda} U_i^c.$$

To put this into words, the complement of a union is the intersection of the complements, and the complement of an intersection is the union of the complements.

Proof:

1. Well firstly, notice $\emptyset^c = X$ and $X^c = \emptyset$ in X. Hence, given $\emptyset$ and X are open sets, $\emptyset$ and X are closed sets.
2. Given A_i are closed, A_i^c is open. Hence, using DeMorgan's Law,

$$\left(\bigcap_{i\in\Lambda} A_i\right)^c = \bigcup_{i\in\Lambda} A_i^c,$$

   and the arbitrary union of open sets is open. Hence, $\bigcap_{i\in\Lambda} A_i$ is closed.
3. We use DeMorgan's Law in exactly the same way to prove that the finite union of closed sets is closed.

□

Lets look at some useful examples:

Example 20

Let (X, d) be a metric space, and let $x \in X$. Then, for any $\epsilon > 0$, $B(x, \epsilon)$ is open in X. In fact, this ball is sometimes referred to as an *open ball*.

Proof: Let $y \in B(x, \epsilon)$. If $x = y$ then this is automatically true, just take $\epsilon' = \frac{\epsilon}{2}$. Suppose that $y \neq 0$, and let $r = \epsilon - d(x, y) > 0$. We want to show that $B(y, r) \subset B(x, \epsilon)$. Let $z \in B(y, r)$. Then

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r = \epsilon.$$

Therefore, $B(y, r) \subset B(x, \epsilon)$, and thus $B(x, \epsilon)$ is an open set. □

Theorem 21

An open subset U in a metric space (X, d) can be written as a union of open balls in X. This is an optional problem on PSET 2.

Remark 22. *Hence, sometimes we will reduce the problem to simply prove propositions.*

Example 23

Let (X, d) be a metric space and $x \in X$. Then, $\{x\}$ is a closed set in X.

Proof: We want to show that for all $y \in X \setminus \{x\}$, there is an open ball around y such that x is not in the ball. Fix $y \in X \setminus \{x\}$; then, $y \neq x$ and thus $d(x, y) > 0$. Let $r = \frac{d(x,y)}{2}$. Hence, consider $B(y, r)$. We know that $x \notin B(y, r)$ as if this were the case, then $d(x, y) < r < d(x, y)$ which is a contradiction. Hence, $B(y, r) \subset X \setminus \{x\}$. Therefore, $X \setminus \{x\}$ is an open set, and thus $\{x\}$ is a closed set in X. □

Remark 24. *One can similarly prove that any finite subset of a metric space is closed.*

Let's now see again how open sets relate to convergence and continuity. To do so, we first observe a fact about convergent sequences in $\mathbb{R}$.

Proposition 25

Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then, $\{x_n\}$ is convergent (and converges to x) if and only if $\forall \epsilon > 0$, all but finitely many terms in $\{x_n\}$ are in $(x - \epsilon, x + \epsilon)$.

Proof: Given $x_n \to x$, given $\epsilon > 0$ there exists an N such that for all $n \geq N$, $|x_n - x| < \epsilon$. Therefore, for all $n \geq N$, $x_n \in (x - \epsilon, x + \epsilon)$. For the other direction, fix arbitrary $\epsilon > 0$ and consider $(x - \epsilon, x + \epsilon)$. Given that all but finitely many terms in $\{x_n\}$ are in $(x - \epsilon, x + \epsilon)$, there exists an M such that for all $n \geq M$, $x_n \in (x - \epsilon, x + \epsilon) = B_\epsilon(x)$. Therefore, x_n is convergent. □

The same can be generally said for metric spaces.

Definition 26 (Neighborhood)

Suppose that $x \in U$ and U is open in X. Then we can U a **neighborhood** of x.

Theorem 27

Let $\{x_n\}$ be a sequence in the metric space (X, d). Then, x_n is convergent and converges to x if and only if for every neighborhood of x, all but finitely many terms in $\{x_n\}$ are not in the neighborhood of x.

Proof: The proof is exactly the same as the proof of $X = \mathbb{R}$, only changing to metric notation. □

Remark 28. *Every closed set has the property that every convergent sequence converges in the set. This will be shown on PSET 2, and gives yet another connection between open/closed sets and convergence.*

We now shift our focus to continuous functions.

Recall (Continuous functions)

Let (X, d_X) and (Y, d_Y) be metric spaces. Then, a function $f : X \supset A \to Y$ is **continuous** if and only if given $x \in A$, $\forall \epsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(x, y) \leq \delta \implies d_Y(f(x), f(y)) \leq \epsilon.$$

We will first show how continuity is related to convergence, and then how continuity is related to open sets.

Theorem 29

Let (X, d_X) and (Y, d_Y) be metric spaces. Then, $f : X \to Y$ is continuous at $c \in X$ if and only if for every sequence $\{x_n\}$ in X with $x_n \to c$, we have that $f(x_n) \to f(c)$.

Proof: Suppose that f is continuous at c. Let $\{x_n\}$ be a sequence in X converging to c. Given $\epsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, c) < \delta \implies d_Y(f(x), f(c)) < \epsilon$. Given $x_n \to c$, there exists an N such that for all $n \geq N$, $d_X(x_n, c) < \delta$. Therefore, $d_Y(f(x_n), f(c)) < \epsilon$. Thus, $f(x_n) \to f(c)$.

Suppose that f is not continuous at c. Let $\epsilon > 0$. Then, for all $n \in \mathbb{N}$, there exists an x_n such that $d(x_n, c) < \frac{1}{n}$ but $d_Y(f(x_n), f(c)) \geq \epsilon$. Then, $x_n \to c\}$ but $f(x_n)$ does not converge to $f(c)$. □

Lemma 30

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous at $c \in X$ if and only if for every open neighborhood U of $f(c)$ in Y, the set $f^{-1}(U)$ contains an open neighborhood of c in X.

Proof: Suppose that f is continuous at c. Let U be an open neighborhood of $f(c)$ in Y. Then, $B_Y(f(c), \epsilon) \subset U$ for some $\epsilon > 0$. By the continuity of f, there exists a $\delta > 0$ such that $d_X(x, c) \implies d_Y(f(x), f(c)) < \epsilon$. Hence,

$$B_X(c, \delta) \subset f^{-1}(B_Y(f(c), \epsilon)) \subset f^{-1}(U)$$

and $B_X(c, \delta)$ is an open neighborhood of c.

For the other direction, let $\epsilon > 0$. If $f^{-1}(B_Y(f(c), \epsilon))$ contains an open neighborhood V of c, then it contains a ball $B_X(c, \delta)$ such that

$$B_X(c, \delta) \subset W \subset f^{-1}(B_Y(f(c), \epsilon)).$$

Therefore, if $d_X(x, c) < \delta \implies d_Y(f(x), f(c)) < \epsilon$. Hence, f is continuous at c. □

Remark 31. *In fact, one can deduce that a function $f : X \to Y$ is continuous if and only if given $U \subset Y$ open, $f^{-1}(U)$ is open in X. This is an optional problem on PSET 2. This idea is once again integral to 18.901.*