Pinned Dot Product Set Estimates

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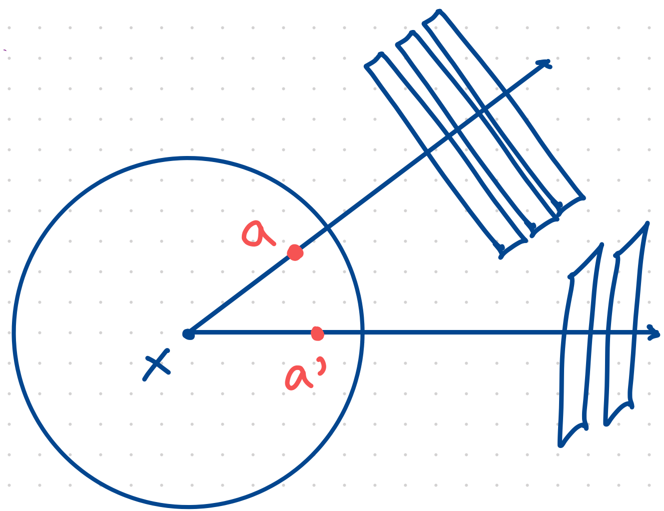
joint w/ C. Marshall + S. Senger



Falconer Distance Conjecture

Given a compact set $A \subseteq \mathbb{R}^n$, define the *distance set*

$$\Delta(A) = \{ |a-y| : a, y \in A \} \subseteq \mathbb{R}.$$

Conjecture (Falconer):

$$\dim A > \frac{n}{2} \Longrightarrow |\Delta(A)| > 0.$$

Best Known Lowerbounds:

$$\dim A \geq \begin{cases} 5/4, & n=2 \\ \frac{n}{2} + \frac{1}{4} - \frac{1}{8n+4}, & n \geq 3 \end{cases}$$

[Guth-Iosevich-Ou-Wang, '20]

[Du-Ou-Ren-Zhang, '23]

In fact, they obtained *pinned results*.

Falconer Distance Conjecture ctd.

Thm: If $\dim A \subseteq \mathbb{R}^n$ and

$$\dim A \geq \begin{cases} 5/4, & n=2 \\ \frac{n}{2} + \frac{1}{4} - \frac{1}{8n+4}, & n \geq 3 \end{cases}$$

[Guth-Iosevich-Ou-Wang, '20]

[Du-Ou-Ren-Zhang, '23],

then $\exists\, a \in A$ with

$$|\{|a-y| : y \in A\}| > 0.$$

Remarks: • Other notions of "size" + "distance"

↪ See Iosevich-Taylor-Uriarte-Tuero

• Applies projection theory (e.g. Orponen, Ren).

A Dot Product Variant

Let $A \subseteq \mathbb{R}^n$ Borel, $a \in \mathbb{R}^n$, and let

$$\pi^a(A) = \{ a \cdot y : y \in A \}.$$

Theorem [B.-Marshall-Senger '24]

Let $A \subseteq \mathbb{R}^n$ Borel. Then, if

- $\dim A > \frac{n+2}{2} \Longrightarrow \exists a \in A$ with $[\pi^a(A)]^\circ \neq \emptyset$.
- $\dim A > \frac{n+1}{2} \Longrightarrow \exists a \in A$ with $|\pi^a(A)| > 0$.
- $\dim A > \frac{n+u}{2} \Longrightarrow \exists a \in A$ with $\dim \pi^a(A) \geq u$.

A Dot Product Variant: Remarks

$\boxed{\begin{array}{l}\text{Theorem [B.-Marshall-Senger '24]}\\ \text{Let } A\subseteq\mathbb{R}^n \text{ Borel. Then, if}\\ \quad\bullet\ \dim A>\frac{n+2}{2}\implies\exists a\in A \text{ with } (\Pi^a(A))^\circ\neq\emptyset.\\ (2)\bullet\ \dim A>\frac{n+1}{2}\implies\exists a\in A \text{ with } |\Pi^a(A)|>0\\ \quad\bullet\ \dim A>\frac{n+u}{2}\implies\exists a\in A \text{ with } \dim\Pi^a(A)\geq u.\end{array}}$

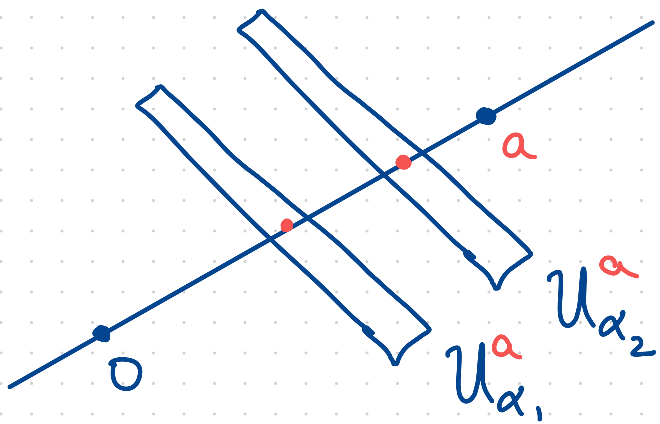
- (2) implied by Iosevich-Taylor-Uriarte-Tuero.

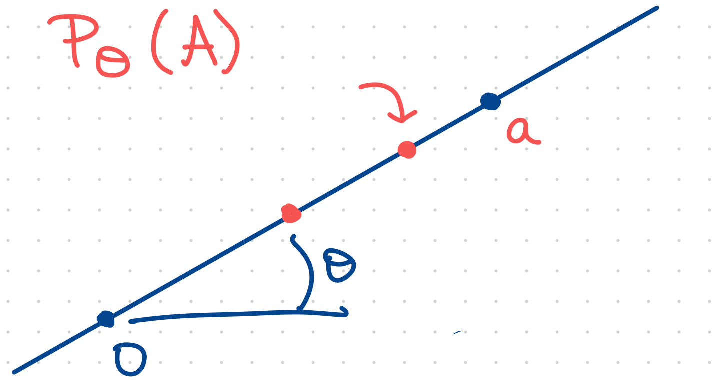
$\hookrightarrow$ Will focus on (2) for proof outline.

- Same proof methodology for all 3.

The Geometry of Dot Products

Let $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, and $U^a_\alpha = \{y \in \mathbb{R}^n : a \cdot y = \alpha\}$.





$\alpha \in \Pi^a(A) \iff U^a_\alpha \cap A \neq \emptyset$

$\mathcal{H}^s(\Pi^a(A)) \sim_{a,s} \mathcal{H}^s(P_\theta(A))$

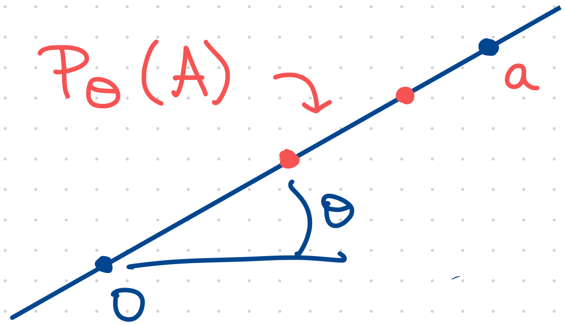
The Geometry of Dot Products

$$\mathcal{H}^s(\pi^a(A)) \sim_{a,s} \mathcal{H}^s(P_\theta(A)), \quad \theta = \frac{a}{|a|}.$$

Key Lemma

- $\pi^a(A)^\circ \neq \emptyset$ if and only if $(P_\theta(A))^\circ \neq \emptyset$.
- $|\pi^a(A)| \sim |P_\theta(A)|$,
- $\dim \pi^a(A) = \dim P_\theta(A)$.

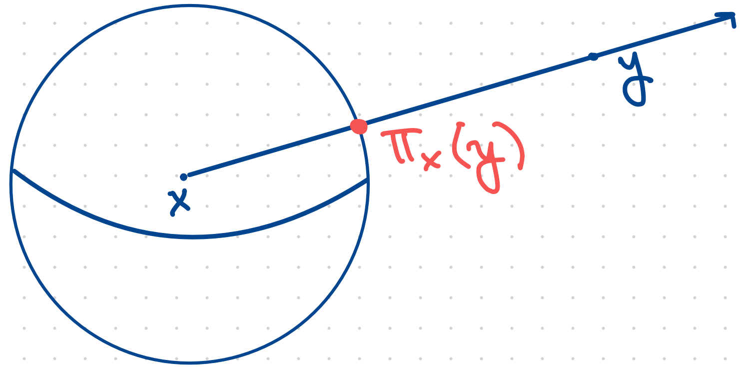
Hence, we have reduced the problem to:



1) How many directions of pins, $\theta = \frac{a}{|a|}$, are there?
↳ radial projection!
2) How many directions are "bad"?
↳ Exceptional Set Estimates!

Radial Projections

$$\pi_x(y) = \frac{y-x}{|y-x|}.$$



$\Rightarrow$ [POSSIBLE Θ] $= \pi_0(A)$, $\dim \pi_0(A) \geq \dim A - 1$.

Exceptional Set Estimates

$\boxed{\begin{array}{l}\text{Thm: [Marstrand Projection Theorem]}\\ \text{Given } A \subseteq \mathbb{R}^n \text{ Borel,}\\ \qquad \dim P_\theta(A) = \min\{\dim A, 1\} \quad \text{a.e. } \theta \in S^{n-1}.\end{array}}$

$\boxed{\begin{array}{l}\text{Thm: [Falconer] If } \dim A > 1, \text{ then}\\ \quad \dim(E(A) := \{\theta \in S^{n-1} : |P_\theta(A)| = 0\}) \leq n - \dim A.\end{array}}$

- Similar statements for different "sizes".

Concluding Proof

Notice, since $\begin{cases} \dim \pi_0(A) \geq \dim A - 1 \\ \dim E(A) \leq n - \dim A \end{cases}$, if

$\dim A > \frac{n+1}{2}$, $\exists\, \theta \in \pi_0(A) \setminus E(A)$! i.e., $\exists a$,

$$|P_\theta(A)| \sim |\pi^a(A)| > 0. \qquad \square$$

Remarks

- In fact, full dimensional set of pins $a \in A$.
- Same proof for all 3 senses of "size".
- Key idea: If [POSSIBLE θ] is larger than [BAD θ], then the result holds.

Q: What if we have better rad proj results?

Answer 1

Let $A \subseteq \mathbb{R}^n$, $a, x \in \mathbb{R}^n$, and let

$$\pi_x^a(A) := \{(a-x)\cdot y : y \in A\}.$$

- [POSSIBLE θ] $= \pi_x(A)$, [BAD θ] $= E(A)$.
- Recent rad. proj results (Ren):

Thm [Ren '23] Let $X, A \subseteq \mathbb{R}^n$ s.t. $\dim X \setminus P^k = \dim X$ $\forall P^k$, then $\forall \varepsilon > 0$, $E_\varepsilon = \{x \in X : \dim \pi_x(A) < \min\{\dim X, \dim A, k\} - \varepsilon\}$ satisfies $\dim X \setminus E_\varepsilon = \dim X$.

Answer 1 ctd.

Thm [B.-Marshall-Senger '24]: Let $A, X \subseteq \mathbb{R}^n$ satisfy $\dim X \geq \dim A := s$ and $\dim X \setminus P^k = \dim X \;\; \forall P^k$.

Then, for sufficiently large k,

- $\dim A > \frac{n+1}{2} \Longrightarrow \exists a, x$ s.t. $[\pi_x^a(A)]^\circ \neq \emptyset$.
- $\dim A > \frac{n}{2} \Longrightarrow \exists a, x$ s.t. $|\pi_x^a(A)| > 0$.
- $\dim A > \frac{n+1-u}{2} \Longrightarrow \exists a, x$ s.t. $\dim \pi_x^a(A) \geq u$.

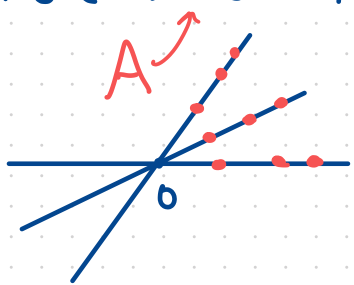
Generally / Heuristically: Better rad proj + exceptional set estimates $\Longrightarrow$ better dot product results!

"Answer" 2

Q: Can we get $\frac{n}{2}$ for untranslated dot prods?

Biggest Hurdle: Does there exist $A \subseteq \mathbb{R}^n$ s.t.

$\dim \pi_0(A) = \dim A - 1$ AND $\dim E(A) = n - \dim A$?



Conj: [B-Marshall-Senger] If $A \subseteq \mathbb{R}^2$, $\dim A > 1$, then $\dim \pi_0(A) > \dim E(A)$.

Thank You!