Recent Developments in Radial Projections

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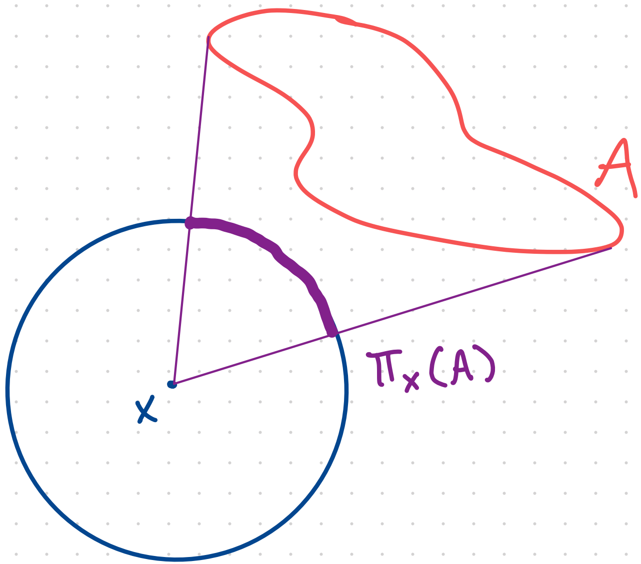


- Liu & Lund-Pham-Thu
- B.-Gan
- Orponen-Shmerkin-Wang (OSW)
- Ren

The story for radial projections is more complicated than for orthogonal projections I find. Still, the applications have proven fruitful

How large is $\{ * \in \mathbb{X} : \text{size of } \pi_*(A) < s \}$?

↳ "Exceptional Set Estimates"

↳ Rmk: Slides for E.S.E. for orthogonal proj. available.

Here, I am being vague about "$\mathbb{X}$" and "size":

- π_x : Radial Projection
- $\mathbb{X}$: $\mathbb{R}^n$ or $\mathbb{F}_q^n$

$$\pi_x(y) = \frac{y-x}{|y-x|} \qquad \pi_x(y) = \text{line through } x \text{ \& } y.$$

- Size: Hausdorff dimension, measure, or cardinality.

Background: Visibility

Let $A \subseteq \mathbb{R}^2$ Borel, with $\mathcal{H}^s(A)$ finite & nonzero for some $s > 1$.

$$\Longrightarrow \underbrace{\mathcal{H}^s(\pi_x(A)) > 0}_{\text{"visible from } \mathcal{H}^s\text{"}} \quad \text{a.e. } x \in \mathbb{R}^2. \quad \text{(Marstrand)}$$

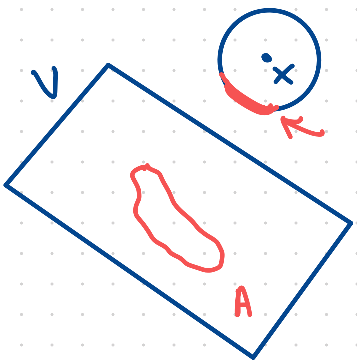
Similarly, for A s.t. $\dim A > n-1$, (Mattila-Orponen + Orponen)

$$\dim \{ x \in \mathbb{R}^n : \underbrace{\mathcal{H}^{n-1}(\pi_x(A)) = 0}_{\text{"invisible"}} \} \leq 2(n-1) - \dim A.$$

Bochen Liu then showed (if $\dim A \in (n-2, n-1]$),

$$\dim \{ x \in \mathbb{R}^n : \dim(\pi_x(A)) < \dim A \} \leq 2(n-1) - \dim A.$$

Note, this estimate is sharp. Let $A \subseteq V \in G(n,n-1)$, $\dim A = n-1$.



If $x \notin V$,

$\dim \pi_x(A) = \dim A$

$\Downarrow$

$\{x : \dim \pi_x(A) < \dim A\} \subseteq V$.

Liu thus conjectured: Given $\dim A \in (k-1,k]$, $k \in \mathbb{N}$,

$$\boxed{\dim\{x \in \mathbb{R}^n : \dim \pi_x(A) < \dim A\} \leq k.}$$

Rmk: This was resolved by B.–Gan & Orponen–Shmerkin–Wang.

Background: Finite Field version

Around the same time, Lund–Pham–Thu studied r. proj over $\mathbb{F}_q$.

Let $A \subseteq \mathbb{F}_q^n$, $|A| = q^a$, and $0 \le s \le n-1$.

$$\boxed{\begin{array}{l}\text{Thm: Given } |A| \gtrsim q^{n-1}, \\ \quad \#\{x \in \mathbb{F}_q^n : |\pi_x(A)| \le q^s\} \lesssim q^{n-1+s-a}.\end{array}}$$

$$\Downarrow$$

Thm ~~Conjecture~~: Given $\dim A > n-1$, $0 \le s \le n-1$,

$\dim\{x \in \mathbb{R}^n : \dim \pi_x(A) < s\} \le n-1+s-\dim A$.

$n=2$: Orponen–Shmerkin

arb. n: B.-Gan and Orponen–Shmerkin–Wang.

In fact more generally, we have

Thm: B.-Lund-Pham: Given $q^{k-1} < |A| \le q^k$

$$\#\{x \in \mathbb{F}_q^n : |\pi_x(A)| \le q^s\} \lessapprox q^{k+s-a}.$$

$\updownarrow$ analogously:

B-Gan + OSW

Thm: Given $\dim A \in (k, k+1]$, $0 \le s \le k$,

$$\dim\{x \in \mathbb{R}^n : \dim \pi_x(A) < s\} \le k + s - \dim A.$$

$\Downarrow$ B.-Gan

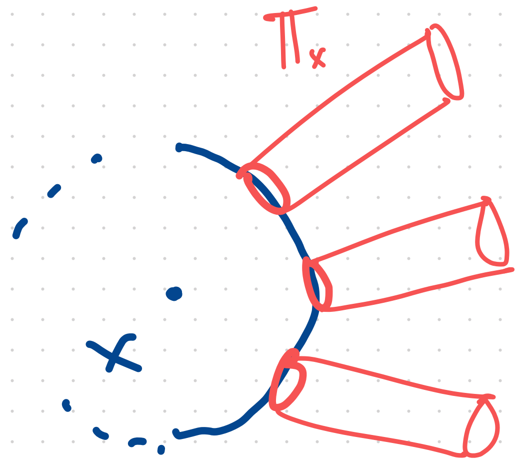
B.-Gan + OSW

Thm: Given $\dim A \in (k-1, k]$,

$$\dim\{x \in \mathbb{R}^n : \dim \pi_x(A) < \dim A\} \le k.$$

Proof Idea (B.-Gan): Use the high-low method!

Discretize, and for all $x \in \{x : \dim \pi_x(A) < s\}$, cover $\pi_x(A)$ by δ-balls ($\lesssim \delta^{-s}$ many). Then, consider its fibres.



$$f_x = \sum_{T \in \Pi_x} \varphi_T \quad , \quad f = \sum_x f_x .$$

Then, consider $\int |f|^2$.

For more, see Exceptional Set Est. for orthogonal proj.

Thm [OSW] Let $\emptyset \neq X, Y \subseteq \mathbb{R}^2$ Borel.

- If $X \not\subseteq$ any line, $\Rightarrow \sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X, \dim Y, 1\}$.
- If $\dim Y > 1$, $\Rightarrow \sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X + \dim Y - 1, 1\}$.

Cor: Given Y, $\dim Y \leq 1 \Rightarrow \dim\{x \in \mathbb{R}^2 : \dim \pi_x(Y) < \dim Y\} \leq 1$.

Pf. Apse FTSOC $\exists \varepsilon > 0$ s.t.

$$\dim \underbrace{\{x \in \mathbb{R}^2 : \dim \pi_x(Y) < \dim Y - \varepsilon\}}_{X} > 1.$$

$$\Rightarrow X \not\subseteq \ell, \text{ so, } \sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X, \dim Y, 1\}$$
$$= \dim Y > \dim Y - \varepsilon \quad \rightarrow\leftarrow.$$

$\square$

Thm [OSW] Let $\emptyset \neq X, Y \subseteq \mathbb{R}^2$ Borel.

- If $X \not\subseteq$ any line, $\Rightarrow \sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X, \dim Y, 1\}$.
- If $\dim Y > 1$, $\Rightarrow \sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X + \dim Y - 1, 1\}$.

Cor: $\dim Y > 1$, then for $0 \leq s < 1$,

$$\dim \underbrace{\{x \in \mathbb{R}^2 : \dim \pi_x(Y) < s\}}_{X} \leq \max\{1 + s - \dim Y, 0\}$$

Pf: FTSOC, suppose $\dim X > \max\{1 + s - \dim Y, 0\} \Rightarrow X \neq \emptyset$.

So, since $\dim Y > 1$,

$$\sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X + \dim Y - 1, 1\}$$
$$> \min\{s, 1\} = s \quad \rightarrow\leftarrow . \quad \square$$

Thm [OSW] Let $\emptyset \neq X, Y \subseteq \mathbb{R}^2$ Borel.

- If $X \not\subseteq$ any line, $\Rightarrow \sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X, \dim Y, 1\}$.

Rmk: Uses ε-improvement for Furstenberg sets in $\mathbb{R}^2$.

Thm: OSW If $k-1 < \dim X \leq k$, but $X \not\subseteq$ any k-plane, given $\dim Y > k - 1/k - \eta$,

$$\sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X, \dim Y\}.$$

Thm: OSW If $k-1 < \dim X \leq k$, but $X \not\subseteq$ any k-plane, given $\dim Y > k - 1/k - \eta$,

$$\sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X, \dim Y\}.$$

Conj: Can have $\dim Y > k-1$.

Then, Ren generalized the Furstenberg result, obtaining

Thm Ren: Given $\emptyset \neq X$, $X \not\subseteq$ any k-plane,

$$\sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X, \dim Y, k\}.$$

$$\boxed{\text{Thm Ren: Given } \emptyset \neq X,\ X \not\subseteq \text{ any } k\text{-plane}, \quad \sup_{x \in X} \dim \pi_x(Y) \geq \min\{\dim X, \dim Y, k\}.}$$

Remarks:

- This implies (via the same proof outline):

$$\boxed{\text{Cor: Given } Y,\ \dim Y \leq k \Rightarrow \dim\{x \in \mathbb{R}^n : \dim \pi_x(Y) < \dim Y\} \leq k.}$$

- This result can be utilized for the Falconer Distance problem (see Du-Ou-Ren-Zhang).

Further applications/papers to look into.

- Generalizing Thm 1.2 of OSW (B.-Fu-Ren)
- Falconer Distance problem Du-Ou-Ren-Zhang
- Applications to orthogonal projections + Furstenberg
  - Orponen-Shmerkin's ABC Sum Product Paper
  - Ren-Wang's Furstenberg set Paper
- Sets of lines spanned by points
  - OSW, B.-Marshall

Sets of lines spanned by points?

See my talk (to occur later this year) for the Harmonic Analysis People's Presentations on YouTube's (HAPPY's) "Hello, World!" Series.

Thank You!