The Restriction Conjecture and Tomas–Stein

By Paige Bright and Dylan Chaussoy

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1 The Restriction Problem

Throughout the last half of the course, we have seen multiple ways in which the Fourier transform of a function relates to the function itself with regards to norms. For instance, we have the Hausdorff–Young inequality which states that for all $p \in [1, 2]$, $q = p'$, and $f \in L^p(\mathbb{R}^n)$,

$$\|\widehat{f}\|_{L^q} \leq \|f\|_{L^p(\mathbb{R}^n)}. \tag{1}$$

But what happens when one starts affecting various parts of this inequality? On the one hand, as we proved in the homework the only range of p where (1) can possible hold is for $p \in [1, 2]$, and by a scaling argument one can see that the only possible value of q must be $q = p'$. On the other hand, if instead of integrating over $\mathbb{R}^n$ we integrated over a compact set (say, a closed ball B), we obtain a similar string of inequalities as a natural corollary: for all $p \in [1, 2]$ and $q \leq p'$,

$$\|\widehat{f}\|_{L^q(B)} \lesssim \|\widehat{f}\|_{L^{p'}(B)} \leq \|f\|_{L^p(B)} \leq \|f\|_{L^p(\mathbb{R}^n)}. \tag{2}$$

Here, we used the fact that B is compact, to apply Hölder's inequality to see that (up to some constant depending on the radius of the ball), (2) holds for all $q \leq p'$.

Both of these statements are the best one can hope for in terms of getting inequalities of the form

$$\|\widehat{f}\|_{L^q(E)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

where E is allowed to be an arbitrary Borel subset of $\mathbb{R}^n$. Analogous questions, however, become much more interesting when we restrict the Fourier transform to some hypersurface S. This is the heart of *restriction theory.*

Definition 1

Let S be a hypersurface embedded in $\mathbb{R}^n$ and let $1 \leq p, q \leq \infty$. We say that $R_S(p \to q)$ holds if

$$\|\widehat{f}\|_{L^q(S)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

This is referred to as a *restriction theorem.*

Remark 2. *Note that it is not a priori clear that the above inequality even makes sense. Indeed, S is a set of measure 0 and one cannot, in general, meaningfully talk about the restriction of a L^q function to a set of measure 0 as they are equivalence classes of functions that are equal up to a measure 0 set. For that reason, when proving that statements of the form "$R_S(p \to q)$ holds," we first prove said estimates for every $f \in \mathscr{S}(\mathbb{R}^n)$. Then, by a density argument (akin to multiple we have seen throughout the course), one can then extend such estimates for every $f \in L^p(\mathbb{R}^n)$. For the arguments throughout this paper, one can notice that every step holds for Schwartz functions.*

Well, if the hypersurface is a hyperplane (e.g. a copy of $\mathbb{R}^{n-1}$), we can't hope for better than (1), but things become much more interesting when S has some *curvature*. For simplicity, throughout the rest of this document (unless stated otherwise) we let $S = \mathbb{S}^{n-1}$.

1.1 Some motivation

One might ask why one might be interested in such restriction theorems. A key motivator for this topic comes in the form of studying partial differential equations (PDEs), as solutions to various types of PDEs lie on nice submanifolds S. An example of such a PDE is the Helmholtz equation:

$$\Delta u + 4\pi^2 u = 0.$$

Notice that by applying the Fourier transform to both sides of this equation, we get that if u is a solution to the Helmhotz equation then

$$4\pi^2(-|\xi|^2 + 1)\widehat{u} = 0.$$

In particular, this implies that if $\widehat{u}(\xi) \neq 0$ then $|\xi|^2 = 1$, i.e. $\widehat{u}$ is supported on the unit sphere. As such, if we knew that $R_S(p \to q)$ held and we knew u was a solution to Helmholtz equation, one can obtain quantitative bounds on the L^q norm of $\widehat{u}$ in terms of the L^p norm of u. This heuristically allows us study how quickly $\widehat{u}$ decays at infinity (morally, the smaller q is, the faster $\widehat{u}$ has to decay).

We note here that one is also able to study *inhomogeneous* and nonlinear PDEs using methods that can be motivated by tools in restriction theory. Take for instance the meson equation

$$\begin{cases} i\partial_t u - \Delta_x u = \lambda |u|^2 u \\ u(0,x) = f(x) \end{cases}$$

where $\lambda \geq 0$ and u is a function of time $t \in \mathbb{R}$ and space $x \in \mathbb{R}^n$. Just as before, one can show that solutions to the homogeneous part of the meson equation are Fourier supported on the paraboloid. Using tools from restriction theory, one can show that for small enough $|\lambda|$ and nice enough initial conditions f, solutions to the meson equation exist for short periods of time. This is done in Notes 4 from a restriction theory course by Terence Tao [6].

Heuristically, it makes sense that restriction theory may be applicable (as solutions to this PDE are Fourier supported on the paraboloid). That said, the proofs that the authors know for proving the short time existence of solutions to the meson equation do not explicitly/directly apply restriction theorems of the form $R_S(p \to q)$. Rather, this statement follows from *techniques* used to study restriction theorems (such as those used to prove the Tomas–Stein theorem discussed in the final section).

1.2 The restriction conjecture

It is interesting to ask what assumptions on p and q are necessary and sufficient to imply $R_S(p \to q)$ holds. As it turns out, the assumptions which must *necessarily* hold can be motivated by a few examples when S is the unit sphere.

Let $d\sigma$ be the surface measure on S. Then, by standard duality arguments, one can show that $R_S(p \to q)$ is equivalent to the inequality

$$\|\widehat{gd\sigma}\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|g\|_{L^{q'}(S)} \qquad \text{for all} \quad g \in L^{q'}(S).$$

The above inequality is known as an *extension* theorem. Using this as a blackbox, notice that if one takes g to be

1 when restricted on the unit sphere, then $R_S(p \to q)$ holds only if

$$\|\widehat{d\sigma}\|_{L^{p'}(\mathbb{R}^n)} \lesssim 1.$$

Hence, a necessary condition for $R_S(p \to q)$ to hold is that $\widehat{d\sigma} \in L^{p'}$. Additionally, $|\widehat{d\sigma}(\xi)|$ behaves like $|\xi|^{-(n-1)/2}$ for large values of $|\xi|$. Thus, $R_S(p \to q)$ holds only if $\widehat{d\sigma} \in L^{p'}$, which holds only if

$$p' > \frac{2n}{n-1}, \qquad \text{or equivalently} \qquad p < \frac{2n}{n+1}. \tag{3}$$

On the other hand, we can show via a scaling argument that $R_S(p \to q)$ holds only if $p' \geq \frac{n+1}{n-1}q$. To see this, suppose that $R_S(p \to q)$ holds and let $\psi \in \mathscr{S}(\mathbb{R}^n)$ such that $\psi \sim 1$ near the origin (e.g. a Gaussian). Then, for $\lambda \gg 1$, define f_λ as

$$f_\lambda(x_1, \ldots, x_n) = \psi(x_1/\lambda^{1/2}, \ldots, x_{n-1}/\lambda^{1/2}, x_n/\lambda).$$

Like ψ, f_λ is also a bump function of height 1, but is concentrated on a tube with dimensions

$$\lambda^{1/2} \times \ldots \times \lambda^{1/2} \times \lambda.$$

Hence, $\|f_\lambda\|_p \approx \lambda^{(n+1)/(2p)}$. On the other hand,

$$\widehat{f_\lambda}(\xi_1, \ldots, \xi_{n-1}, \xi_n) = \lambda^{(n+1)/2}\widehat{\psi}(\lambda^{1/2}\xi_1, \ldots, \lambda^{1/2}\xi_{n-1}, \lambda\xi_n).$$

Thus, $\widehat{f_\lambda}$ has size $\lambda^{(n+1)/2}$ on a cap of S with measure about $\lambda^{-(n-1)/2}$. Combining this information with the assumption that $R_S(p \to q)$ holds, we see that for all $\lambda \gg 1$,

$$\lambda^{\frac{n+1}{2}}\lambda^{-\frac{n-1}{2q}} \lesssim \|\widehat{f_\lambda}\|_{L^{q'}(S)} \lesssim \|f_\lambda\|_p \lesssim \lambda^{\frac{n+1}{2p}}.$$

Thus, $R_S(p \to q)$ holds only if

$$p' \geq \frac{n+1}{n-1}q. \tag{4}$$

Equations (3) and (4) are also conjectured to be sufficient for $R_S(p \to q)$ to hold.

Conjecture 3 (The Restriction Conjecture)
If $S = \mathbb{S}^{n-1}$ is the unit sphere and $1 \leq p, q \leq \infty$, then $R_S(p \to q)$ holds if and only if (3) and (4) both hold.

We end this section with a few remarks. Firstly, the restriction problem has a natural analogue over compact hypersurfaces with certain curvature conditions that is also conjectured to hold. See [6, Notes 1] for more. Furthermore, the restriction conjecture for the sphere is known when $n = 2$ due to work of Fefferman and Stein [4] and is still open in higher dimensions. Lastly, the restriction conjecture is linked to another famous problem in harmonic analysis known as the Kakeya conjecture in the sense that the restriction conjecture implies the Kakeya conjecture due to work of Bourgain [3] (see also Exercise 17 in this blog of Tao). We don't have time to discuss Kakeya sets more here, but mention it for the interested reader.

2 The Tomas–Stein Theorem

Though we won't go into the full proof of the restriction conjecture in two dimensions, we outline the case of the restriction conjecture when $q = 2$ in all dimensions. This is known as the Tomas–Stein theorem, which states:

Theorem 4 (Tomas–Stein, '75)
if $1 \leq p \leq \frac{2(n+1)}{n+3}$, then $R_S(p \to 2)$ holds. I.e., in this range,

$$\|\widehat{f}\|_{L^2(S)} \lesssim \|f\|_{L^p}.$$

The case $p \neq \frac{2(n+1)}{(n+3)}$ was obtained by Tomas in '75 [7] using real interpolation, and in that same year Stein (unpublished) obtained the endpoint case using complex interpolation. The outline of this result follows the much more detailed notes of Wolff [8] and Tao [6].

Firstly, notice that when $q = 2$, we can use the fact that $L^2(\mathbb{R}^n)$ is an inner product space, and in particular,

$$\|\widehat{f}\|^2_{L^2(S)} = \int \widehat{f}(\xi)\widehat{f}(\xi)d\sigma(\xi).$$

Remark 5. *For simplicity, we drop the conjugates in this inner product. Note that the arguments below are not affected by this since $d\sigma$ is real-valued and symmetric.*

The first equality here follows as $|\widehat{f}|^2 = \widehat{f}\overline{\widehat{f}}$, and the second line follows as the surface measure on the sphere is real-valued and symmetric. Additionally, one of the key properties of the Fourier transform that we discussed in class is that the Fourier transform is a unitary operator on L^2, and in particular this implies

$$\int \widehat{f}(\xi)\widehat{f}(\xi)d\sigma(\xi) = \int f(\xi)(\widehat{\widehat{f}(\xi)}d\sigma(\xi))\widehat{\ } \leq \|f\|_p\|\widehat{\widehat{f}d\sigma}\|_{p'}.$$

The second inequality here follows from Hölder's inequality. Additionally, we can simplify the right hand side using convolution, noticing that $\widehat{\widehat{f}d\sigma} = f * \widehat{d\sigma}$. Hence, we have reduced the Tomas–Stein theorem to showing

$$\|f * \widehat{d\sigma}\|_{p'} \lesssim \|f\|_p. \tag{5}$$

The trick we described above is known as the TT^* method, which simplifies showing an operator T is bounded from L^p to L^2 to showing that TT^* is bounded from $L^{p'}$ to L^p. This method of attack makes very particular use of the fact that $q = 2$.

So now the question remains: how can we show (5)? Well, in it's current form, the inequality looks like a prime subject for applying a result like Young's convolution inequality, but in order to be able to apply that result, we need know what L^r space, if any, $\widehat{d\sigma}$ lies in. As it turns out, via the method of stationary phase (discussed in [6, Notes 1]), one can show the following decay of $\widehat{d\sigma}$:

Proposition 6 (Proposition 5.3 in [6])
If $d\sigma$ is the surface measure of the unit sphere, then for all $|\xi| \gg 1$, we have

$$\widehat{d\sigma}(\xi) = C\frac{e^{2\pi i|\xi|}}{|\xi|^{(n-1)/2}} + C\frac{e^{-2\pi i|\xi|}}{|\xi|^{(n-1)/2}} + O(|\xi|^{-n/2}).$$

In particular,

$$|\widehat{d\sigma}(\xi)| \lesssim |\xi|^{-(n-1)/2}. \tag{6}$$

In particular, this allows us to see that

$$\|f * \widehat{d\sigma}\|_{p'} \leq \||f| * |\xi|^{-(n-1)/2}\|_{p'}.$$

Unfortunately, while $\frac{1}{|\xi|^{(n-1)/2}}$ "almost" $L^{2n/(n-1)}$; it isn't quite. Hence, we cannot immediately apply Young's convolution inequality from class. That said, the Hardy–Littlewood–Sobolev inequality deals with this end case.

Lemma 7 (Hardy–Littlewood–Sobolev)
If $0 < \alpha < n$, $1 < p, q < \infty$, and $\frac{1}{q} + 1 = \frac{1}{p} + \frac{\alpha}{n}$, then for all $f \in L^p(\mathbb{R}^n)$,

$$\|f * |x|^{-\alpha}\|_q \lesssim \|f\|_p.$$

One proof of the Hardy–Littlewood–Sobolev inequality is contained in [6], and another can be seen in lecture notes from a course of Guth. Both proofs are essentially the same; the former proof uses distribution functions, while the latter directly makes use of the Hardy–Littlewood maximal operator. Notably, in both proofs, the authors make use of *restricted* weak-type estimates. In particular, an operator T mapping measurable functions on $\mathbb{R}^n$ to itself is of *restricted* weak-type (p, q) if for all Borel $E \subset \mathbb{R}^n$ and $\lambda > 0$,

$$|\{x : |T\mathbf{1}_E(x)| > \lambda\}| \lesssim |E|^{q/p} \lambda^{-q}.$$

While we do not explore this subject further here, the reader may be interested to know that, similar to weak-type estimates, restricted weak-type estimates can also be used to obtain interpolation theorems akin to the Marcinkiewicz interpolation theorem seen in class (which, in turn, can be used to imply Lemma 7).

In either case, using Lemma 7 with Proposition 6, we see that (5) holds for $p \le \frac{4n}{3n+1}$ (notably a smaller range than what is claimed in the statement of Tomas–Stein). To obtain the full range of p, we have to apply interpolation in a smarter way, which is precisely what the work of Tomas and Stein does.

The main part of the argument that was *extremely* lossy came in inequality (6). In particular, $d\sigma$ has a lot of oscillation that is not seen when we bound it by $|d\sigma|$. Hence, we need a way to exploit the amount of oscillation of $d\sigma$, though doing this over all of $\mathbb{R}^n$ at once is quite difficult to do. Rather, using Littlewood–Paley theory, we can break the support of $f * \widehat{d\sigma}$ into different annuli in $\mathbb{R}^n$. This is particularly a good idea to exploit oscillation, as the higher the frequency of ξ (i.e. the further it is from the origin), the more oscillation we expect.

So, let ϕ be a compactly supported radial bump function on $B^n(0, 1)$, and define

$$\psi_k(x) = \phi(2^{-k}x) - \phi(2^{-(k-1)}x).$$

Notice then that ψ_k has size 1 and is (more or less) supported when $|x| \sim 2^k$. Furthermore, notice that we have

$$1 = \phi(x) + \sum_{k>0} \psi_k(x) \qquad \forall x \in \mathbb{R}^n.$$

Therefore, we have

$$f * \widehat{d\sigma} = f * (\phi \widehat{d\sigma}) + \sum_{k>0} f * (\psi_k \widehat{d\sigma}).$$

Hence, by the triangle inequality,

$$\|f * \widehat{d\sigma}\|_{p'} \le \|f * (\phi \widehat{d\sigma})\|_{p'} + \sum_{k>0} \|f * (\psi_k \widehat{d\sigma})\|_{p'}. \tag{7}$$

This approach of decomposing frequency space into disjoint annuli is the key idea of Littlewood–Paley theory, which can be used to, for instance, study the time evolution of solutions to nonlinear PDE [1] and study Fourier multipliers [5].

Let's see why we've made progress towards proving the Tomas–Stein theorem. Firstly, the first term in the above sum can use Young's convolution inequality since ϕ is Schwartz and will decay fast enough to apply this result. To finish, we want bounds on each term in the sum of the form

$$\|\psi_k \widehat{d\sigma}\|_{p'} \lesssim 2^{-\epsilon k}\|f\|_p \tag{8}$$

for some $\epsilon > 0$. Such upper bounds imply that the sum is in fact summable and bounded by $\|f\|_p$ (up to some constant). Obtaining (8) for an arbitrary choice of p' is rather difficult, but as we know $1 \leq p \leq 2$, we have that $2 \leq p' \leq \infty$. Thus, (8) is a prime candidate for applying interpolation theory. In particular, one can find prove (8) when $p' = 2$ and ∞ using the fact that ϕ is Schwartz. Doing so and applying the Marcinkiewicz interpolation theorem shows that (8) holds for all $2 \leq p' \leq \infty$ if $p < \frac{2(n+1)}{n+3}$. In other words, this approach is able to obtain all but the endpoint case of the Tomas–Stein theorem.

To complete the proof of the endpoint case, instead of using the triangle inequality in (7), one can use Stein's complex interpolation theorem (while much of the above proof remains the same). This is all we will say here about what one needs to obtain the endpoint case. That said, one should not dismiss the endpoint case of the Tomas–Stein theorem as such endpoint cases can be quite important. Here is one way to see this. Let S be the paraboloid defined by the equation

$$S = \{(\xi, |\xi|^2) : \xi \in \mathbb{R}^{n-1}\}.$$

One can show via a scaling argument that, since S is not compact, $R_S(p \to q)$ can only hold if $q = \frac{n-1}{n+1}p'$. E.g., if $q = 2$, then p must be $\frac{2(n+1)}{n+3}$ for $R_S(p \to q)$ to hold; just like the endpoint case of Tomas–Stein for the sphere! In fact, $R_S(p \to q)$ holds for such p and q, and one can show this as a corollary of Tomas–Stein (Exercise 9.(vi) in a blog of Tao). In this sense, the endpoint case is of the utmost importance as it is the only case that holds!

Remark 8. *In fact, one can obtain the result for the paraboloid discussed above using the same approach as Tomas–Stein. This can more or less be seen by considering the Fourier decay of the surface measure on the paraboloid. That said, trying to find ways in which the paraboloid is significantly different than the sphere is an important topic for a vast number of problems in harmonic analysis and geometric measure theory.*

However, in regards to the restriction conjecture (Conjecture 3), the endpoint case (when $p = \frac{2n}{n+1}$) does *not* hold, and it furthermore doesn't seem like any weak-type estimate at the endpoint will hold either (see [2]).

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