Sobolev and Isoperimetric Inequalities on Riemannian Manifolds

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Abstract

We provide exposition and proofs of the Sobolev and isoperimetric inequalities on both Euclidean space and submanifolds of $\mathbb{R}^n$, and furthermore prove the logical equivalence of these two inequalities. We then discuss further generalizations to minimal submanifolds and submanifolds of Riemannian manifolds with bounded sectional curvature, following the work of Brendle, Hoffman, and Spruck.

Contents

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1 Introduction

In the study of Riemannian manifolds, many mathematical tools/definitions, theorems, and proofs are motivated by analogous statements in Euclidean space, which are themselves often motivated by rather intuitive concepts. For instance, suppose the derivative of a function is an "approximately" zero. Does this imply that the function is "approximately constant"? Or for instance, suppose the surface area of a subset of $\mathbb{R}^n$ is small. Is the volume of the entire subset small?

These two questions, respectively, motivate what are known as a Sobolev inequality and the isoperimetric inequality. While the motivating questions are conceptually (at least at first) rather separate, the inequalities are in fact logically equivalent as we will prove.

In this expository paper, we will first prove the Sobolev inequality on $\mathbb{R}^n$ in Section 2.1 (following [3]), which will motivate the analogous Sobolev inequality on a Riemannian manifold later on in subection 3.2 (following the proof of [1]). Then, we will spend the majority of the rest of the paper showing the Sobolev and Isoperimetric inequalities are equivalent. In Section 3, we will show that the isoperimetric inequality for Riemannian manifolds (and thus Euclidean space) is implied by the Sobolev inequality. Then, we will see how the isoperimetric inequality implies the Sobolev inequality, thus proving their equivalence. Using this equivalence, we motivate Brendle's proof of the Sobolev inequality on a Riemannian manifold. Note that we do not rewrite his proof here.

In the case of Riemannian manifolds, we will be proving these inequalities for submanifolds embedded into $\mathbb{R}^n$. In Section 4, generalized versions of these theorems will be discussed for submanifolds of an arbitrary Riemannian manifold with some conditions on curvature via the work of [4].

1.1 Preliminaries

Notation 1 (L^p norm)

Given a function $f : M \supset \Omega \to \mathbb{R}$ for a Riemannian manifold M, we denote

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}.$$

We note a standard lemma without proof, which will be used throughout the paper:

Lemma 1 (Hölder's inequality)

Let f, g be measurable real functions on the set $\Omega \subset \mathbb{R}^n$. Then, given $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have that

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Notation 2

The expression $A(x,t) \lesssim_x B(x,t)$ is used to mean that there exists a constant $C(x) \geq 0$ (i.e. that depends only on x) such that

$$A(x,t) \leq C(x)B(x,t).$$

2 A Sobolev inequality on Euclidean Space

2.1 Euclidean space

From calculus, we know that a differentiable function f on $\mathbb{R}^n$ is constant if and only if all of its partial derivatives with respect to any of the n variables is zero. But what if the derivative of f is *approximately* zero? Is the function *approximately* constant? In one dimension, this is conceptually true. Suppose that $f_\epsilon \in C_0^\infty(\mathbb{R})$ and that

$\|f_\epsilon'\|_{L^1(\mathbb{R})} < \epsilon$. Then, for any $a, b \in \mathbb{R}$, we have that

$$|f_\epsilon(b) - f_\epsilon(a)| = \left| \int_a^b f_\epsilon'(x)\,\mathrm{d}x \right| \le \int_a^b |f_\epsilon'(x)|\,\mathrm{d}x \le \int_{-\infty}^{\infty} |f_\epsilon'(x)|\,\mathrm{d}x < \epsilon$$

by the Fundamental Theorem of Calculus. Hence, if the L^1 norm of f_ϵ' is less than epsilon, then the distances between any two points in the image of f_ϵ are also less than epsilon.

The analogue of this in higher dimensions is not true. To see this, let $\epsilon > 0$ and consider a non-negative bump function $f_\epsilon \in C_0^\infty(\mathbb{R}^n)$ (for $n > 1$) with compact support on $B_{2\epsilon}(0)$ such that $f_\epsilon\big|_{B_\epsilon(0)} \equiv 1$. Then, ∇f_ϵ is supported in the region $B_{2\epsilon}(0)$, and thus $\int |\nabla f| \le \frac{2}{\epsilon} \cdot \mathrm{Area}(B_\epsilon(0))$. Hence,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla f_\epsilon(x)|\,\mathrm{d}x &= \int_{0 \le |x| \le 2\epsilon} |\nabla f_\epsilon(x)|\,\mathrm{d}x \\
&= \int_{0 \le |x| \le 2\epsilon} |\nabla f_\epsilon(x)|\,\mathrm{d}x \\
&\lesssim \epsilon^n \cdot \frac{2}{\epsilon} \xrightarrow{\epsilon \to 0} 0.
\end{aligned}$$

However, it is not true that $|f_\epsilon(b) - f_\epsilon(a)| \le 1$ for all $a, b \in \mathbb{R}^n$. In other words, though the "derivative" tends to 0 in an integral sense, the difference $|f_\epsilon(b) - f_\epsilon(a)|$ does not.

That being said, there is a relationship we can find between the integral of a function on an n-dimensional space and the integral of its derivative.

Theorem 2 (Sobolev inequality on $\mathbb{R}^n$)

If $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function on $\mathbb{R}^n$ with compact support, then

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \|\nabla f\|_{L^1}(\mathbb{R}^n).$$

The Sobolev inequality on $\mathbb{R}^n$ is quite a bit nicer to approach since on $\mathbb{R}^n$ we have a nice global orthonormal frame we can take advantage of, though the analogous inequality on Riemannian manifolds will require more work.

To prove Theorem 2, we first prove the Loomis-Whitney inequality to break the integral over $\mathbb{R}^n$ into integrals over $\mathbb{R}$ to which we can apply the one dimensional model case.

Lemma 3 (Loomis-Whitney inequality)

Let $f_1, \ldots, f_n : \mathbb{R}^{n-1} \to \mathbb{R}$ and let $\pi_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection that drops the i-th coordinate (i.e. projection onto a hyper-plane). Then,

$$\left\| \prod_{i=1}^n f_i \circ \pi_i \right\|_{L^1(\mathbb{R}^n)} \le \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

The Loomis-Whitney inequality is a powerful tool that allows us to apply this nice induction by breaking up the integral into lower-dimensional terms.

Before jumping into the proof, let's motivate this statement through an example for the functions f_i. Suppose the f_i's are the indicator functions for $\pi_i(\mathbb{R}^n)$, i.e.

$$f_i(x) = \begin{cases} 1 & x \in \pi_i(\mathbb{R}^n) \\ 0 & \text{otherwise} \end{cases}.$$

Then what this inequality is telling us is that we can bound the volume of an n-dimensional object by the areas of it's $(n-1)$-dimensional projections (albeit with some exponents that make the dimensional analysis play out correctly).

So this theorem is simply giving a way to break the integral of a product of functions ("the volume" of the images of f) via a product of the integrals ("the areas" of it's $(n-1)$-dimensional projections).

Proof. For simplicity, assume the f_i are all non-negative so we don't carry around the absolute values. Integrating over the first coordinate, we have that

$$\int_{\mathbb{R}} \prod_{i=1}^{n} f_i \circ \pi_i \, \mathrm{d}x_i = f_1 \int_{\mathbb{R}} \prod_{i=2}^{n} f_i \circ \pi_i \, \mathrm{d}x_i$$

as $f_1 \circ \pi_1$ is constant as x_i varies. Applying Hölder's inequality to this product, we thus have

$$\int_{\mathbb{R}} \prod_{i=1}^{n} f_i \circ \pi_i \, \mathrm{d}x_i \leq f_1 \prod_{i=2}^{n} \left(\int_{\mathbb{R}} (f_i \circ \pi_i)^{n-1} \, \mathrm{d}x_1 \right)^{1/(n-1)}.$$

Now integrating over the second coordinate, we have

$$\int_{\mathbb{R}^2} \prod_{i=1}^{n} f_i \circ \pi_i \, \mathrm{d}x_1 \mathrm{d}x_2 \leq \left(\int_{\mathbb{R}} (f_2 \circ \pi_2)^{n-1} \, \mathrm{d}x_1 \right)^{1/(n-1)} \int_{\mathbb{R}} (f_1 \circ \pi_1) \prod_{i=3}^{n} \left(\int_{\mathbb{R}} (f_i \circ \pi_i)^{n-1} \, \mathrm{d}x_1 \right)^{1/(n-1)} \mathrm{d}x_2.$$

Applying Hölder's inequality gives

$$\begin{aligned}\int_{\mathbb{R}^2} \prod_{i=1}^{n} f_i \circ \pi_i \, \mathrm{d}x_1 \, \mathrm{d}x_2 \leq \left(\int_{\mathbb{R}} (f_2 \circ \pi_2)^{n-1} \mathrm{d}x_1 \right)^{1/(n-1)} & \left(\int_{\mathbb{R}} (f_1 \circ \pi_1)^{n-1} \mathrm{d}x_2 \right)^{1/(n-1)} \\ & \cdot \prod_{i=3}^{n} \left(\int (f_i \circ \pi_i)^{n-1} \, \mathrm{d}x_1 \mathrm{d}x_2 \right)^{1/(n-1)}.\end{aligned}$$

Reiterating this process $n-2$ more times, we obtain the desired result. □

Proof of Theorem 2. We prove this inequality inductively. In the case of two dimensions, let

$$\begin{aligned}(f_1 \circ \pi_1)(x_1, x_2) &= \int_{\mathbb{R}} |\partial_1 f(x_1, x_2)| \, \mathrm{d}x_1 \\ (f_2 \circ \pi_2)(x_1, x_2) &= \int_{\mathbb{R}} |\partial_2 f(x_1, x_2)| \, \mathrm{d}x_2.\end{aligned}$$

Thus, $f_i \leq |\nabla f|$, and hence by the Loomis-Whitney inequality,

$$\begin{aligned}\int_{\mathbb{R}^2} f^2(x_1, x_2) \, \mathrm{d}x_1 \mathrm{d}x_2 &\leq \int_{\mathbb{R}^2} (f_1 \circ \pi_1)(x_1, x_2)(f_2 \circ \pi_2)(x_1, x_2) \, \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \left(\int_{\mathbb{R}} f_1(x_2) \, \mathrm{d}x_2 \right) \left(\int f_2 \circ (x_1) \, \mathrm{d}x_1 \right) \leq \left(\int_{\mathbb{R}^2} |\nabla f| \right)^2.\end{aligned}$$

Now, in higher dimensions, define

$$(f_i \circ \pi_i)(x_1, \ldots, x_n) = \int |\partial_i f(x_1, \ldots, x_n)| \, \mathrm{d}x_i.$$

Then, $|f| \le f_n \circ \pi_n$ and $\int_{\mathbb{R}^{n-1}} \le \int_{\mathbb{R}^n} |\nabla f|$. Thus,

$$\begin{aligned}
\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} &\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} |f|(f_n \circ \pi_n)^{1/(n-1)} \, \mathrm{d}x_1, \dots, \mathrm{d}x_{n-1} \right) \mathrm{d}x_n \\
&\le \int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} |f|^{\frac{n-1}{n-2}} \right]^{\frac{n-2}{n-1}} \cdot \left[\int_{\mathbb{R}^{n-1}} |f_n \circ \pi_n| \right]^{1/(n-1)} \\
&\le \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\nabla f| \, \mathrm{d}x_1 \dots \mathrm{d}x_{n-1} \mathrm{d}x_n \left(\int_{\mathbb{R}^{n-1}} f_n \circ \pi_n \right)^{1/(n-1)} \\
&\le \left(\int_{\mathbb{R}} |\nabla f| \right)^{\frac{n}{n-1}}.
\end{aligned}$$

which concludes the proof. □

This is the classic proof of the Sobolev inequality on $\mathbb{R}^n$. However, this method will not work on general Riemannian manifolds, where we will most generally have

$$\|f\|_{L^{\frac{n}{n-1}}(\Sigma)} \lesssim \|\nabla f\|_{L^1(\Sigma)} + \|fH\|_{L^1(\Sigma)},$$

where H here is the mean curvature vector field (see subsection 3.2 for a discussion on this).

2.2 Generalized Sobolev Inequality

The Sobolev inequality is also true for other L^p spaces where $p \neq \frac{n}{n-1}$, with some mild adjustments for scaling. This generalization follows as a nice corollary of Theorem 7.

Corollary 4

Let f be a non-negative smooth function with compact support on Σ of dimension n, and let $p \in [1, n)$. Let $q = \frac{np}{n-p}$. Then,

$$\|f\|_{L^q(\Sigma)} \lesssim_n \frac{(n-1)p}{n-p} \|\nabla f\|_{L^p(\Sigma)} + \|fH\|_{L^p(\Sigma)}. \tag{1}$$

Proof. This follows by the Sobolev inequality and Hölder's inequality. Suppose $r > 1$. Then, applying the Sobolev inequality and the chain rule, we have

$$\|f^r\|_{L^{\frac{nr}{n-1}}(\Sigma)} \lesssim \int_{\Sigma} r f^{r-1} |\nabla f| + f^r |H|. \tag{2}$$

Then, applying Hölder's inequality with exponents $\frac{nr}{(n-1)(r-1)}$ and $\frac{nr}{n+r-1}$ gives the following bounds:

$$\begin{aligned}
\int_{\Sigma} f^{r-1} |\nabla f| &\le \left(\int_{\Sigma} f^{\frac{nr}{n-1}} \right)^{\frac{(n-1)(r-1)}{nr}} \left(\int_{\Sigma} |\nabla f|^{\frac{nr}{n+r-1}} \right)^{\frac{n+r-1}{nr}} \\
\int_{\Sigma} f^r |H| &\le \left(\int_{\Sigma} f^{\frac{nr}{n-1}} \right)^{\frac{(n-1)(r-1)}{nr}} \left(\int_{\Sigma} (f|H|)^{\frac{nr}{n+r-1}} \right)^{\frac{n+r-1}{nr}}.
\end{aligned}$$

Applying these two bounds on the right hand side of (2) and dividing through by

$$\left(f^{\frac{nr}{n-1}}\right)^{\frac{(n-1)(r-1)}{nr}} = \|f\|^{r-1}_{L^{\frac{nr}{n-1}}}$$

gives that

$$\|f\|_{L^{\frac{nr}{n-1}}(\Sigma)} \lesssim_{n,p} r \|\nabla f\|_{L^p(\Sigma)} + \|fH\|_{L^p(\Sigma)}.$$

Let $p = \frac{nr}{n+r-1}$ (and thus $r = \frac{(n-1)p}{n-p}$) to conclude the proof. $\square$

In fact, for any value of $p \in [1, n)$, there is a unique q satisfying (1). To see this, consider a smooth bump function η and let $\eta_\lambda(x) = \eta(x/\lambda)$. For simplicity, let $H = 0$ (i.e. just consider the flat case). Then,

$$\frac{\|\eta_\lambda\|_{L^q(\Sigma)}}{\|\nabla \eta_\lambda\|_{L^p(\Sigma)}} = \lambda^{\frac{n}{p} - \frac{n}{q} - 1} \frac{\|\eta\|_{L^q(\Sigma)}}{\|\nabla \eta\|_{L^p(\Sigma)}}.$$

If the Sobolev inequality is to be true (as we have proven above), then it must hold for this entire family of η_λs. Hence, in order for the right hand side to be bounded for all λ, we must have $\frac{n}{p} - \frac{n}{q} - 1 = 0$ i.e. $q = \frac{np}{n-p}$. The constant q here is called the Sobolev conjugate, and is denoted p^*.

3 The Isoperimetric inequality

Suppose one has 10 inches of string on a table, and they are tasked with creating a shape out of the string on the table that maximizes the area of the interior. This goal would be achieved if they made a circle of radius $\frac{10}{2\pi}$. This concept is precisely what motivates the isoperimetric inequality which states that

Theorem 5 (Isoperimetric Inequality)
Let $\Sigma^n \subseteq \mathbb{R}^{n+m}$ be a compact submanifold and let $\Omega \subset \Sigma$ with sufficiently smooth boundary $\partial\Omega$. Then,

$$(\text{Vol}(\Omega))^{\frac{n-1}{n}} \lesssim \text{Area}(\partial\Omega) + \|H\|_{L^1(\Omega)}$$

where the area is in reference to the volume form on the $(n-1)$-dimensional boundary of Ω and H is the mean curvature of Σ with respect to the ambient space $\mathbb{R}^{n+m}$.

Proof. The goal is to create a sequence of piecewise smooth functions that approximate the set Ω (so as to apply the Sobolev inequality) and whose derivatives approximate the area of the boundary. For $\epsilon > 0$, consider the function $f_\epsilon : \Sigma \to \mathbb{R}$ defined piecewise by

$$f_\epsilon(x) = \begin{cases} 1, & x \in \Omega, d(x, \partial\Omega) \geq \epsilon, \\ \frac{d(x,\partial\Omega)}{\epsilon} & x \in \Omega, d(x, \partial\Omega) \leq \epsilon \\ 0, & x \notin \Omega \end{cases}.$$

This is a non-negative function with compact support on Σ that is differentiable everywhere except a set of measure zero. Therefore, we can apply the Sobolev inequality to f_ϵ, obtaining

$$\begin{aligned} \|f_\epsilon\|_{L^{\frac{n}{n-1}}(\Sigma)} &= \left(\int_\Sigma f_\epsilon^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \\ &\lesssim \int_\Sigma (|\nabla f_\epsilon| + |f_\epsilon H|) \, \mathrm{d}\,\text{Vol}_\Sigma . \end{aligned}$$

Notice that $\nabla f_\epsilon(x)$ is zero almost everywhere on $\Sigma \setminus \{x \in \Omega \mid d(x, \partial\Omega) \leq \epsilon\}$, and f_ϵ is non-zero almost everywhere within the ϵ-neighborhood of the boundary. Hence, as $\epsilon \to 0$,

$$\|\nabla f_\epsilon\|_{L^1(\Sigma)} \to \text{Area}(\partial\Omega)$$

where the area is in reference to the volume form on the $(n-1)$-dimensional boundary of Ω.

Similarly, $\|f_\epsilon H\|_{L^1(\Omega)} \to \|H\|_{L^1(\Omega)}$ as $f_\epsilon(x)$ is 1 when $x \in \Omega$ and $d(x, \partial\Omega) \geq \epsilon$. This gives the desired result. $\square$

3.1 Equivalence of Sobolev and Isoperimetric

In fact, one can derive the Sobolev inequality from the isoperimetric inequality, so the two are equivalent. The outline of this proof follows from this blog run by a collection of students from The Chinese University of Hong Kong [5].

Proof. We have shown how the Sobolev inequality implies the Isoperimetric inequality. We now derive the Sobolev inequality from the isoperimetric inequality to prove their logical equivalence. So, suppose that we have Theorem 5.

To prove the Sobolev inequality, we first find an upperbound on $\|f\|_{L^{\frac{n}{n-1}}(\Sigma)}$. We can do so using annuli:

$$\begin{aligned}\|f\|_{L^{\frac{n}{n-1}}(\Sigma)}^{\frac{n-1}{n}} &:= \int_\Sigma |f|^{\frac{n}{n-1}} \,\mathrm{d}\,\mathrm{Vol}_\Sigma \\ &= \int_0^\infty \mathrm{Vol}(\{x \in \Sigma \mid f^{\frac{n}{n-1}}(x) \geq t\})\,\mathrm{d}t\end{aligned}$$

where here we are using the fact that f is non-negative and thus $|f| = f$. Then,

$$\begin{aligned}\|f\|_{L^{\frac{n}{n-1}}(\Sigma)}^{\frac{n-1}{n}} &= \int_0^\infty \mathrm{Vol}(\{x \in \Sigma \mid f(x) \geq t^{\frac{n-1}{n}}\})\,\mathrm{d}t \\ &= \frac{n}{n-1}\int_0^\infty \mathrm{Vol}(\{x \in \Sigma \mid f(x) \geq s\}) s^{\frac{1}{n-1}}\,\mathrm{d}s\end{aligned}$$

using the u-substitution of $s^{\frac{n}{n-1}} = t$. We now compare this to $\|\nabla f\|_{L^1(\Sigma)}$.

To do so, we now find an lowerbound on $\|\nabla f\|_{L^1(\Sigma)}$. Recall the coarea formula, which states:

Lemma 6 (Coarea Formula)
Given f is a Lipschitz function on Σ, we have that

$$\int_\Sigma |\nabla f| \,\mathrm{d}\,\mathrm{Vol}_\Sigma = \int_0^\infty \mathrm{Area}(\{x \in \Sigma \mid f(x) = s\})\,\mathrm{d}s.$$

Thus, we have that

$$\begin{aligned}\|\nabla f\|_{L^1(\Sigma)} &= \int_0^\infty \mathrm{Area}(\{x \in \Sigma \mid f(x) = s\})\,\mathrm{d}s \\ &\geq \int_0^\infty \left(\mathrm{Vol}(\{x \in \Sigma \mid f(x) \geq s\})^{\frac{n-1}{n}} - \|H\|_{L^1(\{x\in\Sigma \mid f(x)\geq s\})}\right)\mathrm{d}s \\ &\geq \int_0^\infty \mathrm{Vol}(\{x \in \Sigma \mid f(x) \geq s\})^{\frac{n-1}{n}}\,\mathrm{d}s - \|fH\|_{L^1(\Sigma)}\end{aligned}$$

where we applied the isoperimetric inequality in the second line.

Comparing the upper bound and lower bound derived so far, we now show that

$$\int_0^\infty \mathrm{Vol}(\{x \in \Sigma \mid f(x) \geq s\})^{\frac{n-1}{n}}\,\mathrm{d}s \gtrsim \left(\frac{n}{n-1}\int_0^\infty \mathrm{Vol}(\{x \in \Sigma \mid f(x) \geq s\}) s^{\frac{1}{n-1}}\,\mathrm{d}s\right)^{\frac{n}{n-1}}$$

to complete the proof. Let $V(s) = \mathrm{Vol}(\{x \in \Sigma \mid f(x) \geq s\})$ for notational simplicity.

Let $f_1(t) = \left(\int_0^t V(s)^{\frac{n-1}{n}}\,\mathrm{d}s\right)^{\frac{n}{n-1}}$ and let $f_2(t) = \frac{n}{n-1}\int_0^t V(s)s^{\frac{1}{n-1}}\,\mathrm{d}s$. We want to show that $f_1(t) \geq f_2(t)$. Notice that $f_1(0) = f_2(0) = 0$. By direct computation $(*)$ we have that $f_1'(t) \geq f_2'(t)$ if and only if

$$\int_0^t V(s)^{\frac{n-1}{n}}\,\mathrm{d}s \geq V(t)^{\frac{n-1}{n}} t.$$

This is true as $V(s)$ is monotone decreasing. To see this, first note that

$$\int_0^t V(s)^{\frac{n-1}{n}}\,\mathrm{d}s \geq \int_0^t V(t)^{\frac{n-1}{n}}\,\mathrm{d}s = V(t)^{\frac{n-1}{n}} t.$$

Therefore, we have that $f_1'(t) \geq f_2'(t)$ for all t. Since $f_1'(t) \geq f_2'(t)$ and $f_1(0) = f_2(0)$, we can conclude that $f_1(t) \geq f_2(t)$

We combine what we have found so far to obtain that

$$\begin{aligned}
\|f\|_{L^{\frac{n}{n-1}}(\Sigma)} - \|fH\|_{L^1(\Sigma)} &\lesssim \left(\frac{n}{n-1}\int_0^\infty \mathrm{Vol}(\{x\in\Sigma \mid f(x)\geq s\})s^{\frac{1}{n-1}}\,\mathrm{d}s\right)^{\frac{n-1}{n}} - \|fH\|_{L^1(\Sigma)} \\
&\lesssim \int_0^\infty \mathrm{Vol}(\{x\in\Sigma \mid f(x)\geq s\})^{\frac{n-1}{n}}\,\mathrm{d}s - \|fH\|_{L^1(\Sigma)} \\
&\leq \int_0^\infty \mathrm{Area}(\{x\in\Sigma \mid f(x)= s\})\,\mathrm{d}s \\
&= \|\nabla f\|_{L^1(\Sigma)}.
\end{aligned}$$

This gives the desired result. □

3.2 A Sobolev inequality on Riemannian Manifolds

The main conceptual idea behind the Sobolev inequality on $\mathbb{R}^n$ turns out to also be true on Riemannian manifolds in general with one slight difference to take into account: curvature. Curvature can affect $|\nabla f|$, so we have another term in the inequality.

Theorem 7 (Sobolev Inequality)
Let $\Sigma^n \subset \mathbb{R}^{n+m}$ (without boundary), and let f be a positive smooth function on Σ. Then,

$$\|f\|_{L^{\frac{n}{n-1}}(\Sigma)} \lesssim_{n,m} \int_\Sigma (|\nabla^\Sigma f| + f|H|) = \|\nabla f\|_{L^1(\Sigma)} + \|fH\|_{L^1(\Sigma)}$$

where H is the mean curvature of Σ with respect to the ambient space $\mathbb{R}^{n+m}$ and ∇^Σ is the induced Riemannian connection on Σ.

This theorem is proven by Brendle [1], whose proof was itself motivated by the Alexandrov-Bakelman-Pucci maximum principle. Rather than reiterate the argument here, we discuss his approach with respect to the equivalence of the Sobolev and Isoperimetric inequalities.

In particular, he notes that through scaling of the function f via multiplication, we may assume that

$$\|\nabla^\Sigma f\|_{L^1(\Sigma)} + \|fH\|_{L^1(\Sigma)} + \|f\|_{\partial\Sigma} = n\|f\|^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(\Sigma)}. \tag{3}$$

This makes the problem slightly more approachable by removing the exponent of $(n-1)/n$ from the right hand side for the time being. Now suppose there existed a g such that

$$g = nf^{\frac{n}{n-1}} - |\nabla^\Sigma f| - |fH|.$$

Then, integrating this equality, we would obtain that

$$\int_\Sigma g = \int_{\partial\Sigma} f$$

by (3). Hence, g must be the divergence of a function, in particular, there must exist a $u \in C^{2,\gamma}$ for each $0 < \gamma < 1$ by elliptic regularity such that $g = \text{div}(f\nabla^{\Sigma}u)$.

So if there is a function g equal to the side of (3) that we wish to understand, it is necessarily the divergence of $f\nabla^{\Sigma}u$. In fact, he goes onto show that this function u is sufficient.

He uses u to cover Σ with balls depending on how steep the gradient of u is. After the covering of these balls, he constructs a bijection between them and the unit ball $B^{n+m} \subset \mathbb{R}^{n+m}$. Conceptually, one can interpret this as "transporting" the mass on Σ to the unit ball in $\mathbb{R}^{n+m}$– in fact, this motivates another proof of the Sobolev inequality by Brendle and Eichmair [2].

The fact that we use the unit ball to understand $\text{div}(f\nabla^{\Sigma}u)$ highlights the relationship between the Sobolev and isoperimetric inequalities. We are using the divergence to understand the surface area integral of f in terms of it's volume ("mass"), which itself we can understand through the isoperimetric inequality. This is all we note of the proof of the Sobolev inequality.

4 Generalizations and Applications

There are a number of directions one can go from here to further these results. Firstly, in the statement of the Sobolev inequality one can also assume that Σ has boundary $\partial\Sigma$, obtaining the stronger result that

$$\|f\|_{L^{\frac{n}{n-1}}(\Sigma)} \lesssim \|\nabla^{\Sigma} f\|_{L^1(\Sigma)} + \|fH\|_{L^1(\Sigma)} + \|f\|_{L^1(\partial\Sigma)}.$$

This can be seen in the work of Simon Brendle [1]. Brendle went through the proof finding an explicit constant which makes the inequality true. In this paper, Brendle proved that this constant is sharp when $\Sigma \subset \mathbb{R}^{n+2}$ has codimension 2, and in fact this implies that Σ is a flat round ball. As a result, if Σ is a compact n-dimensional minimal submanifold of $\mathbb{R}^{n+2}$, then the isoperimetric inequality derived is sharp and equality holds if and only if Σ is a flat round ball.

Furthermore, it is possible to generalize these results to submanifolds Σ^n of a Riemannian manifold M^{n+m} assuming that the sectional curvature of M is bounded. This is done by the work of Hoffman and Spruck [4]. They also discuss some neat corollaries regarding minimal surfaces, and further applications to vector bundles over a Riemannian manifold.

Finally, the Sobolev inequality can directly be used to imply directly imply the embedding of Sobolev spaces into L^p spaces. Specifically: for $p \in [1, n)$, we have

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n).$$

where again p^* is the Sobolev conjugate of p as defined in Section 2.2.

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