Bipartite Beck-type Theorems and Radial Projections

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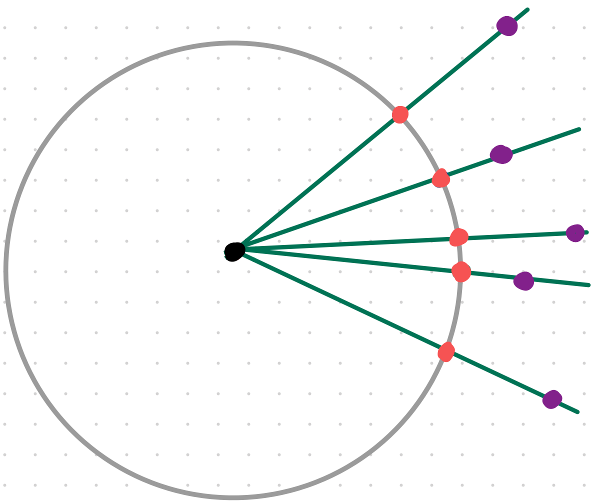
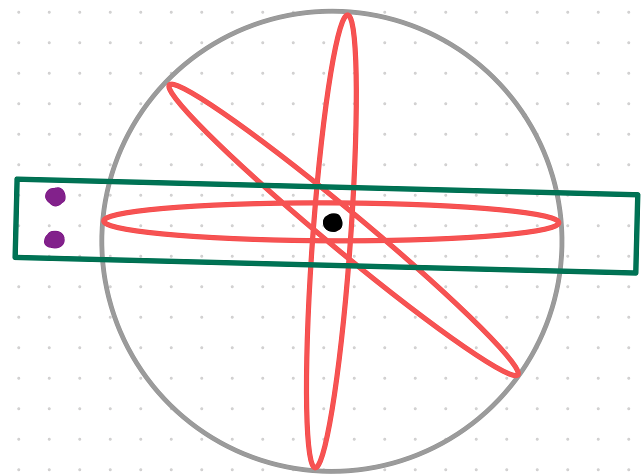




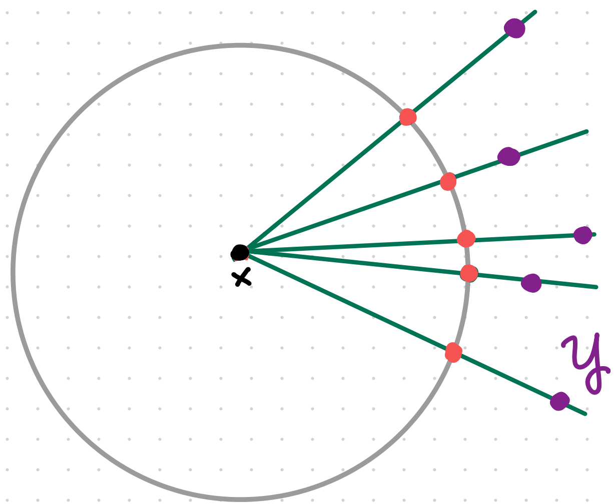
Radial Projections and Lines

Let $x, y \in \mathbb{R}^n$ with $x \neq y$. Then, define

$$\pi_x : \mathbb{R}^n \setminus \{x\} \to S^{n-1}$$
$$y \longmapsto \frac{y-x}{|y-x|}$$

OR

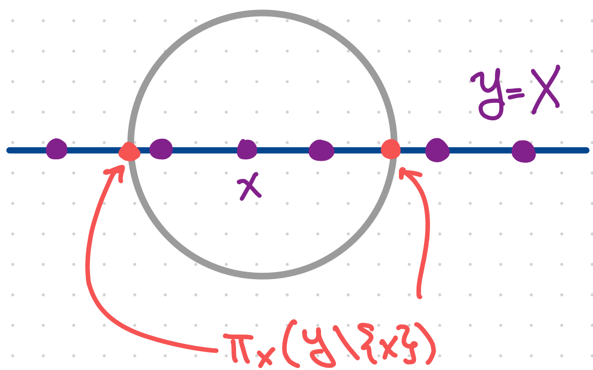
$$\pi_x : \mathbb{R}^n \setminus \{x\} \to G_x(n,1) \simeq S^{n-1}/\pm.$$
$$y \longmapsto \text{line through } x \text{ and } y.$$



Radial Projections in Discrete Geometry

Heuristic: Generically, projections of sets are often large, unless there is an algebraic/arithmetic/geometric reason.

One such "reason" for radial projections is <u>linearity</u>.



Thm [Beck '83] If $X \subseteq \mathbb{R}^2$ is noncollinear and finite,

$$\max_{x \in X} |\pi_x(X \setminus \{x\})| \gtrsim |X|.$$

Radial Projections in Fractal Geometry

40 years later, Orponen–Shmerkin–Wang found a fractal analogue.

Thm [Orponen–Shmerkin–Wang '22] If X Borel and noncollinear,

$$\sup_{x \in X} \dim \pi_x(X \setminus \{x\}) \geq \min\{\dim X, 1\}.$$

In fact, they obtained a bipartite version.

Moreover, for $Y \subseteq \mathbb{R}^2$ Borel,

$$\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \min\{\dim X, \dim Y, 1\}.$$

Radial Projections & Lines ctd.

Corollary [Orponen-Shmerkin-Wang, '22]:

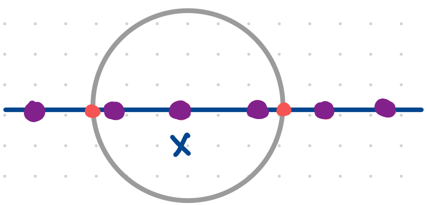
Given $X, Y \subseteq \mathbb{R}^2$ Borel, and

$$\dim X \setminus \ell = \dim X \qquad \forall \ell \in A(2,1),$$

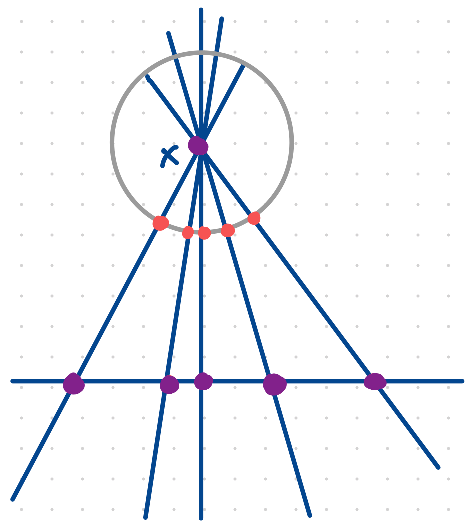
then $\forall \varepsilon > 0$, $\exists X' \subseteq X$ such that $\dim X = \dim X'$ and

$$\dim \pi_x(Y \setminus \{x\}) \geq \min\{\dim X, \dim Y, 1\} - \varepsilon \quad \forall x \in X'.$$

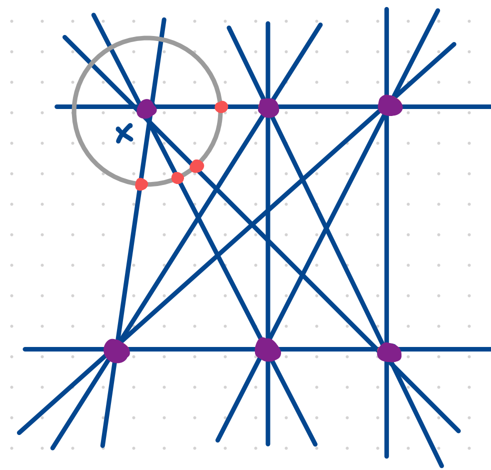
Key Examples



$|\pi_x(X)| \sim 1$



$|\pi_x(X)| \sim |X| \; (1\, x)$



$|\pi_x(X)| \sim |X| \; \forall x$

Beck-type Theorems for Lines

Let $X, Y \subseteq \mathbb{R}^2$, and consider the set

$$\mathcal{L}(X,Y) := \{\ell \in \underbrace{A(2,1)}_{\text{affine lines in } \mathbb{R}^2} : \exists\, x \in X, y \in Y,\ x \neq y, \text{ and } x, y \in \ell\}$$

Theorem [Beck '83]

Let $X \subseteq \mathbb{R}^n$ finite and $|X| = N$. If $|X \setminus \ell| \gtrsim N$ for all lines $\ell \in A(n,1)$, then $|\mathcal{L}(X)| \gtrsim N^2$.

Theorem [Orponen-Shmerkin-Wang '22]

Let $X, Y \subseteq \mathbb{R}^2$ Borel and $\dim(X \setminus \ell) = \dim X$ $\forall \ell \in A(2,1)$.

Then, $\dim \mathcal{L}(X,Y) \geq \min\{2\dim X, 2\dim Y, 2\}$

Beck-type Theorems for Lines Final Remarks

- Obtaining sharp (discrete) bipartite Beck-type theorems would be of great interest for problems in incidences.
- Furthermore, bipartite radial projection results allow us to interrogate the relation between X and Y. E.g.

Thm [Csornyei-Stull '25] If $X, Y \subseteq \mathbb{R}^2$ Borel and not in the same line $\sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \min\{\frac{\dim X + \dim Y}{2}, \dim Y, 1\}$.

- Makes use of the Furstenberg set problem in $\mathbb{R}^2$.

A Hyperplanar Beck's Theorem

Thm [Beck '83] Given $X \subseteq \mathbb{R}^n$ finite, with $|X \setminus H| \gtrsim |X|$ for all hyperplanes H, $|\underbrace{P^{n-1}(X)}_{\text{(n-1)-planes w/ n affinely ind pts of X.}}| \gtrsim |X|^n$.

In 2016, Ben Lund and Thao Do (separately) showed:

Theorem [Lund, Do '16]
For all $\varepsilon > 0$, $\exists\, \gamma_\varepsilon > 0$ such that given $X \subseteq \mathbb{R}^n$ finite, if $|X \setminus \bigcup V_i| \geq (1-\varepsilon)|X|$ for all flats V_i w/ $\sum \dim V_i \leq n-1$,

$$|P^{n-1}(X)| \geq \gamma_\varepsilon |X|^n .$$

A Hyperplanar Beck's Theorem ctd.

Theorem [B.-Ortiz-Zakharov, '25]

Let $X \subseteq \mathbb{R}^n$ Borel s.t.

$\dim(X \setminus \bigcup V_i) = \dim X$ for all flats V_i w/ $\sum \dim V_i \leq n-1$,

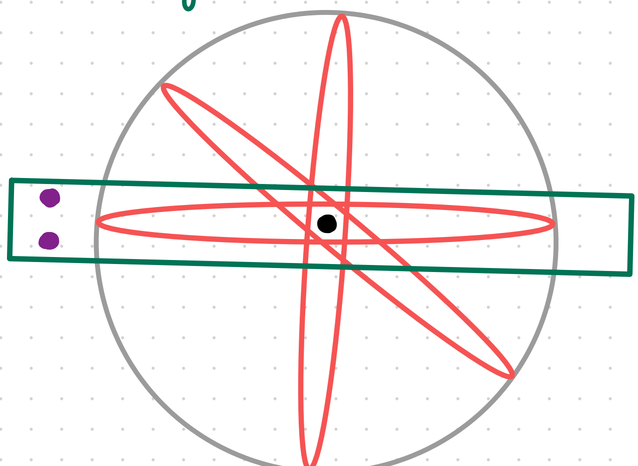
then $\underbrace{\dim P^{n-1}(X)}_{\{P^n \in A(n,n-1) : P \text{ contains } n \text{ affinely ind. pts of } X.} \geq \min\{n \dim X, n\}$.

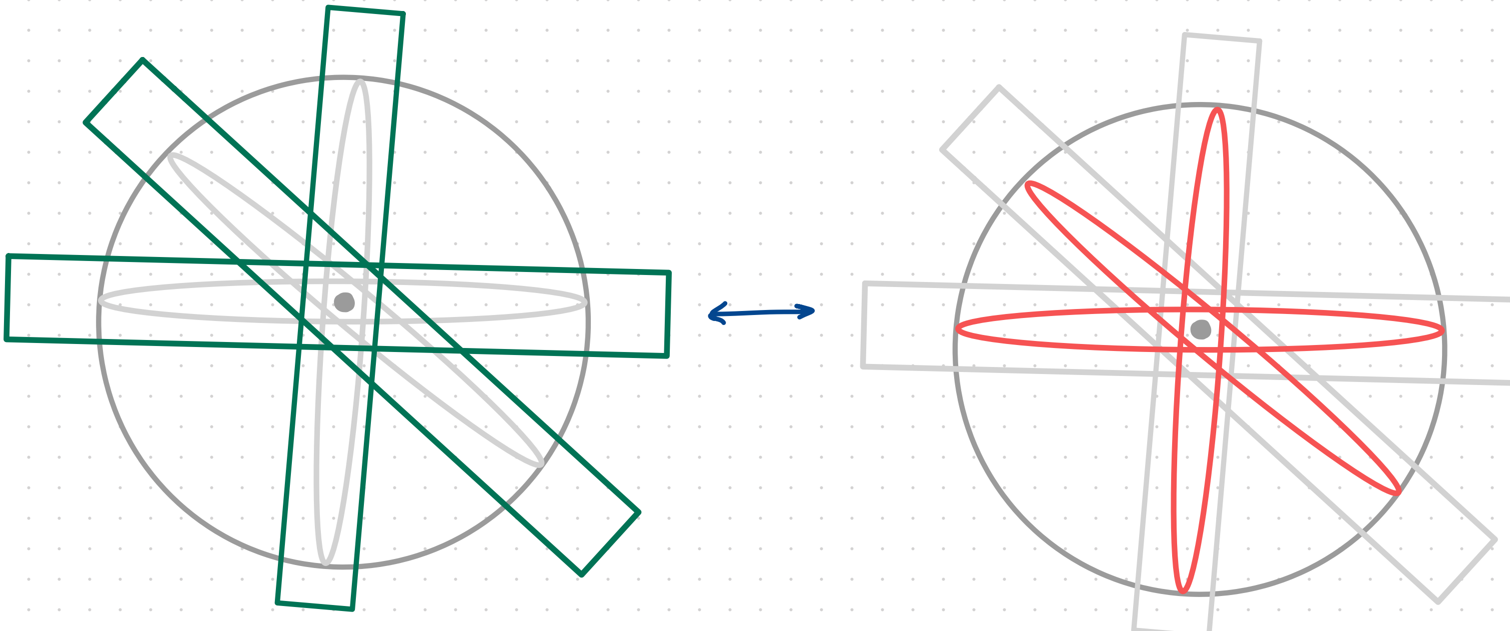
Radial Projections and Hyperplanes

Let $x \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^{n(n-1)}$ such that $(x, y_i : 1 \le i \le n)$ affinely ind.

$$\Pi_x^{n-1} : \mathbb{R}^{n(n-1)} \setminus S^x_{n-1} \longrightarrow G_x(n, n-1) \simeq \mathbb{S}^{n-1}/\pm$$

$$\vec{y} \longmapsto \text{Hyperplane containing } x, y_1, \ldots, y_n.$$





A Hyperplanar Beck's Theorem ctd.

$\boxed{\begin{array}{l}\text{Theorem [B.-Ortiz-Zakharov, '25]}\\ \text{Let } X \subseteq \mathbb{R}^n \text{ Borel s.t.}\\ \quad \dim(X \setminus \bigcup V_i) = \dim X \text{ for all flats } V_i \text{ w/ } \sum \dim V_i \le n-1,\\ \text{then } \sup_{x \in X} \dim \Pi_x^{n-1}(X^{n-1} \setminus S_{n-1}^x) \ge \min\{(n-1)\dim X,\ n-1\}.\end{array}}$

$\boxed{\begin{array}{l}\text{Conjecture: The same lower bound holds for } X \not\subseteq \bigcup V_i\\ \qquad \text{for } \sum \dim V_i \le n-1.\end{array}}$

Question: What about $\Pi_x^{n-1}(Y^{n-1} \setminus S_{n-1}^x)$, $x \in X$?

Thank You!