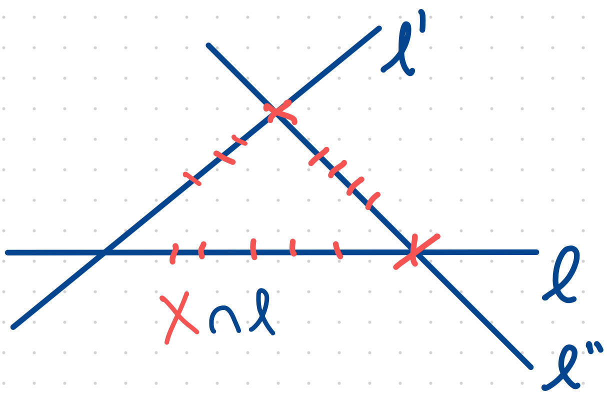
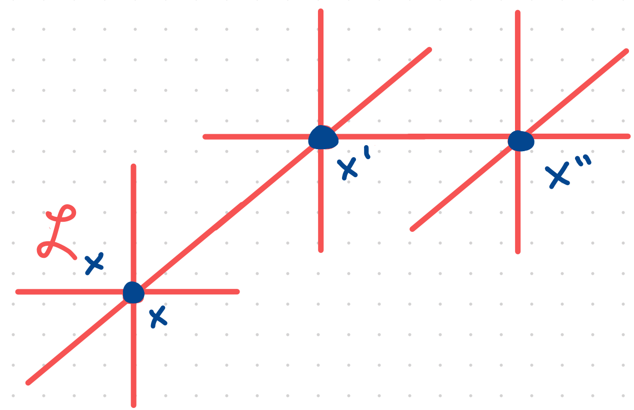
Furstenberg Sets: A Look Forward + Back

By: Paige Bright





Note: Study Guide

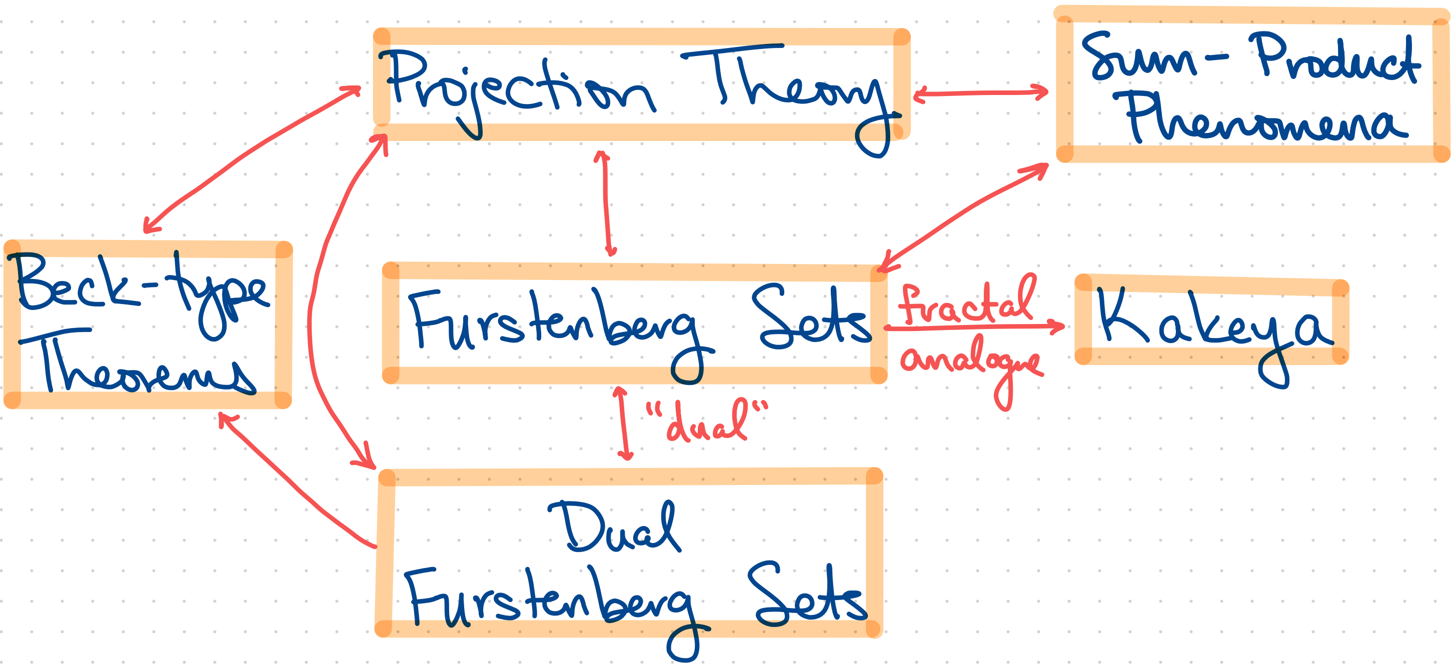
A STUDY GUIDE FOR "ON THE HAUSDORFF DIMENSION OF FURSTENBERG SETS AND ORTHOGONAL PROJECTIONS IN THE PLANE"

AFTER T. ORPONEN AND P. SHMERKIN

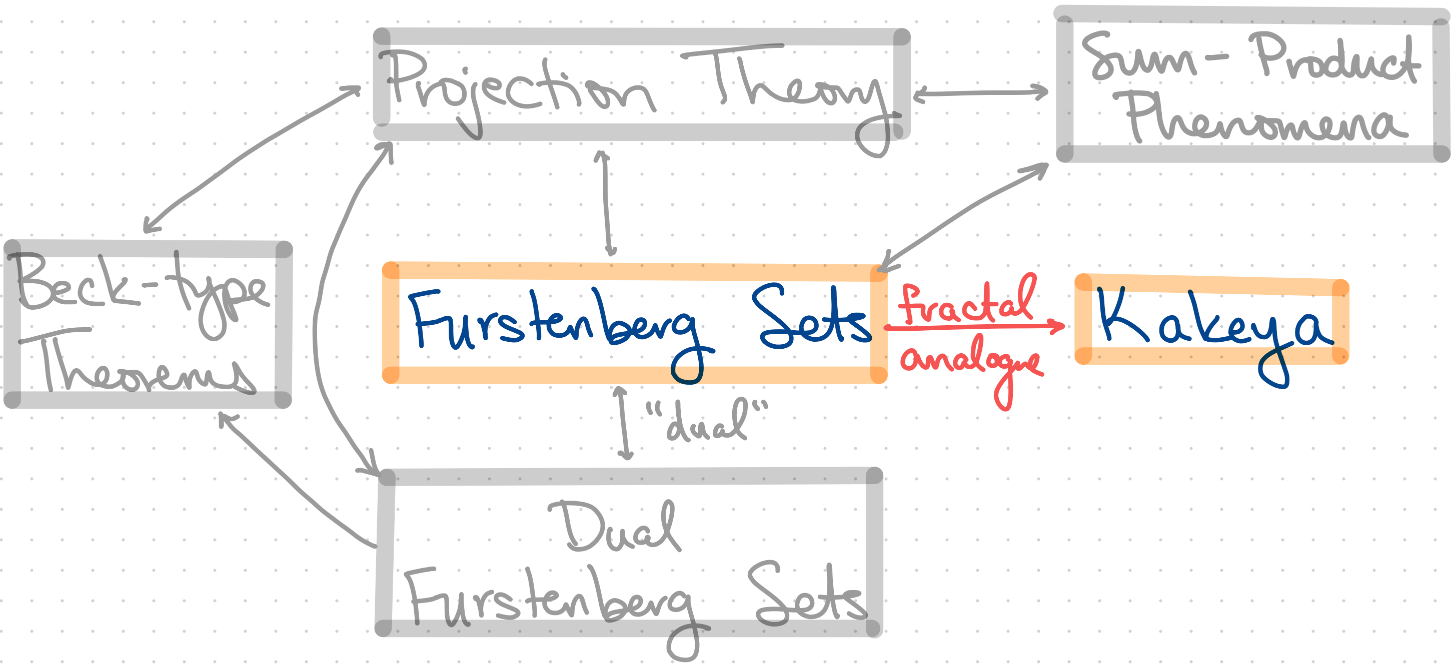
JACOB B. FIEDLER, GUO-DONG HONG, DONGGEUN RYOU, AND SHUKUN WU

Abstract. This article is a study guide for "On the Hausdorff dimension of Furstenberg sets and orthogonal projections in the plane" by Orponen and Shmerkin [OS23a]. We begin by introducing Furstenberg set problem and exceptional set of projections and provide a summary of the proof with the core ideas.

A Snapshot


Projection Theory
Sum-Product Phenomena
Beck-type Theorems
Furstenberg Sets
fractal
analogue
Kakeya
"dual"
Dual Furstenberg Sets

A Snapshot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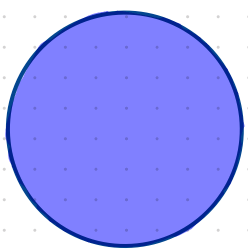
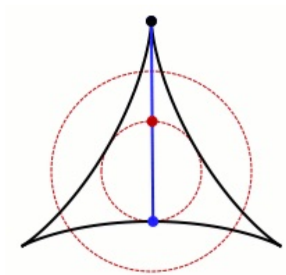
The Kakeya Conjecture

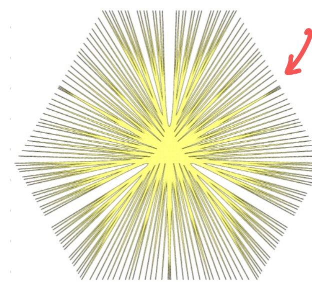
Def: A compact set $K \subseteq \mathbb{R}^n$ is a *Kakeya set*, if for all $\theta \in \mathbb{S}^{n-1}$, K contains a line segment in direction θ.

Conjecture: Every Kakeya set $K \subseteq \mathbb{R}^n$ has $\dim_H K = n$.

↳ Note: Known in $\mathbb{R}^2$ [Davies, '71]. Open for $n \geq 3$.



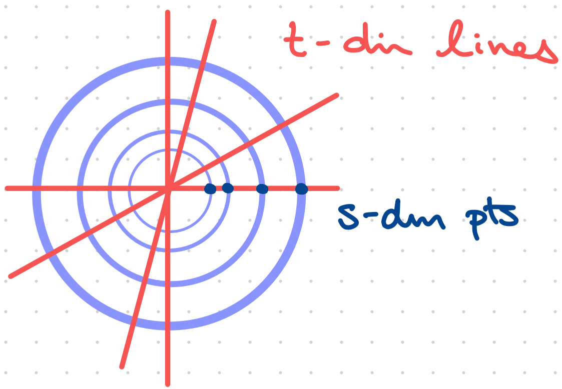




Besicovitch '17

Furstenberg: A fractal Kakeya

Def: A Borel set $F \subseteq \mathbb{R}^2$ is an $\underbrace{(s,t)}_{0\le s\le 1,\ 0\le t\le 2}$-Furstenberg set if there exists a t-dimensional set of lines $\mathcal{L} \subseteq A(2,1)$ such that $\dim F \cap \ell \ge s \quad \forall \ell \in \mathcal{L}$.



Q: How small can $\dim F$ be?

Furstenberg: Some Background

Studied by Wolff '99 with a line in every *direction* $(t=1)$.

- Obtained $\geq \max\{2s, s+\frac{1}{2}\}$ $\xrightarrow[\text{generalizes}]{\text{approach}}$ $\geq \max\{2s, s+\frac{t}{2}\}$.
- Note, the sharp estimate is $\geq \min\{s+t, \frac{3s+t}{2}, s+1\}$.

  ↳ Resolved Orponen-Shmerkin and Ren-Wang '23.
- Discretized incidence / Szemerédi-Trotter heuristic.
- Notice, when $t=2s$, the two terms agree. In fact, over $\mathbb{C}^2$ you cannot "beat" the $2s$ lower bound.

  Ex: $F = \mathbb{R}^2 \subseteq \mathbb{C}^2$, and $\mathcal{L} = A(\mathbb{R}^2, 1) \subseteq A(\mathbb{C}^2, 1)$.

Furstenberg: Discretized Szemerédi–Trotter

Given $P \subseteq \mathbb{R}^n$ points + $\mathcal{L} \subseteq A(n,1)$ lines (finite), let

$$\mathcal{I}(P,\mathcal{L}) := \#\{(p,\ell) \in P \times \mathcal{L} : p \text{ lies on } \ell\}.$$

$$\boxed{\textbf{Theorem [Szemerédi–Trotter '83]} \quad \mathcal{I}(P,\mathcal{L}) \lesssim |P|^{2/3}|\mathcal{L}|^{2/3} + |P| + |\mathcal{L}|.}$$

Heuristically,

$$\begin{cases} \dim \mathcal{L} \geq t \\ \dim(\ell \cap F) \geq s \end{cases} \overset{\text{“ ”}}{\longrightarrow} \begin{cases} |\mathcal{L}|_\delta \approx \delta^{-t} \\ |F \cap \ell|_\delta \approx \delta^{-s}. \end{cases}$$

$$|F \cap \ell|_\delta |\mathcal{L}|_\delta \overset{\text{“ ”}}{\lesssim} \mathcal{I}(F,\mathcal{L}) \overset{\text{“ ”}}{\lesssim} |F|_\delta^{2/3} |\mathcal{L}|_\delta^{2/3} \implies |F|_\delta \gtrsim \delta^{-\left(\frac{3s+t}{2}\right)}$$

Furstenberg: A Rough Timeline

Wolff '99 → Molter-Rela '10 → Lutz-Stull '18 Héra-Shmerkin-Yavicoli '22 → Fu-Ren '23 → Orponen-Shmerkin + Ren-Wang '23

Wolff '99: $\max\{2s, s+\frac{1}{2}\}$ $[t=1]$

Molter-Rela '10: $\max\{2s+t-1, s+\frac{t}{2}\}$

Lutz-Stull '18, Héra-Shmerkin-Yavicoli '22: $\min\{2s, s+t\}$

Fu-Ren '23: $\min\{2s+t-1, s+1\}$ $[0<s\leq 1\leq t\leq 2]$

Orponen-Shmerkin + Ren-Wang '23: $\min\{s+t, \frac{3s+t}{2}, s+1\}$.

ε-improvements

$2s+\varepsilon(s,t)$

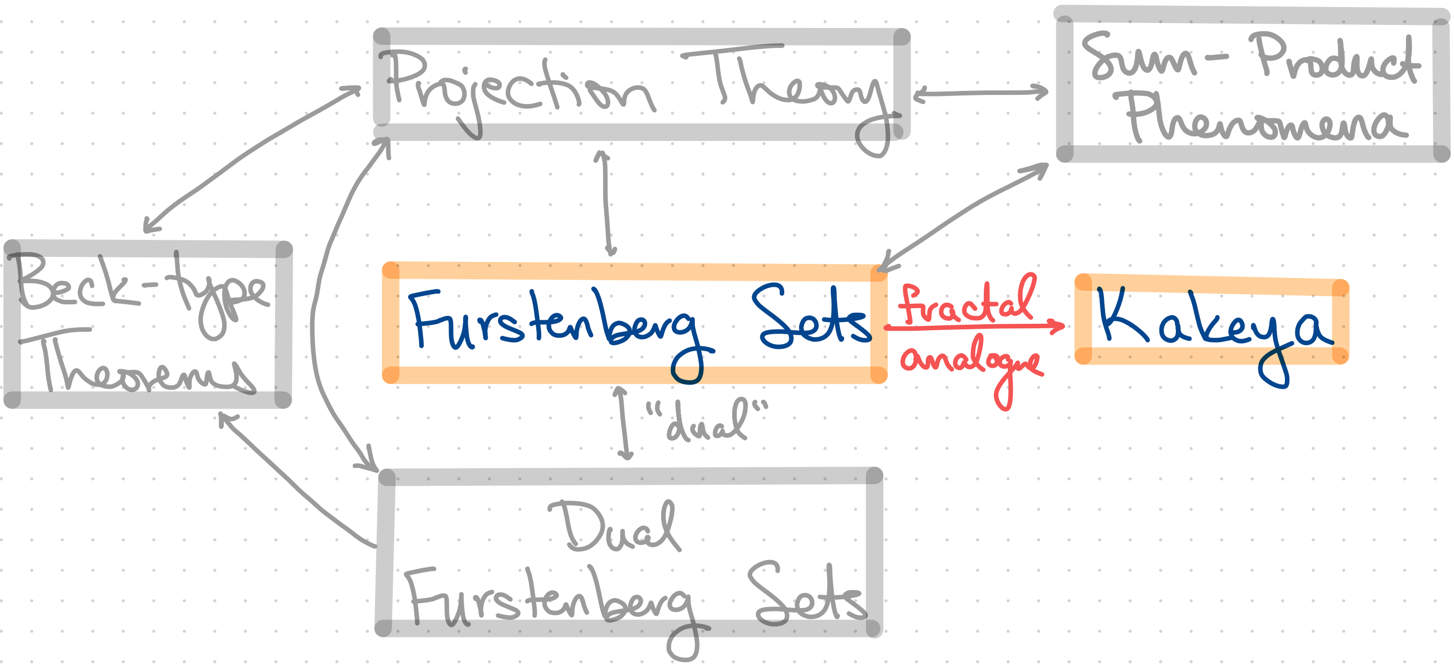
- Katz-Tao '01 + Bourgain '03 $[s=\frac{1}{2}, t=1]$
- Héra-Shmerkin-Yavicoli '21 $[t=2s]$
- Orponen-Shmerkin '21 $[s<t]$

→ Benedetto-Zahl '21 [Quantified]

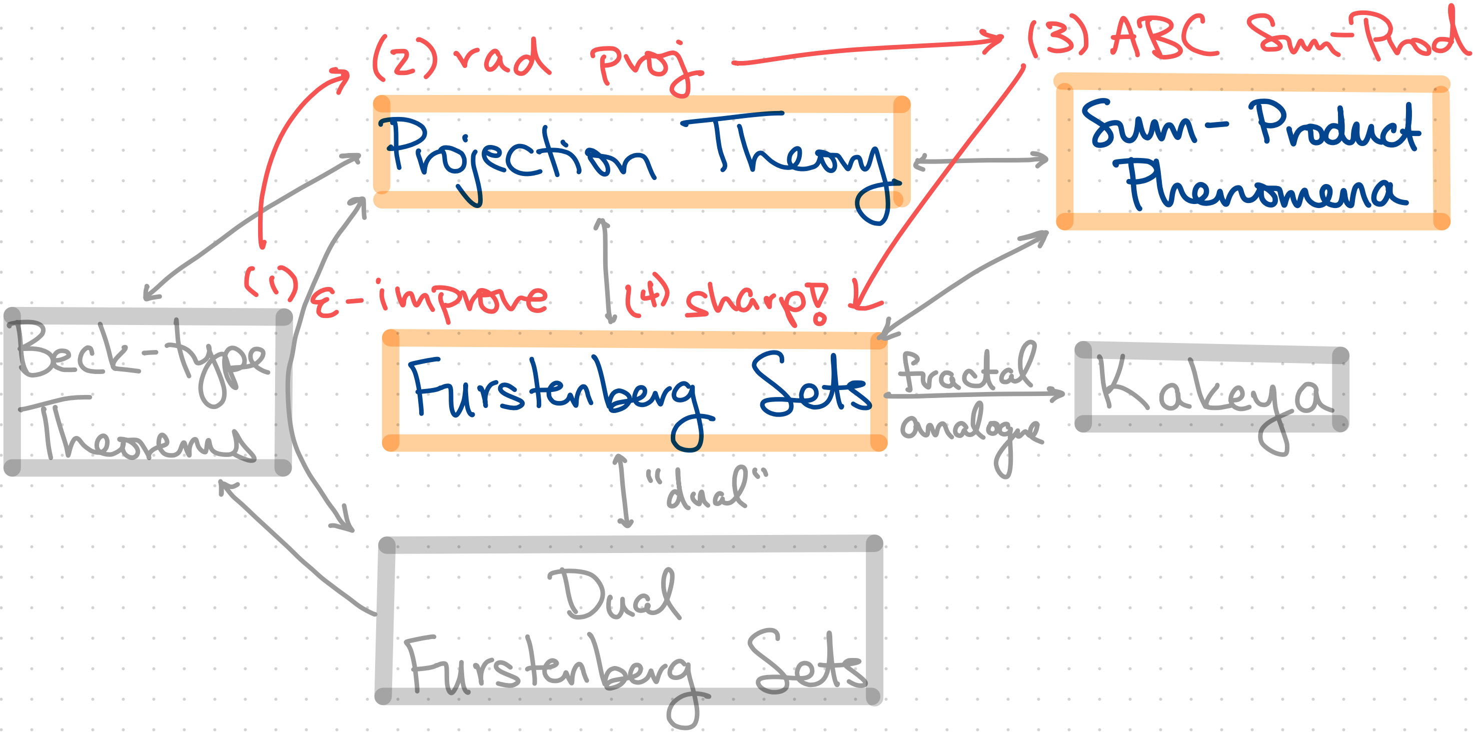
$s+\frac{t}{2}+\varepsilon(s,t)$

- Shmerkin-Wang '22 $[s<t]$

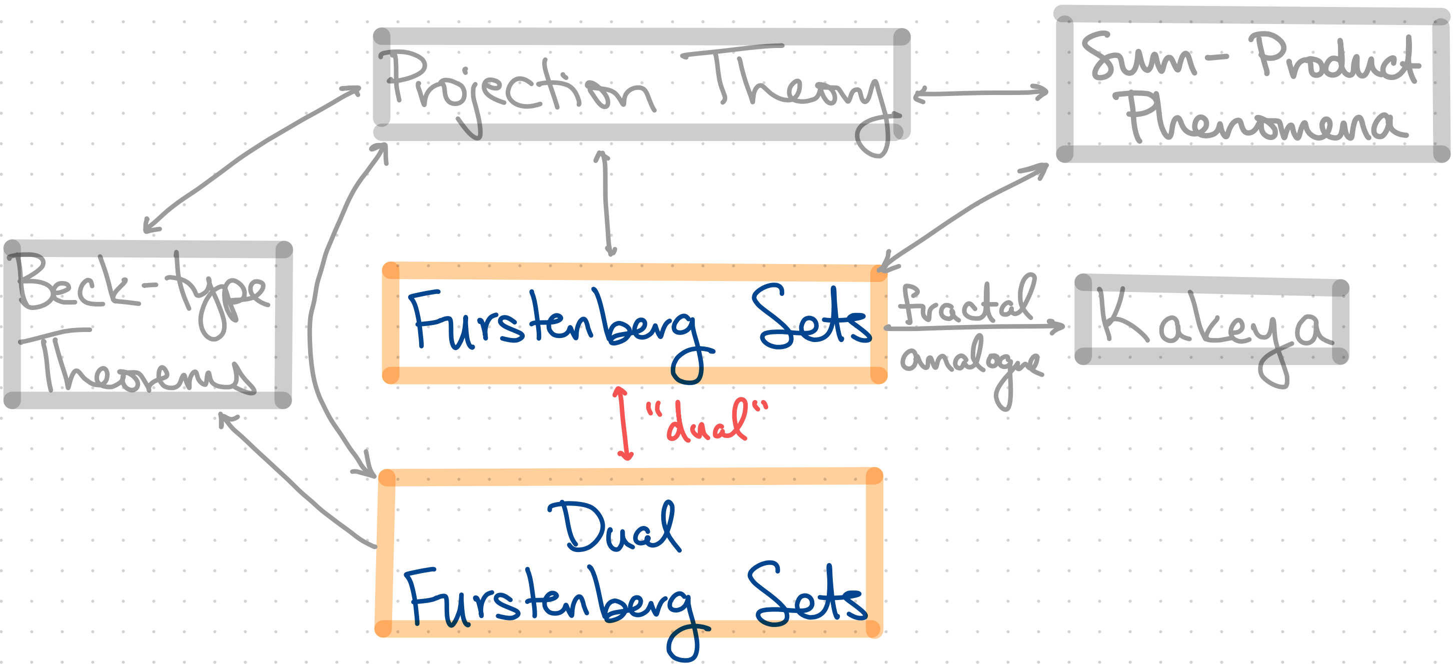
A Snapshot Revisited



A Snapshot Revisited

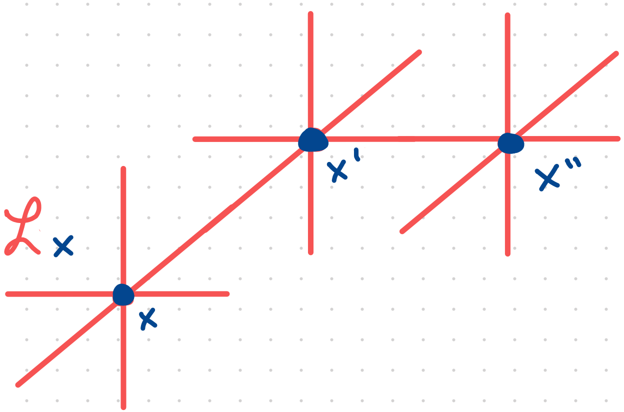


A Snapshot Revisited



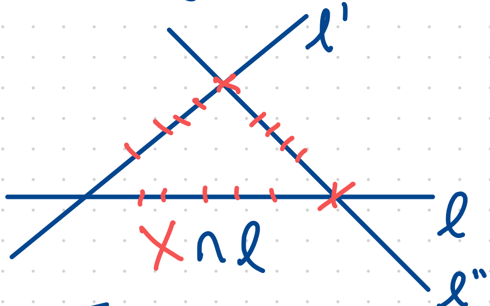
Dual Furstenberg: A Dual Furstenberg(?)

Def: A Borel set $\mathcal{L} \subseteq A(n,1)$ is a *dual* $\underbrace{(s,t)}_{0 \leq s \leq n,\ 0 \leq t \leq (n-1)}$*-Furstenberg set* if there exists a s-dimensional set of points $X \subseteq \mathbb{R}^n$ such that $\dim(\mathcal{L}_x := \{\ell \in \mathcal{L} : x \in \ell\}) \geq t$



Q: How small can $\dim \mathcal{L}$ be?

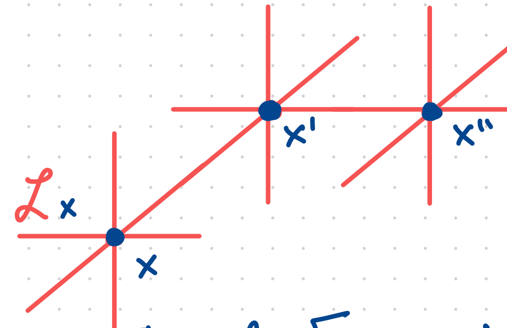
Furstenberg vs Dual Furstenberg



Furstenberg

dual if $n=2$ ⟷

$(m,b) \longleftrightarrow y=mx+b$



Dual Furstenberg

- $X \subseteq \mathbb{R}^n$, $\mathcal{L} \subseteq A(n,1)$ s.t.
  - $\dim \mathcal{L} \geq t$
  - $\dim X \cap \ell \geq s \quad \forall \ell \in \mathcal{L}$

$\Longrightarrow \dim X \geq\ ?$

- $\mathcal{L} \subseteq A(n,1)$, $X \subseteq \mathbb{R}^n$ s.t.
  - $\dim X \geq t$
  - $\dim \mathcal{L}_x \geq s \quad \forall x \in X$

$\Longrightarrow \dim \mathcal{L} \geq\ ?$

A Dual Furstenberg Result

Theorem [B.-Fu-Ren, '24]
Let $\mathcal{L} \subseteq A(n,1)$ be a dual (s,t)-Furstenberg set. Then,

$$\dim \mathcal{L} \geq \min\{2s, s+t\}.$$

Pf Outline: Heuristically: $\begin{cases} \text{Pins: } |X| \approx \delta^{-t} \\ \text{Bushes: } |\mathcal{L}_x| \approx \delta^{-s} \end{cases}$,

and do a double counting argument on

$$J(P,\mathcal{L}) = \#\{(x,x',\ell) \in X^2 \times \mathcal{L} : \ell \in \mathcal{L}_x \cap \mathcal{L}_{x'}\}. \quad \square$$

Incidences, again!

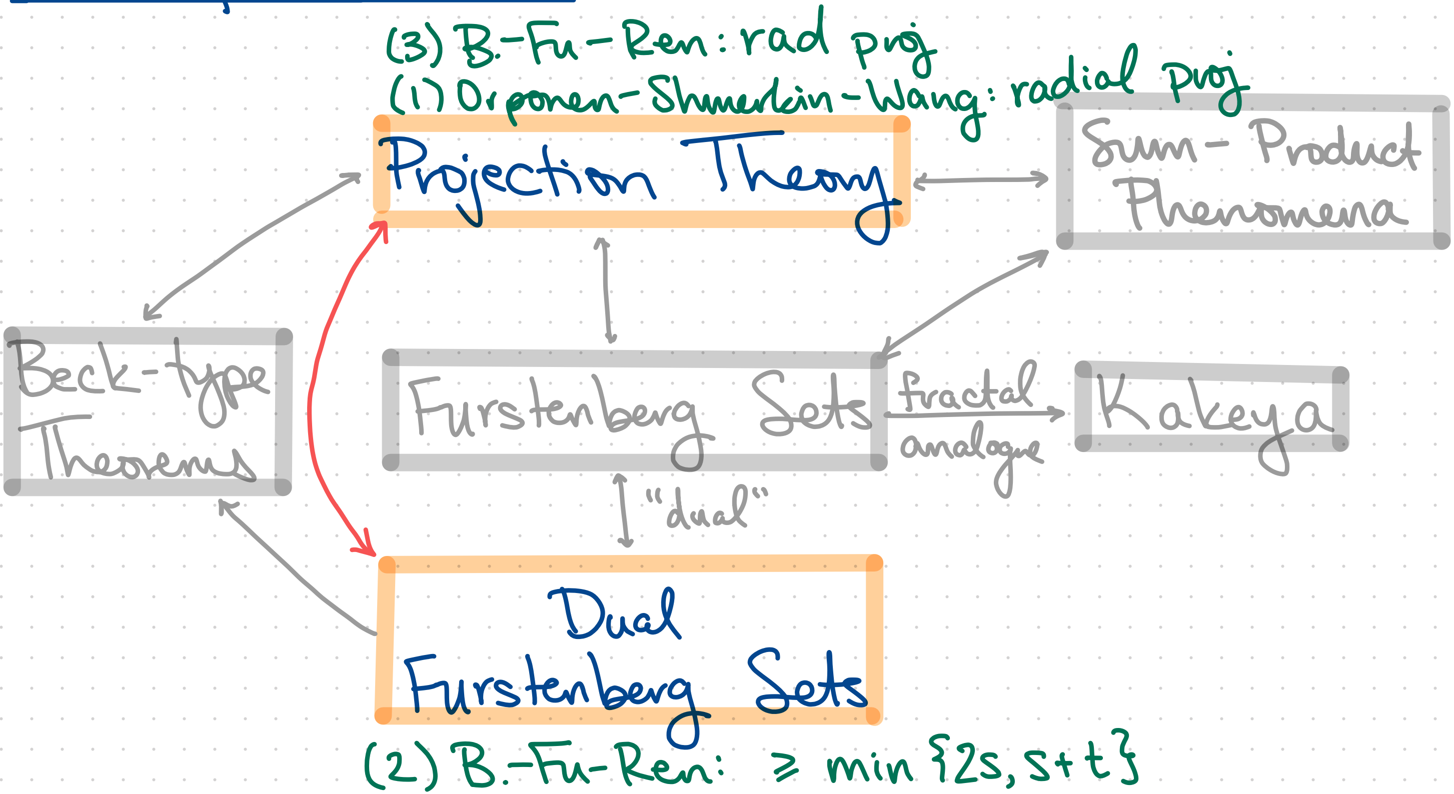
Pinned ~~Dual~~ Furstenberg Remarks

- Dual Furstenberg sets are <u>not</u> dual to Furstenberg sets if $n \geq 3$.

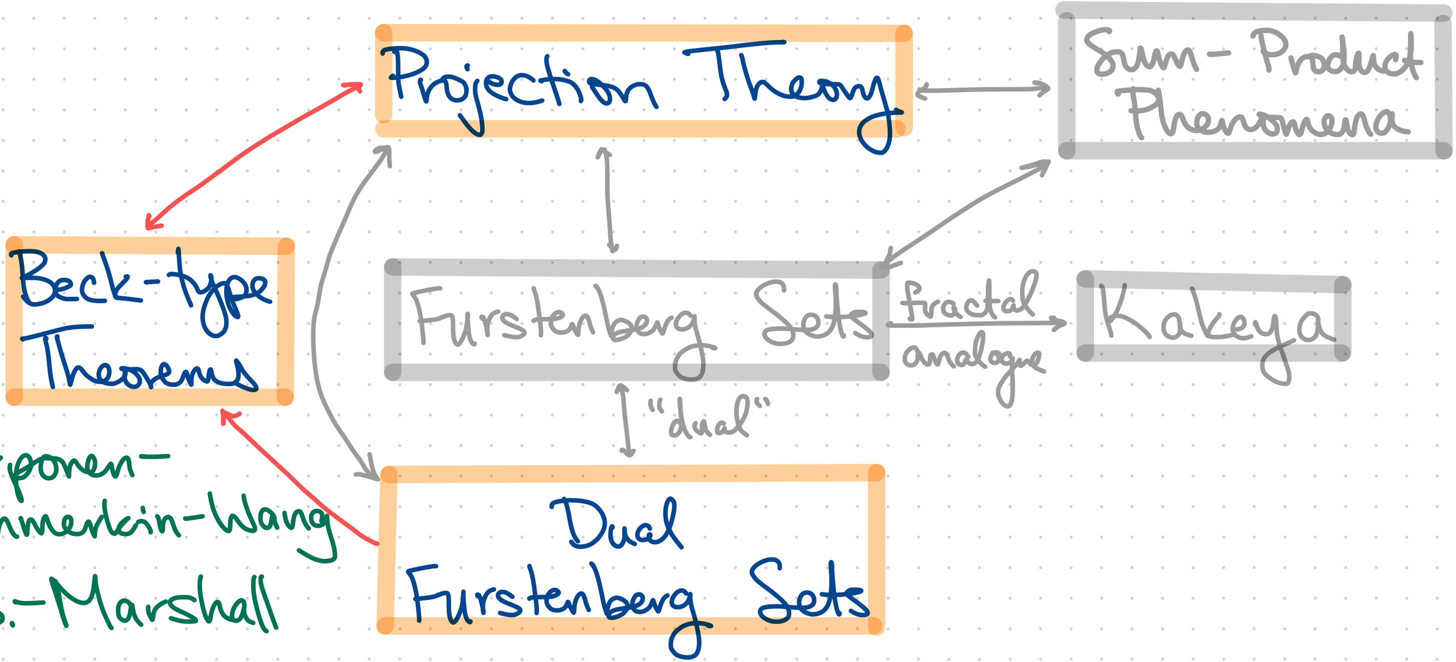
"Dual Furstenberg for Hyperplanes" dual to "Hyperplanar Furstenberg"

- That said, dual Furstenberg sets are interesting in their own right!

A Snapshot: Finale



A Snapshot: Finale



- Orponen-Shmerkin-Wang
- B.-Marshall
- Upcoming: B.-Ortiz-Zakharov

Thank You!