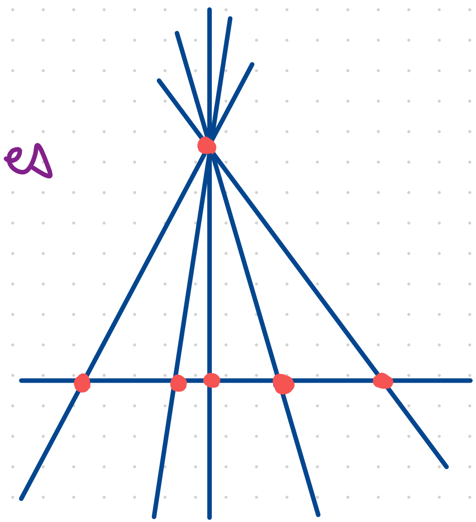
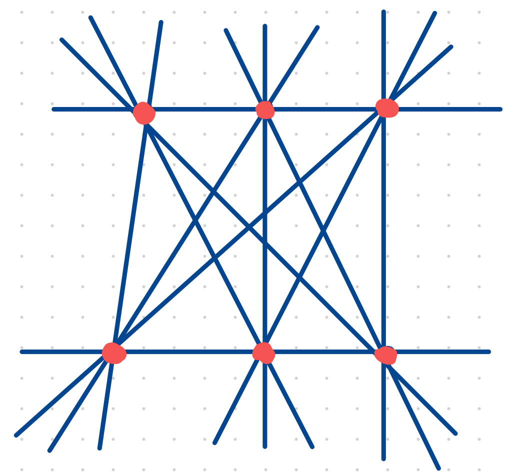
A Continuum Erdős-Beck Theorem

By: Paige Bright
joint work w/ Caleb Marshall

for HAPPY's
"Hello, World!" series





An Overview

- Let $X \subseteq \mathbb{R}^n$, and consider the set

$$\mathcal{L}(X) := \{\ell \in \underbrace{A(n,1)}_{\text{affine lines in } \mathbb{R}^n} : |X \cap \ell| \geq 2\}$$

$=$ "the lines spanned by X".

$\boxed{\text{Q: Given } X \text{ is large (cardinality or Hausdorff dimension), and satisfies (??), how large is } \mathcal{L}(X)\text{?}}$

- Orponen-Shmerkin-Wang (OSW), Ren, B.-Marshall
- Radial projections & (dual) Furstenberg Sets

About Me

$\underbrace{\text{MIT '24,}}_{\text{Undergrad}}$ $\underbrace{\text{UBC '25,}}_{\text{Izabella Łaba, Pablo Shmerkin, Josh Zahl}}$ $\underbrace{\text{MIT '30}}_{\text{Larry Guth}}$

- B.-Gan: "Exceptional set estimates for radial projections in $\mathbb{R}^n$" '22
- OSW: "Kaufmann & Falconer estimates for rad. proj..." '22
- UPenn's Study Guide Workshop '23
- B.-Marshall: "A continuum Erdös-Beck theorem" '24

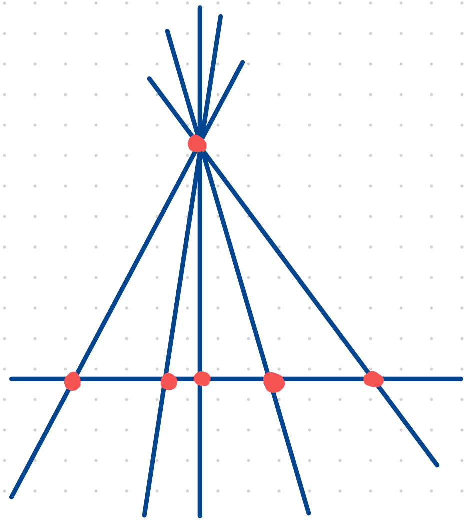
Beck's Theorem

Theorem [Beck '83]
Let $X \subseteq \mathbb{R}^n$ finite and $|X| = N$. If $|X \setminus \ell| \gtrsim N$ for all lines $\ell \in A(n,1)$, then $|\mathcal{L}(X)| \gtrsim N^2$.

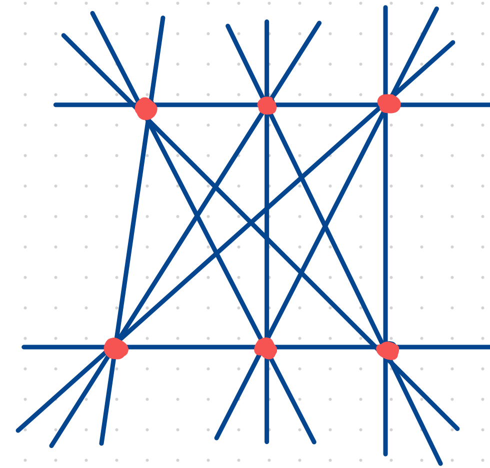
- I.e., if X does not give too much mass to any line, X will span $\gtrsim N^2 \approx \binom{N}{2}$ lines
- Proof follows from an application of Szemeredi-Trotter for k-rich lines (see Wiki).

Examples

$|\mathcal{L}(X)| = 1$



$|\mathcal{L}(X)| \sim |X|$



$|\mathcal{L}(X)| \sim |X|^2$

A Continuum Beck's Theorem

$\boxed{\begin{array}{l}\text{Theorem [Orponen-Shmerkin-Wang '22]}\\ \text{Let } X \subseteq \mathbb{R}^2 \text{ Borel. If } \dim(X \setminus \ell) = \dim X \ \forall \ell \in A(2,1),\\ \text{then } \dim \mathcal{L}(X) \geq \min\{2\dim X, 2\}.\end{array}}$

- I.e., if X does not give too much mass to any line, X will span many lines.
- Heuristic: Covering X by δ balls, $|X|_\delta \approx \delta^{-\dim X} := N$.

$$\Longrightarrow |\mathcal{L}(X)|_\delta \text{ "}\gtrsim\text{" } \binom{N}{2} \approx \delta^{-2\dim X} \Longrightarrow \dim \mathcal{L}(X) \geq 2\dim X.$$

Step 0: Many Large Bushes

- We may always write

$$\mathcal{L}(X) = \bigcup_{x \in X} \mathcal{L}_x \leftarrow \text{lines through } x \text{ in } \mathcal{L}(X).$$

- In fact, using

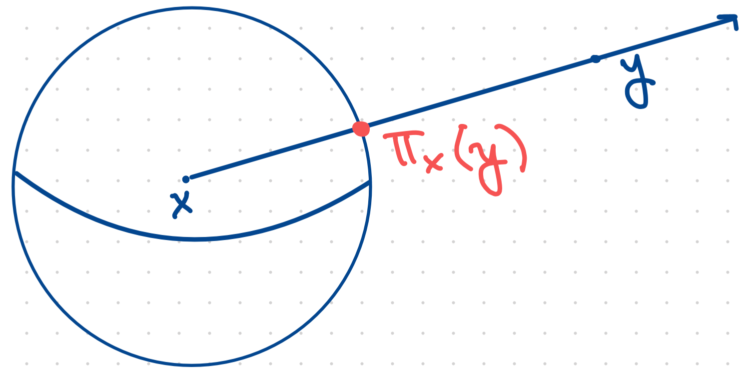
Radial Projections

we can show $\mathcal{L}_x$ is <u>often</u> large.

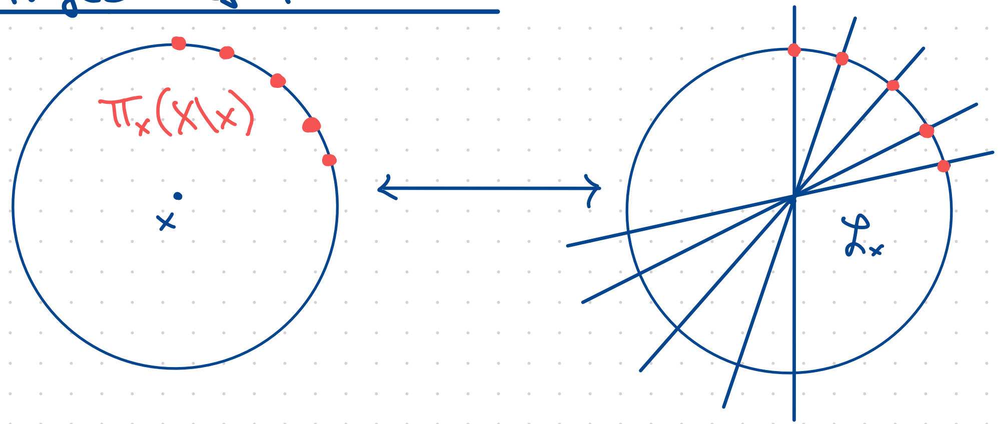
Step 1: Radial Projections

- Let $x, y \in \mathbb{R}^n$ with $x \neq y$. Then, define

$$\pi_x(y) = \frac{y - x}{|y - x|}.$$



Radial Projections & Lines



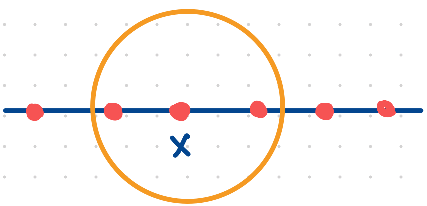
In particular, $\dim \pi_x(X\setminus\{x\}) = \dim \mathcal{L}_x$. So,

How often is $\mathcal{L}_x$ large? $\longleftrightarrow$ How often is $\pi_x(X)$ large?

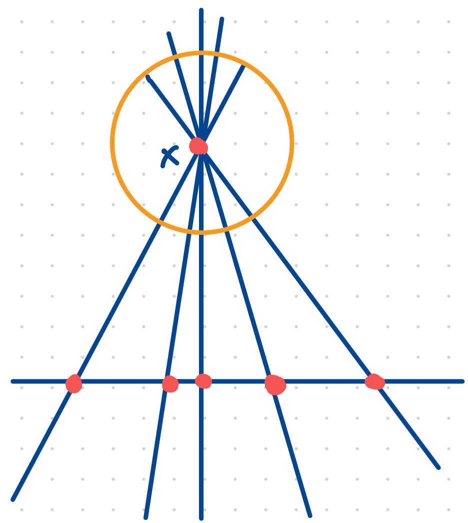
How often is $\pi_x(X)$ small?

Key Examples Revisited

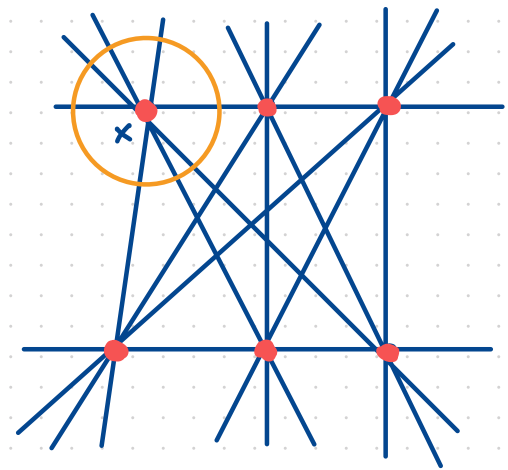
Q: How often/how large can $\pi_x(y)$ be ($y \subseteq \mathbb{R}^n$)?



$|\pi_x(X)| \sim 1$



$|\pi_x(X)| \sim |X|$ ($1\,x$)



$|\pi_x(X)| \sim |X| \;\; \forall x$

Radial Projections & Lines ctd.

$$\boxed{\begin{array}{l}\text{Theorem [Orponen-Shmerkin-Wang '22]} \\ \text{Given } X \not\subseteq \ell, \quad \sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X, \dim Y, 1\}.\end{array}}$$

Let $0 < \sigma < \min\{\dim X, 1\}$ and $B = \{x \in X : \dim \pi_x(X) < \sigma\}$.

Claim: $\dim X \setminus B = \dim X$.

Spse otherwise $\Rightarrow \dim X > \dim X \setminus B$ <u>and</u> $\dim B = \dim X$.

To apply Theorem, either

B is contained in a line or not.

Radial Projections & Lines ctd.

$\boxed{\text{Theorem [Orponen–Shmerkin–Wang '22] Given } X \not\subseteq \ell, \ \sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X, \dim Y, 1\}.}$

If $B \not\subseteq \ell$, then $\sup_{x \in B} \dim \pi_x(X) \geq \min\{\dim X, \dim B, 1\}$

$$= \min\{\dim X, 1\} > \sigma. \ \lightning$$

If $B \subseteq \ell$, then, $\dim X > \dim X \setminus B$

$$\geq \dim X \setminus \ell$$
$$= \dim X. \ \lightning$$

Hence, $\dim X \setminus B = \dim X$, and $\forall \ x \in X \setminus B$,

$$\dim \pi_x(X) = \dim \mathcal{L}_x \geq \sigma.$$

Motivating Dual Furstenberg

Theorem [Orponen–Shmerkin–Wang '22] If $\dim X \setminus \ell = \dim X \ \forall \ell$, then $\dim \mathcal{L}(X) \geq 2\min\{\dim X, 1\}$.

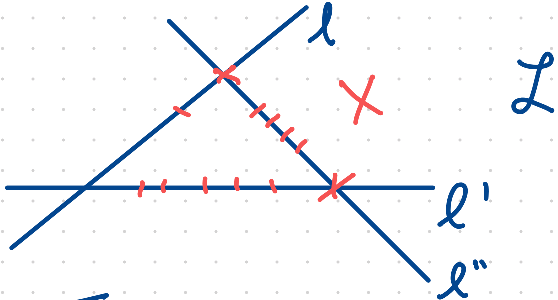
Hence, at this point, we have shown

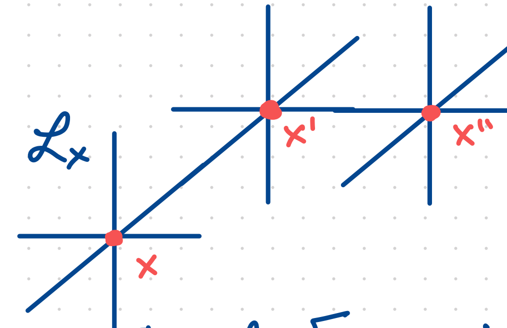
$$(*) \quad \mathcal{L}(X) \supseteq \bigcup_{\underbrace{x \in X \setminus B}_{\geq \dim X\text{-dimensional} \geq \sigma\text{-dim}}} \overbrace{\mathcal{L}_x}^{\geq \sigma\text{-dim}}$$

Theorem [B.–Fu–Ren '24] $\dim \bigcup_{\underbrace{x \in X}_{\geq s}} \overbrace{\mathcal{L}_x}^{\geq t} \geq s + \min\{s, t\}$.

Therefore, $\dim \mathcal{L}(X) \geq 2\sigma$. Send $\sigma \nearrow \min\{\dim X, 1\}$. $\square$

Furstenberg vs Dual Furstenberg





Furstenberg $\xleftrightarrow[\text{if } n=2]{\text{dual}}$ Dual Furstenberg

- $X \subseteq \mathbb{R}^n$, $\mathcal{L} \subseteq A(n,1)$ s.t.
  - $\dim \mathcal{L} \geq t$
  - $\dim X \cap \ell \geq s \quad \forall \ell \in \mathcal{L}$

$\Longrightarrow \dim X \geq ?$

- $\mathcal{L} \subseteq A(n,1)$, $X \subseteq \mathbb{R}^n$ s.t.
  - $\dim X \geq s$
  - $\dim \mathcal{L}_x \geq t \quad \forall x \in X$

$\Longrightarrow \dim \mathcal{L} \geq ?$

Line Sets in $\mathbb{R}^n$

$$\boxed{\text{Theorem [Ren '23] Let } X \not\subseteq P^k. \text{ Then, } \sup_{x \in X} \dim \pi_x(Y \setminus \{x\}) \geq \min\{\dim X, \dim Y, k\}}$$

$\Downarrow$ [B.-Fu-Ren '24]

$$\boxed{\text{Cor: Let } X \subseteq \mathbb{R}^n \text{ s.t. } \dim X \setminus P^k = \dim X \ \forall P^k \in \mathcal{A}(n,k). \text{ Then, } \dim \mathcal{L}(X) \geq \min\{2\dim X, 2k\}.}$$

Erdős-Beck Theorem

$\boxed{\begin{array}{l}\text{Theorem [Erdős-Beck]}\\ \text{Let } X \subseteq \mathbb{R}^n \text{ finite and } |X| = N. \text{ If } |X \setminus \ell| \gtrsim t \text{ for all}\\ \text{lines } \ell \in A(n,1), \text{ then } |\mathcal{L}(X)| \gtrsim Nt. \ (0 < t \leq N)\end{array}}$

- I.e., if X does not give <u>too</u> much mass to any line, X will span $\gtrsim Nt$ lines

$\boxed{\begin{array}{l}\text{Q: Is there a continuum analogue of this result?}\\ \text{A: B.-Marshall '24: Yes!}\end{array}}$

A Continuum Erdős-Beck Theorem

Theorem [B.-Marshall '24]

Let $X \subseteq \mathbb{R}^n$ & fix $k \in \{1, \dots, n-1\}$.

1) If $\dim X \setminus P^k = \dim X \;\; \forall P^k$

$$\Rightarrow \dim \mathcal{L}(X) \geq \min\{2\dim X, 2k\}$$

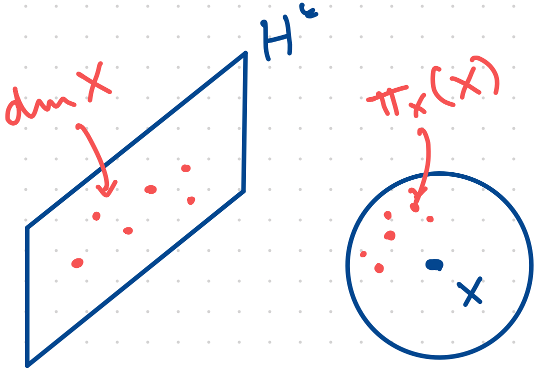
2) If not, let $0 < t < \dim X$ be s.t. $\dim X \setminus P^k \geq t$. $\forall P^k$

$$\Longrightarrow \dim \mathcal{L}(X) \geq \dim X + t.$$

Proof Outline

In 2), $\exists\, H^k \in A(n,k)$ with $\dim X \setminus H^k < \dim X$

$$\Longrightarrow \dim X \cap H^k = \dim X.$$



For all $x \in X \setminus H^k$,

$$\dim \pi_x(X) = \dim \mathcal{L}_x \geq \dim X.$$

$$\Longrightarrow \mathcal{L}(X) \supseteq \bigcup_{x \in \underbrace{X \setminus H^k}_{\geq t}} \overset{\geq \dim X}{\mathcal{L}_x}$$

By B.-Fu-Ren'24, $\dim \mathcal{L}(X) \geq \dim X + \min\{\cancel{\dim} X, t\}$. $\square$

Thank You!