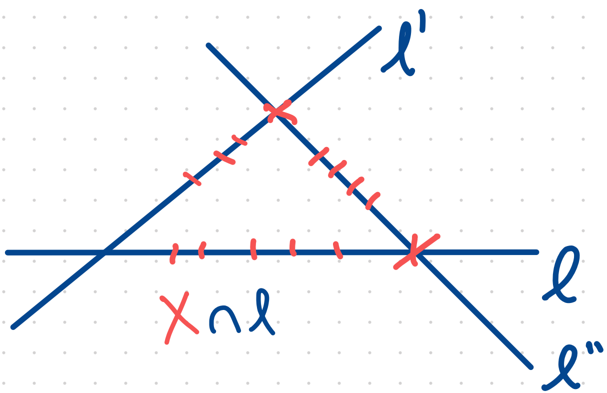
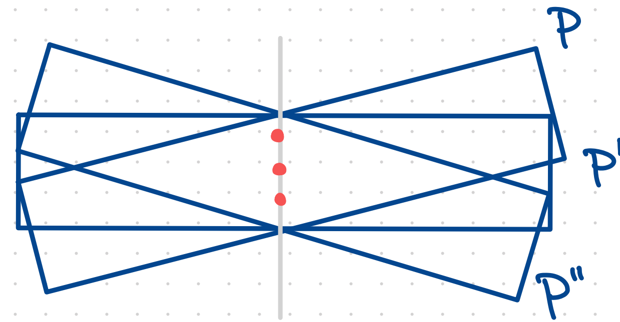
An Introduction to Furstenberg Sets

joint work w/ Manik Dhar

By: Paige Bright



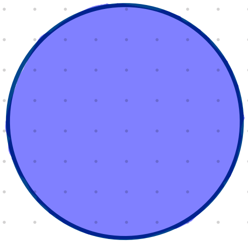


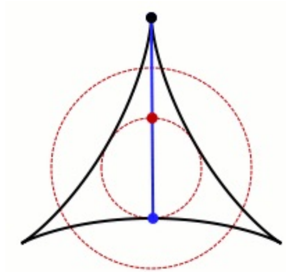
The Kakeya Conjecture

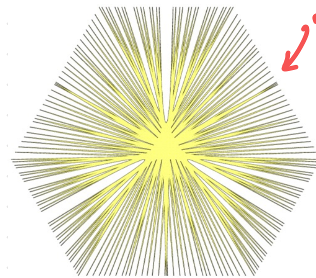
Def: A compact set $K \subseteq \mathbb{R}^n$ is a *Kakeya set*, if for all $\theta \in \mathbb{S}^{n-1}$, K contains a line segment in direction θ.

Conjecture: Every Kakeya set $K \subseteq \mathbb{R}^n$ has $\dim_H K = n$.

- $n=2$ [Davies '71], $n=3$ [Wang–Zahl '25], Open for $n \geq 4$.



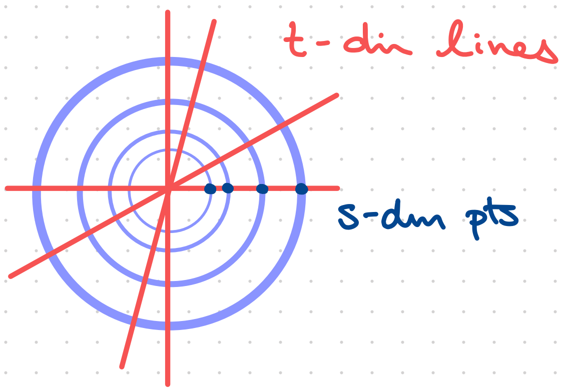




Besicovitch 1917

Furstenberg: A fractal Kakeya

Def: A Borel set $X \subseteq \mathbb{R}^2$ is an $\underbrace{(s,t)}_{0 \le s \le 1,\ 0 \le t \le 2}$- Furstenberg set if there exists a t-dimensional set of lines $\mathcal{L} \subseteq A(2,1)$ such that $\dim X \cap \ell \ge s \quad \forall \ell \in \mathcal{L}$.



Q: How small can $\dim X$ be?

Furstenberg: An Incidence Approach

Def: A finite set $X \subseteq \mathbb{R}^2$ is an (s,t)-discrete Furstenberg set if there exists t lines $\mathcal{L} \subseteq A(2,1)$ such that
$$|X \cap \ell| \geq s \quad \forall \ell \in \mathcal{L}.$$

Q: How small can $|X|$ be?

Consider the set of incidences between X and $\mathcal{L}$:
$$\mathcal{I}(X,\mathcal{L}) = \{(x,\ell) \in X \times \mathcal{L} : x \text{ lies on } \ell\}.$$

Q: How large can $|\mathcal{I}(X,\mathcal{L})|$ be?

Point-Line Incidence Bounds

$$\boxed{\begin{array}{l}\text{Setup: } X \subseteq \mathbb{R}^2 \text{ finite, } \mathcal{L} \subseteq A(2,1) \text{ finite, and} \\ \qquad \mathcal{I}(X,\mathcal{L}) = \{(x,\ell) \in X \times \mathcal{L} : x \text{ lies on } \ell\}.\end{array}}$$

- Trivial Bound: $|\mathcal{I}(X,\mathcal{L})| \leq |X||\mathcal{L}|$
  ↳ Note: If equality, every point lies on every line.
- Double Counting: $|\mathcal{I}(X,\mathcal{L})| \leq |X|^{1/2}|\mathcal{L}| + |X|$
  ↳ Every 2 pts determine a line
- Szemerédi-Trotter '83: $|\mathcal{I}(X,\mathcal{L})| \lesssim |X|^{2/3}|\mathcal{L}|^{2/3} + |X| + |\mathcal{L}|$
  ↳ Uses topology of $\mathbb{R}$.

Furstenberg: An Incidence Approach

Setup: $\cdot\ |\mathcal{L}| = t$, $|X \cap \ell| \geq s\ \forall \ell$

$\cdot\ \mathcal{I}(X,\mathcal{L}) \lesssim \begin{cases} |X|^{1/2}|\mathcal{L}| + |X| & \text{(Double Counting)} \\ |X|^{2/3}|\mathcal{L}|^{2/3} + |X| + |\mathcal{L}| & \text{(Szemerédi–Trotter)} \end{cases}$

Notice, given the Setup, $st \leq |X \cap \ell||\mathcal{L}| \leq |\mathcal{I}(X,\mathcal{L})|$.

$$\boxed{\text{Prop: Given Setup, } |X| \gtrsim \begin{cases} \min\{s^2, st\} \\ \min\{st, s^{3/2}t^{1/2}\} \end{cases}}$$

Furstenberg + Discretized Incidences

Setup: $\cdot X \subseteq \mathbb{R}^2$, $\mathcal{L} \subseteq A(2,1)$, $\dim X \cap \ell \geq s$, $\dim \mathcal{L} \geq t$.

$$\boxed{\underline{\text{Heuristically}},\quad \begin{cases} \dim \mathcal{L} \geq t \\ \dim (X \cap \ell) \geq s \end{cases} \text{“}\longrightarrow\text{”} \begin{cases} |\mathcal{L}|_\delta \approx \delta^{-t} \\ |X \cap \ell|_\delta \approx \delta^{-s}. \end{cases}}$$

$$|X \cap \ell|_\delta |\mathcal{L}|_\delta \text{“}\lessapprox\text{”} \mathcal{I}(X_\delta, \mathcal{L}_\delta) \text{“}\lessapprox\text{”} \begin{cases} |X|_\delta^{1/2} |\mathcal{L}|_\delta + |X|_\delta \\ |X|_\delta^{2/3} |\mathcal{L}|_\delta^{2/3} + |X|_\delta + |\mathcal{L}|_\delta \end{cases}$$

Theorem: If $X \subseteq \mathbb{R}^2$ is an (s,t)-Furstenberg set,

1) $\dim X \geq \min\{2s, s+t\}$ Héra-Shmerkin-Yavicoli, Lutz-Stull
2) $\dim X \geq \min\{s+t, \frac{3s+t}{2}, s+1\}$. Orponen-Shmerkin, Ren-Wang

Furstenberg in Different Directions?

Recall that Kakeya sets have line segments in every direction.
Manik Dhar and I explored similar conditions on Furstenberg sets.

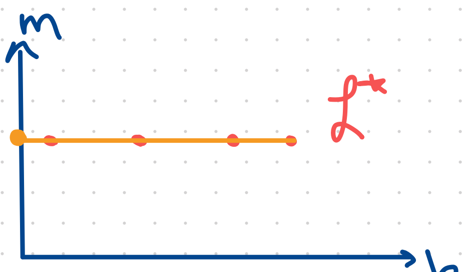
Def: A Borel set $X \subseteq \mathbb{R}^2$ is an $\underbrace{(s,t)}_{0 \leq s \leq 1,\ 0 \leq t \leq 1}$-spread Furstenberg set if there exists a t-dimensional set of subspaces $\mathcal{L} \subseteq G(2,1)$ such that $\dim[(X+a_\ell) \cap \ell] \geq s$ $\forall\, \ell \in \mathcal{L}$ for some $a_\ell \in \mathbb{R}^2$.

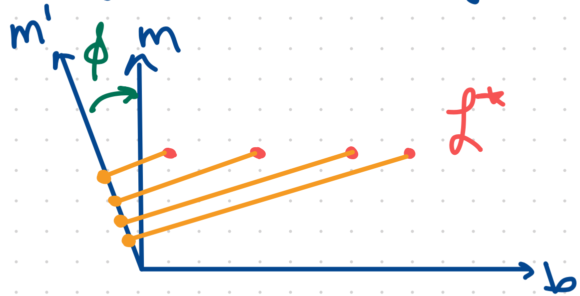
Our results are more interesting in higher dimensions, but I want to highlight why spread/not spread are not so different.

Furstenberg $\longrightarrow$ Spread Furstenberg

Prop: If $F \subseteq \mathbb{R}^2$ is an (s,t)-Furstenberg set, then $\exists$ proj. trans. $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(F)$ is $(s, \min\{t,1\})$-spread.

Pf Sketch: Write each line $y = mx + b \xrightarrow[\text{duality}]{\text{point-line}} (m,b) = \ell^*$.





A good m' comes from Marstrand's Projection Theorem. $\square$

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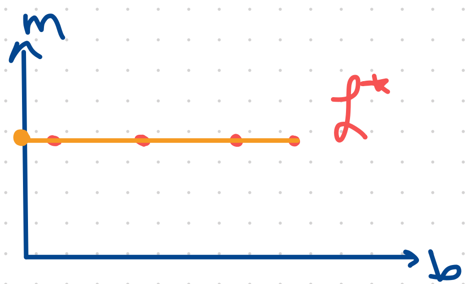

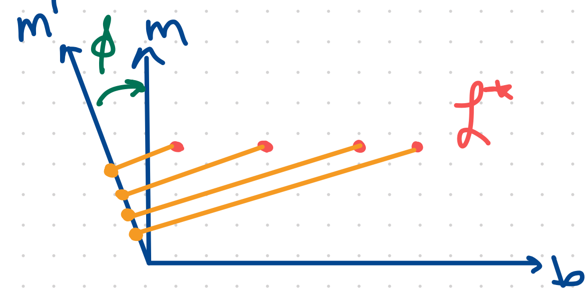

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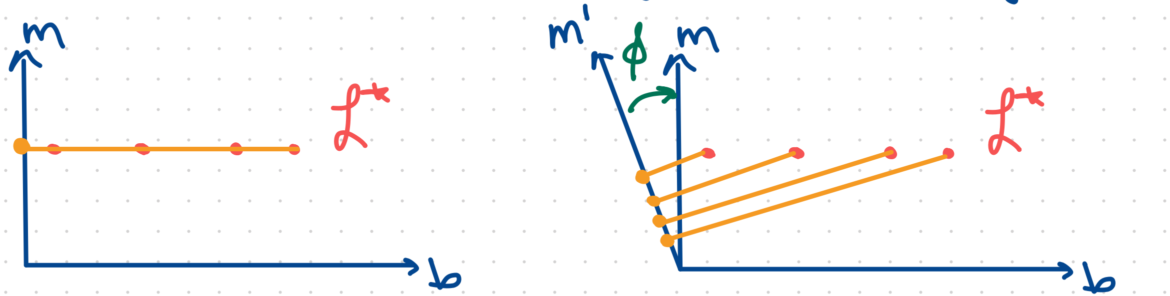

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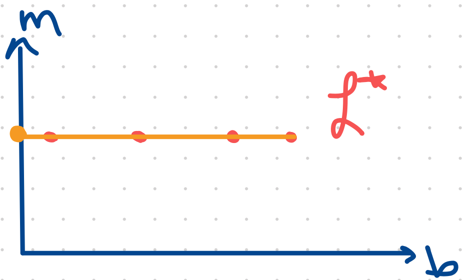

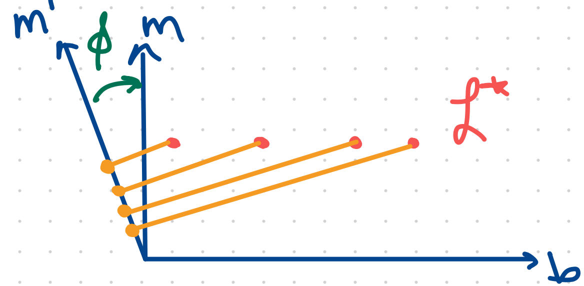

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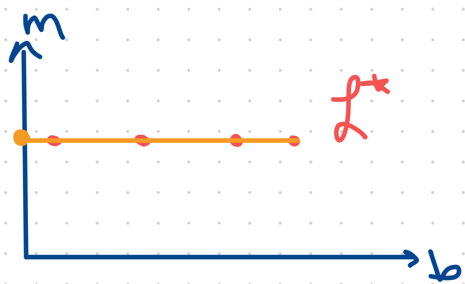

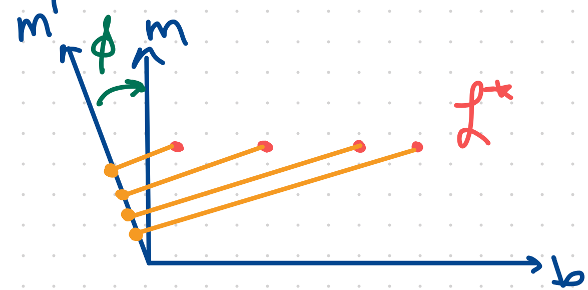

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Furstenberg in Higher Dimensions?

We can ask Furstenberg set-type questions in $\mathbb{R}^n$.

Def: A finite set $X \subseteq \mathbb{R}^n$ is an (s,t)-discrete Furstenberg set if there exists t lines $\mathcal{L} \subseteq A(n,1)$ such that

$$|X \cap \ell| \geq s \quad \forall\, \ell \in \mathcal{L}.$$

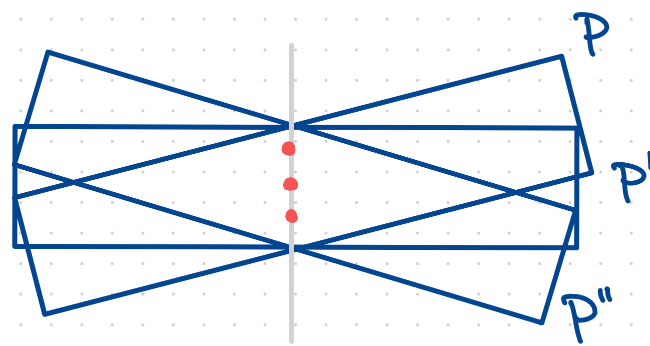
However, Szemerédi–Trotter is sharp and holds for all n by orthogonally projecting onto $\mathbb{R}^2$.

Remark: That said, one can do better if there are more conditions on $\mathcal{L}$, see Guth-Katz.

Furstenberg in Higher Dimensions?

Def: A finite set $X \subseteq \mathbb{R}^n$ is an $(s,t;k)$-discrete Furstenberg set if there exist t k-planes $\mathcal{P} \subseteq A(n,k)$ such that
$$|X \cap P| \geq s \quad \forall P \in \mathcal{P}.$$

However, in the discrete setting, $|X| \geq s$ for $k \geq 2$ is sharp.



Furstenberg in Higher Dimensions?

Def: A Borel set $X \subseteq \mathbb{R}^n$ is an $\underbrace{(s,t;}\ k)$-Furstenberg set

$0 \le s \le k,\ 0 \le t \le (k+1)(n-k),\ 1 \le k \le n-1$

if there exists a t-dimensional set of k-planes $\mathcal{P} \subseteq A(n,k)$ such that $\dim X \cap P \ge s \quad \forall P \in \mathcal{P}$.

Some of the aforementioned proof methods generalize to $\mathbb{R}^n$:

Theorem: Héra, Héra-Keleti-Máthé

$$\dim X \ge \max\left\{2s + \min\{t,1\} - k,\ s + \frac{t - (k - \lceil s \rceil)(n-k)}{\lceil s \rceil + 1}\right\}.$$

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My work with Manik Dhar gives

1) Lower bds to $\dim X$ for all $k \geq k_0(n)+1$, $s > k_0$.
2) Prop: X $(s,t;n-1)$-Furstenberg $\Rightarrow$ $\Phi(X)$ $(s, \min\{t, n-1\}; n-1)$-spread using point-hyperplane duality + Marstrand's Proj Thm.

Thank You!