Analysis Summer UROP

Paige Dote

with Professor Guth and Yuqiu Fu

Contents

Chapter 1

Pre-Project

1.1 June 03-09

June 03

Hello Professor Guth! Hope all is well with you. Yuqiu and I met for the first time today to begin planning out a schedule for reading this summer. The current tentative plan is to meet weekly (on Thursdays), and to read 2 chapters a week. Thus, over the next week the expectation is to have read through chapters 1 and 2. As we go through the chapters I will work on completing the exercises, and if time allows over the week, the extra problems at the end of each chapter. I will further use this LaTeXdocument to show solutions and comments on exercises done thus far. I was able to get a bit of a head-start on chapter 1, and thus some of these exercises are included in today's update.

Exercise 1.1: This one and the next are straightforward, using the distributive and commutative properties of addition.

$$\begin{aligned}\sum_{k=1}^{N} Ca_k &= Ca_1 + Ca_2 + \cdots + Ca_N \\ &= C(a_1 + a_2 + \cdots + a_N) \\ &= C\sum_{k=1}^{N} a_k.\end{aligned}$$

□

Exercise 1.2:

$$\begin{aligned}\sum_{k=1}^{N}(a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_N + b_N) \\ &= (a_1 + a_2 + \cdots + a_N) + (b_1 + b_2 + \cdots + b_N) \\ &= \sum_{k=1}^{N} a_k + \sum_{k=1}^{N} b_k.\end{aligned}$$

□

Exercise 1.3: Yuqiu very astutely pointed out that this exercise is flawed– equality also holds if $a_k = \alpha b_k$ for all

k and $\alpha \in \mathbb{R}$ such that $\alpha \geq 0$. This is shown below:

$$\sum_{k=1}^{N} a_k b_k = \sum_{k=1}^{N} \alpha b_k^2 = \left(\sum_{k=1}^{N} \alpha^2 b_k^2\right)^{1/2} \left(\sum_{k=1}^{N} b_k^2\right)^{1/2} = \left(\sum_{k=1}^{N} a_k^2\right)^{1/2} \left(\sum_{k=1}^{N} b_k^2\right)^{1/2}. \tag{1.1}$$

Thus, amending this question, I will show that the Cauchy-Schwarz inequality is an equality if and only if $a_k = \alpha b_k$ for $\alpha \in [0, \infty]$.

As we have shown in (1), if $a_k = \alpha b_k$, then $\sum_{k=1}^{N} a_k b_k = \left(\sum_{k=1}^{N} a_k^2\right)^{1/2} \left(\sum_{k=1}^{N} b_k^2\right)^{1/2}$. Next, I will show that if $a_k \neq \alpha b_k$ for all $\alpha \in [0, \infty]$ then equality does not hold.

June 04

I begin by finishing Exercise 1.3. Given $a_k \neq \alpha b_k$ for all $\alpha \in [0, \infty]$,

$$\sum_{k=1}^{N} (a_k - \alpha b_k)^2 > 0$$

for all $\alpha \in [0, \infty]$. Expanding this expression, we get that

$$0 < \sum_{k=1}^{N} a_k^2 - 2\alpha \sum_{k=1}^{N} a_k b_k + \alpha^2 \sum_{k=1}^{N} b_k^2.$$

Let $\alpha = \frac{\sum_{k=1}^{N} a_k b_k}{\sum_{k=1}^{N} b_k^2}$. Then,

$$\begin{aligned} 0 &< \sum_{k=1}^{N} a_k^2 - 2\left(\frac{\sum_{k=1}^{N} a_k b_k}{\sum_{k=1}^{N} b_k^2}\right) \sum_{k=1}^{N} a_k b_k + \left(\frac{\sum_{k=1}^{N} a_k b_k}{\sum_{k=1}^{N} b_k^2}\right)^2 \sum_{k=1}^{N} b_k^2 \\ &= \sum_{k=1}^{N} a_k^2 - \frac{\left(\sum_{k=1}^{N} a_k b_k\right)^2}{\sum_{k=1}^{N} b_k^2}. \end{aligned}$$

Thus, we can conclude

$$\sum_{k=1}^{N} a_k b_k < \left(\sum_{k=1}^{N} a_k^2\right)^{1/2} \left(\sum_{k=1}^{N} b_k^2\right)^{1/2}.$$

□

Exercise 1.4: We actually did this problem in 18.102, though I will nonetheless include the solution here. Consider the sequences of real numbers a_k, b_k. We may assume that both $\sum_{k=1}^{N} a_k^p$ and $\sum_{k=1}^{N} b_k^{p'}$ are nonzero, as otherwise Hölder's Inequality is trivial. We replace a_k and b_k with $a'_k = \frac{a_k}{\left(\sum_{k=1}^{N} a_k^p\right)^{1/p}}$ and $b'_k = \frac{b_k}{\left(\sum_{k=1}^{N} b_k^{p'}\right)^{1/p'}}$ respectively such that $\sum_{k=1}^{N} (a'_k)^p = \sum_{k=1}^{N} (b'_k)^{p'} = 1$. Thus, assuming (1.8) in the book (where it is further proven),

$$\begin{aligned} \sum_{k=1}^{N} a'_k b'_k &\leq \sum_{k=1}^{N} \frac{1}{p} (a'_k)^p + \frac{1}{q} (b'_k)^{p'} \\ &= \frac{1}{p} + \frac{1}{p'} \\ &= 1. \end{aligned}$$

Hölder's Inequality follows. □

Exercise 1.5: Consider $\langle a-tb, a-tb\rangle$ for $t \in [0,1]$. We will minimize this inner product using basic calculus:

$$\begin{aligned}\langle a-tb, a-tb\rangle &= (a_1-tb_1)^2+(a_2-tb_2)^2+\cdots+(a_N-tb_N)^2\\ &= \sum_{k=1}^{N} a_k^2 - 2t\sum_{k=1}^{N} a_kb_k + t^2\sum_{k=1}^{M} b_k^2\\ &= ||a||^2+t^2||b||^2-2t\langle a,b\rangle.\end{aligned}$$

It is evident that the critical point of this equation is located at $t=\frac{\langle a,b\rangle}{||b||^2}$. It is furthermore clear that this is where the minimum of the equation is. Therefore, we get the following:

$$\left\langle a-\frac{\langle a,b\rangle}{||b||^2}b, a-\frac{\langle a,b\rangle}{||b||^2}b\right\rangle = ||a||^2-\frac{\langle a,b\rangle^2}{||b||^2}.$$

Multiplying by $||b||^2$ on both sides, we can conclude

$$0 \le \big|\big|||b||^2a-\langle a,b\rangle b\big|\big|^2 = \big\langle||b||^2a-\langle a,b\rangle b, ||b||^2a-\langle a,b\rangle b\big\rangle = ||a||^2||b||^2-\langle a,b\rangle^2 \implies \langle a,b\rangle \le ||a||\cdot||b||.$$

$\square$

June 05

This day I read through chapter 2 (with full intention still to return to the remaining exercises in chapter 1). I only have a few questions on the content itself:

Question 1. *In section 2.1, the book states: "one of these projections must contain at least $C\sqrt{N}$ points". Is the C a typo? I am figuring that it is based on the proof that followed.*

– Yes it is a typo

Question 2. *Is the '#' notation for set cardinality standard in certain fields of math?*

Question 3. *What is the Exercise 2.5 described on page 13? Specifically, below the line that states*

$$\chi_{S_N}(x_1,x_2) \le \chi_{\pi_1(S_N)}(x_2)\cdot\chi_{\pi_2(S_N)}(x_1).$$

Question 4. *I am confused on the $I \times II$ notation on page 14. Is this just a way to "abbreviate" the previous equation?*

– Yes this is just a way to abbreviate the equation to deal with it in smaller chunks

It is also notable that the author misspells the "Cauchy-Schwarz" theorem a few times. More notes and exercises to follow.

June 06-07

I spent these two days working on the exercises from Chapter 1:

Exercise 1.6: I did this exercise using the "another approach" suggested, but am interested in how the first proof method would go. Using the convexity of the exponential function described in this new approach, letting $t_j=\frac{1}{n}\ge 0$ for all $n\in\mathbb{N}$ and thus $\sum_{j=1}^{n} t_j = 1$,

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} = \left(\prod_{i=1}^{n} e^{\ln(x_i)}\right)^{1/n} = \prod_{i=1}^{n} e^{\frac{\ln(x_i)}{n}} \le \frac{1}{n}\sum_{i=1}^{n} e^{\ln(x_i)} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

□

Exercise 1.8: I have started this exercise but have yet to complete it yet. Note the following:

$$\begin{aligned}\nabla f(a_1,\dots,a_n) &= \langle a_1^{p_1-1},\dots,a_i^{p_i-1},\dots,a_n^{p_n-1}\rangle\\ \nabla g(a_1,\dots,a_n) &= \langle a_2a_3\dots a_n, a_1a_3\dots a_n,\dots,a_1a_2\dots a_{n-1}\rangle.\end{aligned}$$

Thus, we have the following system of equations if $\nabla f = \lambda\nabla g$ for $\lambda\in\mathbb{R}$ and $g(a_1,\dots,a_n) = c > 0$:

$$\begin{cases} a_i^{p_i-1} = \lambda\prod_{j=1}^{i-1} a_j\prod_{j=i+1}^{n} a_j \forall i\in\{1,\dots,n\}\\ \prod_{j=1}^{n} a_j = c\end{cases} \implies a_1^{p_1} = a_2^{p_2} = \cdots = a_n^{p_n}.$$

Note that we dismiss the trivial case of $\lambda = 0$ as this would imply the trivially true inequality $0\le 0$. Evaluating f at this point, we get

$$f(a_1,a_2,\dots,a_n) = \sum_{i=1}^{n}\frac{a_i^{p_i}}{p_i} = \sum_{i=1}^{n}\frac{a_1^{p_1}}{p_i} = a_1^{p_1}.$$

This is where I got stuck. I am hoping that from here we can imply the inequality that we want, but am uncertain how to. Any suggestions? My best bet is that I am missing a key component of the second equation gained from the Lagrange multiplier process (namely, $g(a_1,\dots,a_n) = c$). For now, I moved onto exercises from chapter 2.

Exercise 2.2: For now, I skipped Exercise 2.1. From here I go through a thorough verification of the steps of the proof as requested. There are a few parts of this proof that I am left confused about, but I will explain what I understand so far.

Firstly, the Cauchy-Schwarz inequality was used to say that

$$\sum_{x_1,x_2}\chi_{\pi_3(S_N)}(x_1,x_2)\sum_{x_3}\chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3)\le I\times II.$$

Furthermore, $\chi^2_{\pi_j(S_N)}(x) = \chi_{\pi_j(S_N)}(x)$ as

$$\chi^2_{\pi_j(S_N)}(x) = \begin{cases}1^2 & x\in\pi_j(S_N)\\ 0^2 & \text{otherwise}\end{cases} = \begin{cases}1 & x\in\pi_j(S_N)\\ 0 & \text{otherwise}\end{cases} = \chi_{\pi_j(S_N)}(x).$$

I imagine that the use of this fact (that χ only takes on values of 1 and 0) proves some of the other equalities in this proof whose verification eludes me.

Finally, through set theory it is clear that

$$\sum_{x_1,x_2}\sum_{x_3}\sum_{x_3'}\chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3)\chi_{\pi_1(S_N)}(x_2,x_3')\chi_{\pi_2(S_N)}(x_1,x_3')\le$$
$$\sum_{x_1,x_2}\sum_{x_3}\sum_{x_3'}\chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3'),$$

as

$$\begin{aligned}A &= \{(x_2,x_3)\mid\chi_{\pi_1(S_N)}(x_2,x_3)=0\}\cup\{(x_1,x_3')\mid\chi_{\pi_2(S_N)}(x_1,x_3')=0\}\subset\\ B &= \{(x_2,x_3)\mid\chi_{\pi_1(S_N)}(x_2,x_3)=0\}\cup\{(x_1,x_3)\mid\chi_{\pi_2(S_N)}(x_1,x_3)=0\}\cup\\ &\qquad\{(x_2,x_3')\mid\chi_{\pi_1(S_N)}(x_2,x_3')=0\}\cup\{(x_1,x_3')\mid\chi_{\pi_2(S_N)}(x_1,x_3')=0\}.\end{aligned}$$

Hence, $B^c\subset A^c$.

Currently, the parts of the proof that confuse me are the equalities.

June 08

I have sent my notes thus far to Yuqiu. So far, in chapters 1 and 2, I have the following left to complete:

1. Exercises 1.7, 1.8, 2.1, and 2.3
2. Additional "difficult questions" at the end of both chapters.

Between now and our meeting on Thursday, I will read chapters 3 and 4, and work on these problems (in that order).

June 09

Over the course of the last day, I was able to better understand one of the two equalities on page 13:

$$\sum_{x_1,x_2} \chi_{\pi_1(S_N)}(x_2)\chi_{\pi_2(S_N)}(x_1) = \left(\sum_{x_1} \chi_{\pi_2(S_N)}(x_1)\right)\left(\sum_{x_2} \chi_{\pi_1(S_N)}(x_2)\right).$$

The way I understood this best was by considering the two sets:

$$A = \{x \mid x \in \chi_{\pi_1(S_N)}(S_N)\}$$
$$B = \{x \mid x \in \chi_{\pi_2(S_N)}(S_N)\}.$$

Fix some $x \in A$ (we know that one exists if $N \neq 0$. If $N = 0$ this equality is trivially true). Then, running through elements of the form (x, y) such that $y \in B$, we get that there are $|B|$ elements with the first element fixed. Thus, running through the $|A|$ elements with this same process, we get that

$$\sum_{x_1,x_2} \chi_{\pi_1(S_N)}(x_2)\chi_{\pi_2(S_N)}(x_1) = |A| \cdot |B|$$

which clearly equates the right hand side of the equality. A similar process can be used to understand the first major equality on page 14.

I finished up this day by doing a preliminary reading of Chapters 3 and 4 to lightly discuss with Yuqiu.

1.2 June 10-16

June 10

Good evening Professor Guth! Yuqiu and I had a very productive meeting today in which we discussed various exercises and questions I had from the reading. Such exercises, questions, and general comments are included below.
Exercise 1.8: We finished up this problem, with there being only two steps I was missing. Firstly, note that the reason we dismiss the case of $\lambda = 0$ is as this implies $a_i = 0$ for all i,, and thus it cannot be the case that $a_1a_2 \dots a_n = c > 0$. Furthermore, we can show that $c = a_1^{p_1}$, and thus we have found our minimum. Using $a_1^{p_1} = \cdots = a_i^{p_i} = a_n^{p_n}$, we get

$$\begin{aligned} c &= a_1a_2 \dots a_n \\ &= a_1 a_1^{\frac{p_1}{p_2}} \dots a_1^{\frac{p_1}{p_n}} \\ &= a_1^{p_1\left(\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_n}\right)} = a_1^{p_1}. \end{aligned}$$

Therefore, our minimum *should* be at $f(a_1, a_1^{p_1/p_2}, \dots, a_n^{p_1/p_n}) = a_1^{p_1} = c = a_1 a_n \dots a_n$. However, the Lagrange multipliers theorem requires that we show that f **has** a minimum/maximum. We know that this is the case however for the following two reasons:

1. f is a continuous function, and we know that on the closed and bounded interval $a_i \in [-n, n]$ (for each a_i and $n \in \mathbb{N}$), f has a minimum and maximum.
2. For any i, as $a_i \to \infty$, $f \to \infty$.

Therefore, f has a minimum. I am still in the process of finding a specific theorem that states this, but conceptually this is how we thought through showing f has a minimum value.

Next, we looked back at Exercise 1.6, working through the first hint (the one I didn't use to solve this problem):
Exercise 1.6: We will prove this is true for $n = 2^k$ using induction:
Base case: $k = 1$. For all $x_1, x_2 \in \mathbb{R}$, we have the following line of reasoning:

$$\begin{aligned} 0 &\le (x_1 - x_2)^2 \\ 2x_1x_2 &\le x_1^2 + x_2^2 \\ 4x_1x_2 &\le x_1^2 + 2x_1x_2 + x_2^2 \\ x_1x_2 &\le \frac{(x_1+x_2)^2}{4} \\ (x_1x_2)^{1/2} &\le \frac{x_1+x_2}{2}. \end{aligned}$$

Inductive Hypothesis: Assume for some arbitrary $k \in \mathbb{N}$ that the inequality is true for $n = 2^k$. We will show that this implies it is true for 2^{k+1}.

$$(x_1x_2 \dots x_{2^{k+1}})^{-2^{k+1}} = \left((x_1x_2\dots x_{2^k})^{-2^k}(x_{2^k+1}\dots x_{2^{k+1}})^{-2^k}\right)^{1/2}.$$

Applying the base case:

$$\le \frac{(x_1x_2\dots x_{2^k})^{-2^k} + (x_{2^k+1}\dots x_{2^{k+1}})^{-2^k}}{2}.$$

Applying the inductive hypothesis,

$$\le \frac{\sum_{i=1}^{2^{k+1}} x_i}{2^{k+1}}.$$

Hence, we have proven the inequality for $n = 2^k$. Now, we 'fill in the gaps'. For all $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $2^{k-1} < n \le 2^k$. Then, consider the following:

$$\begin{aligned} \left(\sum_{i=1}^n x_i\right)\prod_{i=1}^n x_i &= \left(\left(\left(\sum_{i=1}^n x_i\right)\prod_{i=1}^n x_i\right)^{1/2^k}\right)^{2^k} \\ &\le \left(\frac{2\cdot\sum_{i=1}^n x_i}{2^k}\right)^{2^k} = \left(\frac{\sum_{i=1}^n x_i}{2^{k-1}}\right)^{2^k}. \end{aligned}$$

Therefore,

$$\prod_{i=1}^n x_i \le \frac{\left(\sum_{i=1}^n x_i\right)^{2^k-1}}{(2^{k-1})^{2^k}}.$$

I need to work on this problem a little bit more to finish it off– I thought I had a complete proof, but in the example

Yuqiu and I worked on during the meeting we had used this process for $n = 3$ (which worked for this value as it is one less than 2^2, but this process does not immediately carry over to values such as 5 which is not one less than 2^k for $k \in \mathbb{N}$). Nonetheless, I found it very interesting to begin looking into this alternate approach to this proof.

Exercise 2.2: Here, Yuqiu discussed how the third line of this proof on page 14 is similar to Fubini's theorem on a discrete measure space. We also discussed other ways to understand why this equality is true, but it was interesting to see how Fubini's theorem could possibly be applied in a discrete way. This concept further applies to the equalities on page 15. Thus, I have finished writing out/thinking through this proof.

Exercise 2.3: We discussed how this problem can be thought through, and I will work on writing up the proof later this week. However, it seems like it should just be almost exactly the same as the proof done in exercise 2.2, only with a triple integral for the volume as opposed to sums– this problem will likely involve the regular Fubini's theorem as we know it to be from 18.02. We also believe that this theorem *should* be true even if the subset of $\mathbb{R}^3$ is not convex. Is this the case? I will see if this comes up in my write up of this theorem.

Exercise 3.1: To make the right hand side much bigger than the left, I considered the set $S_N = \{(x, x, x, x) \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq N\}$. This results in a distinct shadow for all x under projection to each plane. Therefore, for N points, we get N shadows on each of the 6 planes, and thus we have

$$N \leq \pi_{i<j;1\leq i,j\leq 4} \# \pi_{ij}(S_N))^{\frac{1}{3}} = (N^6)^{\frac{1}{3}} = N^2.$$

This should be the biggest we can make the right hand side for each value of N.

To construct the set for which he two sides are about equal, we consider a hypercube. For simplicity, let $N \in \mathbb{N}$ such that $N^{1/4} \in \mathbb{N}$. Then, consider the set $S_N = \{(w, x, y, z) \mid 1 \leq w, x, y, z \leq N^{1/4}\}$. This is a hypercube of side length $N^{1/4}$. Then, there are $N^{1/2}$ shadows under projection to each of the 6 planes, and thus we have

$$N \leq \pi_{i<j;1\leq i,j\leq 4} \# \pi_{ij}(S_N))^{\frac{1}{3}} = (N^{1/2})^{6\cdot\frac{1}{3}} = N.$$

This is what Iosevich meant when he said that the symmetric case was the most optimal on page 12. What makes this quote more interesting however, as Yuqiu pointed out, is that the set I constructed to make the right hand side *much bigger* also involved some nice symmetry.

After this, we discussed parts of Theorem 3.2 that I was confused about, namely that $F_1 = C_1$ and $F_2 = C_2$ (using notation given in the book). I plan on rigorously going through this proof later, making sure that I fully understand it like in Exercise 2.2. We also discussed parts of the major proof for Chapter 4, however we will get back to this next week after Yuqiu has more time to read through the material. Problems left to be done in the chapters read so far include: [Note that I cross things out on this list as they are completed.]

1. ~~Exercise 1.6, specifically filling in the gaps, and~~ Exercise 1.7. 1.7 has piqued both Yuqiu's interest and my own, though we have not finished the problem yet.
2. ~~Exercise 2.1 and 2.3~~.
3. Exercise~~s 3.2 and~~ 3.3. ~~I also want to go through the proof in this chapter more closely.~~
4. ~~Chapter 4 exercises. I am also interested in the case where d is odd, as described in the Notes, Remarks, and difficult questions on page 36.~~

Between now and next week, I plan to focus on problems from Chapters 2, 3 and 4, work through the major proofs in Chapters 2, 3, and 4, and give a preliminary read of chapters 5 and 6.

June 11-12

Over these two days I worked on digesting the proofs from Chapters 2 and 3, and in a week from now I will try to present these proofs (as well as the ones in Chapter 4 hopefully) to Yuqiu to see if I get stuck at any particular points (as Larry suggested doing). As of right now, the only portion of the 4-Dimensional proof in chapter 3 I am finding confusing is the scaling portion (equation (3.15)).

Besides this, I think I might have a way to finish off Exercise the argument for 1.6:

Exercise 1.6: Last we left off, I realized there was a slight issue with the proof– the proof Yuqiu and I had come up with only works for integers one less than a power of 2. in other words, if $n = 2^k - 1$ for $k \in \mathbb{N}$, then we have

$$\prod_{i=1}^{n} x_i \le \frac{\left(\sum_{i=1}^{n} x_i\right)^{2^k-1}}{\left(2^{k-1}\right)^{2^k}} \le \frac{\left(\sum_{i=1}^{n} x_i\right)^{n}}{n}.$$

Now consider natural numbers of the form $n = 2^k - 2$ for $k \in \mathbb{N}$. Then, similarly,

$$\begin{aligned}\left(\sum_{i=1}^{n} x_i\right)\prod_{i=1}^{n} x_i &= \left(\left(\left(\sum_{i=1}^{n} x_i\right)\prod_{i=1}^{n} x_i\right)^{1/(2^k-1)}\right)^{2^k-1} \\ &\le \left(\frac{2\cdot\sum_{i=1}^{n} x_i}{2^k-1}\right)^{2^k-1} \le \left(\frac{\sum_{i=1}^{n} x_i}{2^{k-1}}\right)^{2^k-1}.\end{aligned}$$

Therefore,

$$\prod_{i=1}^{n} x_i \le \frac{\left(\sum_{i=1}^{n} x_i\right)^{2^k-2}}{\left(2^{k-1}\right)^{2^k-1}} \le \frac{\left(\sum_{i=1}^{n} x_i\right)^{n}}{n}.$$

Reiterating this process, we have substantially 'filled in the gaps'. Hence, for all $n \in \mathbb{N}$,

$$\prod_{i=1}^{n} x_i \le \frac{\left(\sum_{i=1}^{n} x_i\right)^{n}}{n}.$$

□

Besides equation (3.15) that I am having trouble understanding, I plan to **not** look at these proofs again until this Thursday.

June 13

Yuqiu was able to help me understand (3.15)! I had most of the right idea but his suggestion helped secure my understanding. What was most confusing to me was that the scaling felt eerily similar to Exercise 1.4 (the proof of Hölder's Inequality). However, here we were able to scale two separate sequences to get the desired result, whereas here we only had a_j. Hence, I was thinking we could say something along the lines of "Let $a_j' = \frac{a_j}{C_1}$", but this doesn't lead to a denominator of C_2 that we want. Thus, Yuqiu suggested doing a more *general* version of the proof in this chapter, by considering

$$a_j^{s-\alpha} b_j^{\alpha}.$$

By 1.8,

$$a_j^{s-\alpha} b_j^{\alpha} \le \frac{a_j^{p(s-\alpha)}}{p} + \frac{b_j^{p'\alpha}}{p'}.$$

Letting $\alpha = 2(s-1)$ and $p = \frac{1}{2-s}$, we get

$$a_j^{s-\alpha} b_j^{\alpha} \leq \frac{a_j}{p} + \frac{b_j^2}{p'}.$$

Relabel a_j and b_j with $a_j' = \frac{a_j}{C_1}$ and $b_j = \frac{a_j}{C_2}$. Then,

$$\begin{aligned} \frac{a_j^s}{C_1^{1/p} \cdot C_2^{2/p'}} &= \left(\frac{a_j}{C_1}\right)^{s-\alpha} \left(\frac{a_j}{C_2}\right)^{\alpha} \\ &= \left(\frac{a_j}{C_1}\right)^{1/p} \left(\frac{a_j}{C_2}\right)^{2/p'} \\ &\leq \frac{a_j}{pC_1} + \frac{a_j^2}{p'C_2^2}. \end{aligned}$$

Summing this over j,

$$\frac{\sum a_j^s}{C_1^{1/p} \cdot C_2^{2/p'}} \leq \frac{1}{p} + \frac{1}{p'} = 1.$$

This gives us the conclusion we were hoping for.

I also had the following question:

Question 5. *For Theorem 3.16, why must* $1 < s < 2$*?*

I was able to pinpoint the line of the proof that leads to this conclusion:

$$p = \frac{1}{2-s}.$$

If $s > 2$, then $p < 0$, which is a contradiction. Furthermore, if $s < 1$, $p < 1 \implies \frac{1}{p} > 1$ which implies there does not exist a positive p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. It is further clear that $s \neq 1$ and 2 as if $s = 1$, p' must be infinity, and if $s = 2$, p is undefined. This was a very satisfying part of the proof to figure out.

June 14-15

Firstly, I started considering Exercise 2.3. Yuqiu and I had started to discuss the concepts in this problem when we met last week– noting that the proof likely follows the outline of the major proof done in this chapter. This is reflected in my write up of the problem. I am also wondering the following:

Question 6. *Must* Ω *be a convex subset?*

Though I am currently unsure of the answer to this problem, I do have a theory as to why this detail is included. I think that convexity may be required to let χ_{π_i} be a continuous function over Ω, such that we can apply Fubini's Theorem. I am still uncertain if this is the case however. In any case, my write up is done below (with possible notation issues regarding the integrals). Let $A_i = \sqrt{area(\pi_i(\Omega))}$. Furthermore, note that $\int$ denotes $\int_{\mathbb{R}}$ unless stated otherwise.

Exercise 2.3: Let Ω be a convex subset of $\mathbb{R}^3$. Then,

$$\begin{aligned} vol(\Omega) &= \int_{\mathbb{R}^3} \chi_\Omega(x)\,\mathrm{d}x \\ &\leq \iiint \chi_{\pi_1(\Omega)}(x_2, x_3)\chi_{\pi_2(\Omega)}(x_1, x_3)\chi_{\pi_3(\Omega)}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ &= \iint \chi_{\pi_3(\Omega)}(x_1, x_2)\left(\int \chi_{\pi_1(\Omega)}(x_2, x_3)\chi_{\pi_2(\Omega)}(x_1, x_3)\,\mathrm{d}x_3\right)\mathrm{d}x_1\,\mathrm{d}x_2. \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\leq \left(\iint \chi^2_{\pi_3(\Omega)}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \left(\iint \left(\int \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\,\mathrm{d}x_3\right)^2 \mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \\
&\leq A_3^{1/2}\left(\iiiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\chi_{\pi_1(\Omega)}(x_2,x_3')\chi_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_3\,\mathrm{d}x_3'\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \\
&\leq A_3^{1/2}\left(\iiiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_3\,\mathrm{d}x_3'\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \\
&= A_3^{1/2}\left(\iint \chi^2_{\pi_1(\Omega)}(x_2,x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3\right)^{1/2}\left(\iint \chi^2_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_1\,\mathrm{d}x_3'\right)^{1/2} = A_1^{1/2}A_2^{1/2}A_3^{1/2}.
\end{aligned}
$$

□

Exercise 3.2: I was able to figure this exercise out by starting to look at the same problem but in lower dimensions. For now, consider the inequality on the first line of page 13 which is conceptually the same as this problem. For simplicity, as opposed to focusing on N points, we will focus on infinitely many. Consider the set

$$S = \{(x,y) \mid x^2 + y^2 = 1\}.$$

The projection onto the x and y planes is the interval $[-1,1]$. Now consider the set $B = \{(x,y) \mid x^2 + y^1 < 1.\}$ For all $(x,y) \in B$, $x \in [-1,1]$ and $y \in [-1,1]$, and thus, the product on the right hand side is 1, while the left hand side of the inequality is 0. If we wanted specifically N points, we could picture a 'lower resolution' version of the circle, instead looking at the set

$$S = \{(0,0),(0,1),(0,2),(1,2),(2,2),(2,1),(2,0),(1,0)\}$$

which is not sharp given the point $(1,1)$. We can consider this same concept in higher dimensions, (for instance, inequality (2.1) is not sharp for hollow 3D-spheres). For our purposes of problem 3.2, we would need to consider a set akin to the shell of a 4-dimensional hypersphere. The points on the interior of this shell would make the inequality not sharp. To get a finite number of points instead of infinitely many, one can once again picture a lower-resolution version of this picture. It is notable that these shapes do not need to be perfect spheres, just topologically equivalent to them. In fact, it isn't fully necessary for the shape to be a closed surface (picture if we moved the translated the top half of the unit circle up by 1/2). This is just one way to *start* to picture what sets make this inequality not sharp.

Now, the sets that make this inequality always sharp satisfy the following condition: for all $x \in S$ there does not exist $y_1, y_2, \ldots, y_6 \in S$ (not necessarily distinct, but all not equal to x), such that $\pi_{12}(x) = \pi_{12}(y_1)$, $\pi_{13}(x) = \pi_{13}(y_2), \ldots, \pi_{34}(x) = \pi_{34}(y_6)$. I wonder if there is a less rigorous way to state this that still encapsulates the idea.

The concepts used in this problem also apply to **Exercise 2.1**:. □

To be honest, there are some parts of chapter 4 that confuse me (specifically pertaining to calculus and higher dimensions) that I am interested in talking about on Thursday. I am also interested in the following question:

Question 7. *Given $a_j \geq 0$, and*

$$\left(\sum_j a_j^k\right)^{1/k} \leq C_k \quad \textit{and} \quad \left(\sum_j a_j^{k+1}\right)^{1/(k+1)} \leq C_{k+1},$$

what can we say about

$$\left(\sum_j a_j^s\right)^{1/s}$$

for $s \in (k, k+1)$*? Does the estimate get better as* k *increases, or worse? Furthermore, how is this concept (of interpolation of estimates) used in harmonic analysis?*

My goal between now and Thursday is to work on this problem, and depending on how quickly I finish it, reread Chapter 4 closely to better understand the chapter. I have sent the notes so far to Yuqiu so we can be best prepared for Thursday.

June 16

Today I worked on and completed Question 7, it followed the same process as the example in the book for $k = 1$.

Let

$$\begin{cases} p(s-\alpha) = k \\ p'\alpha = k+1 \end{cases}.$$

This is analogous to the beginning of page 23. Then,

$$\alpha = (s-k)(k+1) \quad \text{and} \quad p = \frac{k}{s-\alpha} = \frac{k}{s-(s-k)(k+1)}.$$

Then, plugging everything into (3.14) and scaling, we get

$$\sum_j a_j^s \leq C_k^{k/p} \cdot C_{k+1}^{(k+1)/p'} = C_k^{s-(s-k)(k+1)} \cdot C_{k+1}^{(s-k)(k+1)}.$$

Hence,

$$\left(\sum_j a_j^s\right)^{1/s} \leq C_k^{1-(1-\frac{k}{s})(k+1)} \cdot C_{k+1}^{(1-\frac{k}{s})(k+1)} = C_k^{\frac{k^2+k}{s}-k} \cdot C_{k+1}^{k+1-\frac{k^2+k}{s}}.$$

This generalized form agrees at $k = 1$. □

I finished of the day by rereading Chapter 4. I am still a bit confused with the calculus parts, namely the concept of surface measure.

1.3 June 17-23

June 17

Hello Larry! I hope all is well with you. Once again, we have had a very productive week going through the material. Today, I presented the major proofs from Chapters 2 and 3 (2D, 3D, and 4D projections onto lower dimensions). These went very well. I found your suggestion to read through the material and work through it again later to be very useful. It helped me think through the problems more closely, specifically at which terms Iosevich chooses to keep and throw away (i.e. the top line of page 22 when he throws out 3 of the terms). There were also parts of the problem that I hadn't thought through super closely, and that I was able to work through the previous week.

After presenting the proofs, we talked about Chapter 4. The first approach to finding the volume of a ball of radius R in $\mathbb{R}^d$ seems to be more difficult than the one discussed in Exercise 4.2. My goal later today is to work through Exercise 4.2– hopefully this way makes more sense for me.

Yuqiu also added another possible question to explore (as an add-on to Question 7 done on June 16):

Question 8. *Given $a_j \geq 0$, and $m \in \mathbb{N}$ with*

$$\left(\sum_j a_j^k\right)^{1/k} \leq C_k \quad \textit{and} \quad \left(\sum_j a_j^{k+m}\right)^{1/(k+m)} \leq C_{k+m},$$

what can we say about

$$\left(\sum_j a_j^s\right)^{1/s}$$

for $s \in (k, k+m)$? Presumably this estimation is not as close as between two integers, but how does it compare?

Over the next week, I plan to do the following:

1. ~~Answer Question 8.~~
2. ~~Evaluate $\int_0^{\frac{\pi}{2}} \sin^d \theta \, \mathrm{d}\theta$ for $d \in \mathbb{N}$. This problem comes up in Exercise 4.2.~~
3. ~~Closely working through the rest of Chapter 4 (namely the estimation portion), as well as reading through Chapters 5 and 6.~~

I have still not figured out Question 6– if Ω must be a convex subset. My idea moving forward is to consider possible counterexamples. The goal is to maximize the total number of "repeated" shadows when projected onto each plane thus decreasing the areas but increasing the volume. Perhaps something akin to a Rubik's cube with space between each individual cube. However, the Rubik's Cube example does raise some red flags, namely that the area and volumes will be the same as one larger (convex) cube. I think this will continue to be an issue, which raises another question:

Question 9. *Can any concave subset of $\mathbb{R}^3$ be transformed into a convex subset of $\mathbb{R}^3$ with the same volume and areas under projection? Similar to how we can move the individual cubes in the Rubik's Cube example together to form one large convex cube.*

If this is the case, then this would show that *any* subset of $\mathbb{R}^3$ satisfies this inequality. I think this problem should prove interesting. In two weeks from now, I hope to be able to present the major proofs in chapter 4, as well as possibly some from Chapters 5 and 6.

I hope your summer is going well so far!

Now, I answer question 8. Starting again with the system of equations

$$\begin{cases} p(s-\alpha) = k \\ p'\alpha = k+m \end{cases} \implies \alpha = \frac{(k+m)(s-k)}{m} \quad \text{and} \quad p = \frac{m}{(k+m)-s}.$$

Thus,

$$\begin{aligned} a_j^s &= a_j^{s-\alpha} a_j^{\alpha} = a_j^{k/p} a_j^{(k+m)/p'} \\ &\leq \frac{a_j^k}{p} + \frac{a_j^{k+m}}{p'}. \end{aligned}$$

Scaling this equation and summing, we get

$$\begin{aligned}\sum_j \frac{a_j^s}{C_k^{k/p} C_{k+m}^{(k+m)/p'}} &= \left(\frac{a_j}{C_k}\right)^{k/p} \left(\frac{a_j}{C_{k+m}}\right)^{(k+m)/p'} \\ &\leq \sum_j \frac{a_j^k}{p C_k^k} + \frac{a_j^{k+m}}{p' C_{k+m}^{(k+m)}} \\ &\leq \frac{1}{p} + \frac{1}{p'} = 1 \\ &\boxed{\sum_j a_j^s \leq C_k^{k/p} C_{k+m}^{(k+m)/p'}.}\end{aligned}$$

Plugging in p and p', and raising to the $1/s$ power, we get (for $s \in (k, k+m)$)

$$\left(\sum_j a_j^s\right)^{1/s} \leq C_k^{\frac{k(k+m-s)}{ms}} C_{k+m}^{\frac{(k+m)(s-k)}{ms}}.$$

This agrees with the equation found in Question 7 (with $m = 1$). $\square$

Exercise 4.2: Firstly, we check that $\omega_d(R) = R^d \omega_d(1)$. However, this is not terribly hard to do. Let $B_r = \{x \in \mathbb{R}^d \mid |x| \leq r\}$. Then,

$$\omega_d(R) = \int_{B_R} \mathrm{d}x.$$

Let $y = \frac{x}{R}$. Then, $\mathrm{d}y = \mathrm{d}y_1 \mathrm{d}y_2 \ldots \mathrm{d}y_d = \frac{1}{R^d} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_d = \frac{\mathrm{d}x}{R^d}$. Hence, using this change of variable, we have

$$\omega_d(R) = \int_{B_R} \mathrm{d}x = \int_{B_1} R^d \mathrm{d}y = R^d \omega_d(1)$$

as $B_1 = \{y \in \mathbb{R}^d \mid |y| = \frac{|x|}{R} \leq 1\}$.

Now we want to evaluate $\omega_d(1)$. The following lines come from the book:

$$\begin{aligned}\omega_d(1) &= \int_0^1 \int_{x_1^2 + \cdots + x_{d-1}^2 \leq 1 - t^2} \mathrm{d}x' \, \mathrm{d}t \\ &= \int_0^1 \omega_{d-1}(\sqrt{1-t^2}) \, \mathrm{d}t.\end{aligned}$$

This is clear, as the set of $x \in \mathbb{R}^{d-1}$ such that $x_1^2 + \cdots + x_{d-1}^2 \leq 1 - t^2$ is the ball of radius $\sqrt{1-t^2}$ in $\mathbb{R}^{d-1}$. Then,

$$= \omega_{d-1}(1) \int_0^1 (1-t^2)^{\frac{d-1}{2}} \, \mathrm{d}t.$$

Let $t = \cos\theta$, and thus $\mathrm{d}t = -\sin\theta \, \mathrm{d}\theta$. Hence, we have

$$\begin{aligned}&= \omega_{d-1}(1) \int_{\frac{\pi}{2}}^0 -(1-\cos^2\theta)^{\frac{d-1}{2}} \sin\theta \, \mathrm{d}\theta \\ &= \omega_{d-1}(1) \int_0^{\frac{\pi}{2}} \sin^d\theta \, \mathrm{d}\theta.\end{aligned}$$

Though this integral is interesting to prove, it is ultimately a calculus problem and thus for our purposes I will just state what it is equal to:

$$\int_0^{\frac{\pi}{2}} \sin^d \theta \, \mathrm{d}\theta = \begin{cases} \frac{(d-1)!!}{d!!} \cdot \frac{\pi}{2} & d \text{ is even} \\ \frac{(d-1)!!}{d!!} & d \text{ is odd} \end{cases}.$$

where $n!! = n(n-2)(n-4)\ldots$. Therefore, we will split this problem into two cases.
d is even: Let $d = 2n$. Then,

$$\begin{aligned}
\omega_{2n}(1) &= \omega_{2n-1}(1) \int_0^{\frac{\pi}{2}} \sin^{2n} \theta \, \mathrm{d}\theta \\
&= \omega_{2n-1}(1) \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \\
&= \omega_{2n-2}(1) \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \sin^{2n-1} \theta \, \mathrm{d}\theta \\
&= \omega_{2n-2}(1) \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \cdot \frac{(2n-2)!!}{(2n-1)!!} \\
&= \omega_{2n-2}(1) \cdot \frac{(2n-2)!!}{(2n)!!} \cdot \frac{\pi}{2} \\
&\ \ \vdots \\
&= \omega_2(1) \frac{2}{(2n)!!} \left(\frac{\pi}{2}\right)^{n-1} \\
&= \pi \cdot \frac{2}{(2n)!!} \left(\frac{\pi}{2}\right)^{n-1} = \frac{\pi^n}{2^{n-2}(2n)!!}.
\end{aligned}$$

This is where I am slightly confused, as I do not think this should be the right answer, but have gotten the same answer multiple times. Recall that $\omega_d(1)$ is the volume of a ball with radius 1 in $\mathbb{R}^d$. Therefore, we want to find R_d such that $\omega_d(R_d) = 1$, which is fairly straight forward to do (at least by the calculations done thus far, if the calculations are correct):

$$\omega_d(R_d) = R_d^d \cdot \omega_d(1) = R_d^d \cdot \frac{\pi^n}{2^{n-2}(d)!!} = 1 \implies R_d = \left(\frac{2^{n-2}(d)!!}{\pi^n}\right)^{1/d}.$$

Based on what we showed on page 29, it should be the case that

$$R_d = \left(\frac{d}{|S_{d-1}|}\right)^{1/d} \implies |S_{d-1}| = \frac{\pi^n}{2^{n-2}(d-2)!!}.$$

However, on page 31 we state that $|S_{d-1}| = \frac{\pi^n}{n!}$. This should be a contradiction, which means I am likely making an error with my calculation of $\omega_{2n}(1)$. Any suggestions? It is close to the right form with π^n, which makes me think I am on the right track.

June 18

At this point I emailed my work to Yuqiu to ask for possible suggestions about how to move forward (as we had talked about this problem that day). In the meantime, I started to work on reading the second half of chapter 4, of which I have a few questions:

Question 10. *Why must $s \geq 1$? I thought the problem we were looking at required $s = 1$ (which would satisfy this inequality, but still want to be sure).*

– We let s be the largest side length of a unit cube such that when the ball is projected onto the k dimensional plane, the unit ball fits into the unit cube. Thus, s is not necessarily 1, but is at least greater than or equal to it.

Question 11. *At the bottom of page 34, it says "In other words, we do not need to project the ball very far down in order to be able to fit the unit cube inside." How does this follow from the calculations done thus far?*

– What I missed was that we let k be the largest plane we can project the unit ball onto such that it fits into the unit cube.

Exercise 4.4: Let $a_n = \frac{\log(n)}{n}$. Then, $a_n' = \frac{1-\log n}{n^2}$. For $n > e$, we have $f'(n) < 1$. Hence, the sequence $b_n = a_{n+3}$ is monotonically decreasing. Furthermore, b_n is bounded, as $a_n = \frac{\log(n)}{n} \leq 2$ for all n (as $n \leq e^{2n}$ for all n). Therefore, by the monotone convergence theorem, the sequence converges. Furthermore, by Bolzano-Weierstrass there exists a convergence subsequence b_{n_k}, which converges to the same limit as b_n. Let $n_k = 2^k$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{k\to\infty} b_{n_k} = \lim_{n\to\infty} \frac{\log(2^k)}{2^k} = \log(2) \lim_{k\to\infty} \frac{k}{2^k} = 0.$$

This last limit is clearly 0. This follows from L'Hospital's Rule, or Combinatorics (the combinatorics technique is done at the bottom of the page). □

One thing that feels slightly weird about this problem is that basic calculus would have proved this from the beginning. Is there a reason we tried to show this in a different way? Maybe it is just a segue into the combinatorics section of this book.

June 19-21

Sincerely for the slight delay in notes (as you can see this section covers 3 days)– I was busy with birthday things. Nonetheless I am back on top of it! Over these days, I read chapters 5 and 6, and continued on the exercises from chapter 4. Yuqiu got back to my work on Exercise 4.2, and he isn't quite certain where it goes wrong (if it indeed does). I will revisit this problem later, as I have made some more progress on other aspects of the material.

Exercise 4.5: Let $\alpha > 0$. Then, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \alpha$. Then, consider the following:

$$\begin{aligned}\log n &= \int_1^n \frac{\mathrm{d}t}{t} \\ &= \sum_{k=0}^{N-1} \int_{R_k}^{R_{k+1}} \frac{\mathrm{d}t}{t}\end{aligned}$$

where $1 = R_0 \leq R_1 \leq \cdots \leq R_N = n$. Then,

$$\leq \sum_{k=0}^{N-1} \frac{R_{k+1} - R_k}{R_k}.$$

Let $R_k = n^{k/N}$.

$$\begin{aligned}&= \sum_{k=0}^{N-1} \frac{n^{(k+1)/N} - n^{k/N}}{n^{k/N}} \\ &= \sum_{k=0}^{N-1} n^{1/N} - 1 \\ &= N(n^{1/N} - 1).\end{aligned}$$

Therefore,

$$\frac{\log n}{n^\alpha} \leq \frac{N(n^{1/N} - 1)}{n^\alpha} \to 0 \text{ as } n \to \infty,$$

given that $\frac{1}{N} < \alpha$. □

This follows the same logic as the argument on page 34, just generalized.

Exercise 4.6: Let $I_n = \int_0^\infty e^{-t}t^n \, \mathrm{d}t$. Then, integrating by parts, we get the following:

$$\begin{aligned} I_n &= -t^n e^{-t}\big|_{t=0}^{\infty} + n\int_0^\infty e^{-t}t^{n-1} \, \mathrm{d}t \\ &= n \cdot I_{n-1}. \end{aligned}$$

Reiterating this process, we get

$$\begin{aligned} &= n(n-1)(n-2)\dots(2)(1)I_0 \\ &= n!\int_0^\infty e^{-t} \, \mathrm{d}t \\ &= n!\left(-e^{-t}\big|_{t=0}^{\infty}\right) \\ &= n!. \end{aligned}$$

It is also clear that the critical point of the integrand is at $t = n$, however I am uncertain how to progress from here to finish the approximation.

Exercise 6.1: I will show this using induction on d. Firstly, consider $\mathbb{F}_q$. This clearly has q elements:

$$\{0, 1, \dots, q-1\}.$$

Then, assume for some $k \in \mathbb{N}$, $|\mathbb{F}_q^k| = q^k$. Now we will show that $|\mathbb{F}_q^{k+1}| = q^{k+1}$. However, this is clear as we can simply write each coordinate in $\mathbb{F}_q^{k+1}$ as a k-tuple with a 1-tuple (i.e. as $\mathbb{F}_q^k \times \mathbb{F}_q$). Then, by assumption, there are q^k elements in $\mathbb{F}_q^k$ and q elements in $\mathbb{F}_q$ (as we showed in the base case). Therefore,

$$|\mathbb{F}_q^{k+1}| = |\mathbb{F}_q^k||\mathbb{F}_q| = q^k \cdot q = q^{k+1}.$$

□

Exercise 6.2: Showing that the three lines are the same is fairly straight forward, but I will explain what is going on. Firstly, $(2,2) = 2(1,1)$, and note that $2\{0,1,2\} = \{0,2,4\} = \{0,2,1\}$. This explains why the first 2 are equivalent. Furthermore, note that $(1,2) = (0,1) + (1,1)$, and thus

$$\begin{aligned} L((1,2),(1,1)) &= \{(1,2) + t(1,1) \mid t = 0,1,2\} \\ &= \{(0,1) + (t+1)(1,1) \mid t = 0,1,2\} \\ &= \{(0,1) + t(1,1) \mid t = 1,2,3\} \\ &= \{(0,1) + t(1,1) \mid t = 0,1,2\} = L((0,1),(1,1)). \end{aligned}$$

□

Exercise 6.3: This is, in some ways, a more general proof of exercise 6.2. Let $v = \lambda v'$. Firstly, note that

$$\{0, 1, \dots, q-1\} = \lambda\{0, 1, \dots, q-1\}$$

for $0 \neq \lambda \in \mathbb{F}_q$ (this also uses the fact that q is prime). Thus,

$$L(x, v) = \{x + tv \mid t = 0, 1, \dots, q-1\}.$$

Let $t = \lambda t'$. Then

$$\begin{aligned}
&= \{x + \lambda t' v \mid \lambda t' = 0, 1, \ldots, q-1\} \\
&= \{x + t'v' \mid \lambda t' = 0, \lambda, \ldots, \lambda(q-1)\} \\
&= \{x + t'v' \mid t' = 0, 1, \ldots, q-1\} = L(x, v').
\end{aligned}$$

I am still working on the other direction of the proof, but it is proving a bit tricky. So far, my idea is that if $L(x, v) = L(x, v')$ for $v \neq v' \neq \vec{0}$, then there exists $t, t' \in \{0, 1, \ldots, q-1\}$ such that

$$x + tv = x + t'v' \implies tv = t'v'.$$

The idea from here would be to let $t' = \lambda +$ *something*, and utilize the fact that $v - v' \neq \vec{0}$. However, I have yet to find a t' that works out in this way.

I am sincerely sorry for being so behind on the material this week. Tomorrow, I will do the following:

1. ~~Show that $\mathbb{F}_q$ is a field for q prime.~~
2. ~~Exercise 6.6.~~
3. ~~Chapter 5 problems.~~ Exercise 5.8.

The Chapter 5 problems on graph theory seem interesting.

June 22-23

Firstly we will show that $\mathbb{F}_q$ is a field for q prime. Note that the multiplicative inverses property has already been proven in the book. Most of this follows directly from the associativity, commutativity, and distributivity of plain addition. Let $a_1, a_2, a_3 \in \mathbb{F}_q$. Then, there exists $k_i \in \mathbb{Z}$ such that $a_i = a_i' + qk_i$ for each i. Hence:

1. Associativity of addition:

$$\begin{aligned}
a_1 + (a_2 + a_3) &= a_1' + qk_1 + (a_2' + qk_2 + a_3' + qk_3) \\
&= (a_1' + qk_1 + a_2' + qk_2) + a_3' + qk_3 \\
&= (a_1 + a_2) + a_3.
\end{aligned}$$

   Associativity of multiplication:

$$\begin{aligned}
a_1(a_2a_3) &= (a_1' + qk_1)([a_2' + qk_2][a_3' + qk_3]) \\
&= ([a_1' + qk_1][a_2' + qk_2])(a_3' + qk_3) \\
&= (a_1a_2)a_3.
\end{aligned}$$

2. The other properties (commutative, distributive) follow similarly.
3. The only other properties of note is that the additive identity is 0 as usual, the multiplicative identity is 1 as usual, and the additive inverse for $a \in \mathbb{F}_q$ is $q - a$ (as $a + q - a = q = 0$).

The reason I am so quick to dismiss showing all of these properties is because $\mathbb{F}_q$ is a subset of $\mathbb{F}_q[\sqrt{2}]$ that is closed under addition, multiplication, and contains the additive and multiplicative inverses. Thus, the fact that $\mathbb{F}_q$ is a field follows from exercise 6.6:

Exercise 6.6: The biggest thing to show here is the multiplicative inverse, which is not all that bad. Consider

$a + b\sqrt{2}$ for $a, b \in \mathbb{F}_q$. Note that $a^2 - 2b^2 \neq 0$ for all $a, b \in \mathbb{F}_q$. Hence,

$$(a + b\sqrt{2}) \left(\frac{a - b\sqrt{2}}{a^2 - 2b^2} \right) = 1.$$

The other properties can be shown by the same process used in proving $\mathbb{F}_q$ is a field. □

I was able to make some more progress on Exercise 4.2! I will copy and paste that section of the notes here, and go back through the problems.

Exercise 4.2: (continued) Let $d = 2n$. Then,

$$\begin{aligned}
\omega_{2n}(1) &= \omega_{2n-1}(1) \int_0^{\frac{\pi}{2}} \sin^{2n} \theta \, \mathrm{d}\theta \\
&= \omega_{2n-1}(1) \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \\
&= \omega_{2n-2}(1) \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \sin^{2n-1} \theta \, \mathrm{d}\theta \\
&= \omega_{2n-2}(1) \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \cdot \frac{(2n-2)!!}{(2n-1)!!} \\
&= \omega_{2n-2}(1) \cdot \frac{(2n-2)!!}{(2n)!!} \cdot \frac{\pi}{2} \\
&\vdots \\
&= \omega_2(1) \frac{2}{(2n)!!} \left(\frac{\pi}{2} \right)^{n-1} \\
&= \frac{0!!}{(2n)!!} \left(\frac{\pi}{2} \right)^{n} \\
&= \frac{\pi^n}{2^{2n}(n!)}.
\end{aligned}$$

This utilizes the fact that $(2n)!! = 2^n(n!)$, a fact I had not realized. Hence, we get that

$$\omega_{2n}(R_d) = R_d^d \left(\frac{\pi^n}{2^{2n}(n!)} \right) = 1 \implies R_d = \frac{(n!)^{\frac{1}{2n}}}{2\sqrt{\pi}}.$$

Now it is simply off by this factor of 2 that I am uncertain how to fix, but I am extremely close to getting the right answer! After I figure out where this extra factor of 2 comes from, I will consider the odd case.

On the 23rd I reread chapters 4, 5, and 6 in preparation for Thursday.

1.4 June 24-30

June 24

Hello Larry! I hope your vacation is treating you well. This was a very exciting meeting because we were able to finally figure out what was going wrong with Exercise 4.2. This and more is attached below.

Exercise 4.2:: (continued) The calculations I was doing on Exercise 4.2 were not wrong, it was an equation on page 31 that was incorrect: the integral should go from -1 to 1, not 0 to 1:

$$\omega_d(1) = \int_{-1}^{1} \int_{x_1^2 + \cdots + x_{d-1}^2 \leq 1 - t^2} \mathrm{d}x' \mathrm{d}t = 2 \int_0^1 \int_{x_1^2 + \cdots + x_{d-1}^2 \leq 1 - t^2} \mathrm{d}x' \mathrm{d}t.$$

Thus, we get the following:

$$\begin{aligned}
\omega_{2n}(1) &= 2\omega_{2n-1}(1)\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\,\mathrm{d}\theta \\
&= 2\cdot\omega_{2n-1}(1)\cdot\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2} \\
&= \pi\cdot\frac{(2n-1)!!}{(2n)!!}\cdot\omega_{2n-2}(1)\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\,\mathrm{d}\theta \\
&= 2\pi\cdot\frac{(2n-2)!!}{(2n)!!}\omega_{2n-2}(1).
\end{aligned}$$

Repeating this process,

$$\begin{aligned}
&= 2^{n-1}\pi^{n-1}\cdot\frac{2}{(2n)!!}\cdot\omega_2(1) \\
&= \pi^n\cdot\frac{2^n}{(2n)!!} \\
&= \frac{\pi^n}{n!}.
\end{aligned}$$

Hence,

$$\omega_{2n}(R_{2n}) = R_{2n}^{2n}\cdot\frac{\pi^n}{n!} = 1 \implies R_{2n} = \frac{(n!)^{\frac{1}{2n}}}{\sqrt{\pi}}.$$

Similarly,

$$\omega_{2n+1}(1) = 2\omega_{2n}(1)\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\,\mathrm{d}\theta = 2\cdot\pi^n\cdot\frac{2^n}{(2n)!!}\cdot\frac{(2n)!!}{(2n+1)!!} = \frac{\pi^n\cdot 2^{n+1}}{(2n+1)!!}.$$

Therefore,

$$R_{2n+1} = \left(\frac{(2n+1)!!}{\pi^n\cdot 2^{n+1}}\right)^{\frac{1}{2n+1}}.$$

In regards to the rest of the problem (i.e., how far down you'd have to project an odd-dimensional ball of volume 1 down for it to fit inside the unit cube), Yuqiu astutely pointed out that the answer should also be roughly $\frac{d}{4}$ as you can project it one dimension down, and you once again get an even-dimensional ball.

Exercise 6.3: I had part of the right idea originally for 6.3, but Yuqiu helped put together the final piece. For all t there exists a t' such that $x + tv = x + t'v'$. Hence, let $t = 1$ and $t' = \lambda$. Therefore, $v = \lambda v'$ and then, $L(x, v) = L(x, v')$ if and only if $v = \lambda v'$ for $\lambda \in \mathbb{F}_q$. $\square$

There is still more left to do in this exercise, but at the very least this part of the problem is now complete.

We then worked on the first problem in the Notes, remarks, and difficult questions in Chapter 5. Let $A \subset \mathbb{Z}$. Consider the $(\#A)^2$ lines $(at, t + a')$ for all $a, a' \in A$, and the set of $\#(A\cdot A)\cdot\#(A + A)$ points. Yuqiu pointed out that every line contains at least $\#A$ points (let $t \in A$). Hence, using the Szemeredi-Trotter incidence theorem, we get that

$$(\#A)^3 \leq I(n) \leq C(\#(A\cdot A)\cdot\#(A+A) + (\#A)^2 + (\#(A\cdot A)\cdot\#(A+A)\cdot(\#A)^2)^{2/3}).$$

Let $B = \max\{\#(A\cdot A), \#(A + A)\}$. Then,

$$(\#A)^3 \leq C(B^2 + (\#A)^2 + (B^2\cdot(\#A)^2)^{2/3}).$$

Therefore, one of the following cases must be true by the pigeonhole principle:

1. $\frac{(\#\mathbf{A})^{\mathbf{3}}}{\mathbf{3}} \leq \mathbf{CB^2}$. Then, $\frac{(\#A)^{\frac{5}{4}}}{\sqrt{3C}} \leq \frac{(\#A)^{\frac{3}{2}}}{\sqrt{3C}} \leq B$.
2. $\frac{(\#\mathbf{A})^{\mathbf{3}}}{\mathbf{3}} \leq \mathbf{C(\#A)^2}$. Then, $(\#A) \leq 3C \implies (\#A)^{\frac{1}{4}} \leq (3C)^{\frac{1}{4}}$. Trivially, $(\#A) \leq B$. Hence, multiplying these two inequalities together, we get $\frac{(\#A)^{\frac{5}{4}}}{(3C)^{\frac{1}{4}}} \leq B$.

or 3. $\frac{(\#\mathbf{A})^{\mathbf{3}}}{\mathbf{3}} \leq \mathbf{CB}^{\frac{\mathbf{4}}{\mathbf{3}}}(\#\mathbf{A})^{\frac{\mathbf{4}}{\mathbf{3}}}$. Thus, $\frac{(\#A)^{\frac{5}{3}}}{3C} \leq B^{\frac{4}{3}} \implies \frac{(\#A)^{\frac{5}{4}}}{(3C)^{\frac{3}{4}}} \leq B$.

Thus, $B \geq C'(\#A)^{\frac{5}{4}}$ for some $C' > 0$. □

One thing that confused me in this proof is that I thought the C here was the same as the one in the Szemeredi-Trotter incidence theorem, however Yuqiu said this is not the case.

The goal over the next week is as follows:

1. ~~Be able to prove the major theorems in Chapters 4 and 5 next Thursday.~~
2. ~~The graph theory problems in Chapter 5.~~
3. ~~Read and work on Chapters 7 and 8.~~
4. ~~I will also go back through these notes and make a summarized version.~~

This version of notes is nice to see the thought process, but for ease and simplicity I figure that a summarized version could be helpful. This will include exercises, as well as general questions on the content.

June 25

I started today by reading the main proofs in Chapters 4 and 5. I only had one question which I asked Yuqiu:

Question 12. *On page 32, above equation (4.6), Iosevich states "we must have $s \geq 1$". Why must this be true? I thought that we were trying to fit the unit sphere inside a unit cube, which would imply $s = 1$ right? This wouldn't make this inequality wrong, but it does make it slightly confusing.*

– He quickly responded later this day, stating: by s he might mean the largest side-length of a cube that is centered at the center of the k-dimensional ball and can fit into that ball. So s could any positive number but s has to be ≥ 1 for a unit side-length cube to fit into that ball.

Exercise 4.3: This doesn't change too much, with the exception that if $s = \epsilon$, then $k \leq \frac{2d}{\pi e \epsilon^2}$. Thus, k can be much larger if the side length is smaller. This make sense, as fitting a smaller object into another object with a static volume should be easier the smaller the first object is. I am uncertain if there are more conclusions to possibly be reached in this exercise that Iosevich is alluding to. □

I finished off this day by reading through Chapters 7 and 8. My goal for this weekend is to work through this material more closely and work on better understanding the material.

June 26-27

I spent this weekend reading through chapters 7 and 8. Below are the main questions I have from this chapter.

Question 13. *On Page 54 Iosevich states "It is pretty clear that the total number of points in B is at least $(q+1) \cdot \frac{q}{2}$". The one part I am slightly confused about is the $\frac{1}{2}$ and wanted to check my understanding: We divide by 2 to avoid double counting right?*

– Yes! We can pick a line L_1 at random, of which there are q points from other lines on it. However, pick one of these q points, which by assumption lies on both the chosen line and another, L_2. Then, L_2 has q points, but one of them has already been counted by L_1. This is the 'double counting' that occurs, and thus we have a factor of $\frac{1}{2}$.

Question 14. *In a similar way I am confused on the factor of 2 used in (7.1), in which we state*

$$\sum_{i=1}^{q+1} \sum_{j=1}^{q+1} \#(L_i \cap L_j) = 2q(q+1).$$

The reason this confuses me is because I thought it should be $q(q+1)$ based on this line of reasoning:

$$\begin{aligned}\sum_{i=1}^{q+1}\sum_{j=1}^{q+1}\#(L_i\cap L_j) &= \#\{(i,j)\mid i\neq j \text{ and } 1\leq i,j\leq q+1\}\\ &=(q+1)^2-(q+1)\\ &=q(q+1).\end{aligned}$$

Is there something I am missing here?

– I was able to figure this out a bit later. I was completely dismissing the case in which $i=j$, when in fact if $i=j$, $\#L_i\cap L_j=\#L_i=q$ which gives us the additional $q(q+1)$ that results in the factor of 2.

Question 15. *I am particularly confused in chapter 8. We start off the first proof by saying "Suppose that $\#B\leq \frac{q^{\frac{d+1}{2}}}{4}$", and later show that $\#B\geq \frac{q^{\frac{d+1}{2}}}{4}$ on page 61. Doesn't this imply that $\#B=\frac{q^{\frac{d+1}{2}}}{4}$? I think I am missing something. In general, I am confused on why we can/would want to simply assume the size of B to begin with. What happens if $\#B$ is bigger than we initially supposed? Do we simply not care because then the inequality is trivially true?*

Exercise 8.1: Note that each line has q points. Thus, for all i, $\sum_{x\in B'}\chi_{L_i}(x)=q$, and there are k lines. Hence,

$$\sum_{x\in B'}\sum_{i=1}^{k}\chi_{L_i}(x)=\sum_{i=1}^{k}\sum_{x\in B'}\chi_{L_i}(x)=\sum_{i=1}^{k}q=qk.$$

This is why the first line in (8.2) is true. Additionally we used the Cauchy-Schwarz inequality in the second line:

$$\left(\sum_{x\in B'}\left(\sum_{i=1}^{k}\chi_{L_i}(x)\right)\right)^2=\left(\sum_{x\in B'}\left(\sum_{i=1}^{k}\chi_{L_i}(x)\right)\cdot 1\right)^2.$$

Using the C-S Inequality here,

$$\begin{aligned}&=\left(\left(\sum_{x\in B'}1^2\right)^{1/2}\left(\sum_{x\in B'}\left(\sum_{i=1}^{k}\chi_{L_i}(x)\right)^2\right)^{1/2}\right)^2\\ &=\left(\sum_{x\in B'}1^2\right)\cdot\sum_{x\in B'}\left(\sum_{i=1}^{k}\chi_{L_i}(x)\right)^2\\ &=\#B'\sum_{x\in B'}\left(\sum_{i=1}^{k}\chi_{L_i}(x)\right)^2.\end{aligned}$$

□

My goal moving forward with this week is to attempt the chapter 5 graph theory problem, as well as work on Exercise 3.3 in a possible attempt to generalize this concept in higher dimensions.

June 28-29

What follows is a (rough) attempt to do Exercises 5.5-5.9.

Exercise 5.5: I imagine this can be done through induction on n and f, however I opted to do induction on e, the number of edges. This problem initially really confused me: if $n=1$, then there are no edges, and no faces right? However, after talking to a friend of mine who is interested in Graph Theory, he pointed out that there is 1 face:

the infinite unbounded space that remains outside of that vertex. Hence, our base case is complete: if $e = 0$, then $n = f = 1$, and $n - e + f = 2$. For some $k \in \mathbb{N}$, assume that for $e = k$ it is always true that $n - e + f = 2$.

Now consider a graph with $k + 1$ edges. There are 2 cases that can follow: whether or not there exists a vertex that is only connected to one edge. If there exists a vertex that is only connected to one edge, consider a graph G' that takes out both this vertex and this edge. Then, there are k edges and $n - 1$ vertices. Hence, by assumption: $(n - 1) - (k + 1 - 1) + f = 2 \implies n - (k + 1) + f = 2$. If there does not exist a vertex that is connected to only one edge, then there must exist a loop in the graph G (as G is connected). Pick one of the edges in this loop, and consider a graph with this edge removed. Then, the number of edges goes down by 1, and the number of faces goes down by 1 as the face created by this loop is combined with another face in the graph. Then, by assumption, $n - (k + 1 - 1) + (f - 1) = 2$ and thus $n - (k + 1) + f = 2$. $\square$

Exercise 5.6: I am uncertain how we would make this completely rigorous, but the first inequality follows from the fact that each face is bounded by at least 3 edges, and an edge (at most) has two faces on either side of it. Then we have

$$\begin{aligned} 2 &= n - e + f \\ 2 &\leq n - e + \frac{2}{3}f \\ 2 &\leq n - \frac{1}{3}e \\ e &\leq 3n - 6. \end{aligned}$$

$\square$

Exercise 5.7: Consider a graph G with crossing number $cr(G)$, and n vertices, e edges, and f faces. Then, for each crossing, remove one of the edges in the crossing to result in a new connected graph G'. Now, G' is planar, with n vertices and $e - cr(G)$ edges. Using what we showed in **Exercise 5.6**, this implies $e - cr(G) \leq 3n - 6$, which implies (5.3). $\square$

Exercise 5.8: Let G' be the subgraph of G described at the bottom of page 45. Then,

$$cr(G') \geq e' - 3n' + 6 \geq e' - 3n'$$

where e' is the number of edges in G' and similarly n' is the number of vertices in G'. Then, letting $\mathbb{E}$ denote expected value, we get

$$\mathbb{E}(cr(G')) \geq \mathbb{E}(e') - 3\mathbb{E}(n') \implies p^4 cr(G) \geq ep^2 - 3np.$$

Optimizing in p (by letting $p = \frac{4n}{e}$), we obtain

$$cr(G) \geq \frac{e^3}{16n^2} - 3\frac{e^3}{16n^2} = \frac{e^3}{64n^2}.$$

$\square$

Question 16. *Is there a way to make this inequality sharper? How else can this probability method be used to solve problems? Why is $\mathbb{E}(cr(G')) = p^4 cr(G)$?*

– As per the last question, my best guess is that this follows from the fact that crossing points must arise from two edges, each of comes from 2 vertices, and p is the probability of each of these four vertices surviving.

While I am still a bit confused on **Exercise 5.8**, I certainly understand this outline a lot more than I had. On Wednesday I will work on Exercise 3.3, and start reviewing material from Chapters 2 and 3 as Yuqiu suggested (in preparation for Larry's big project idea for this summer). Speaking of which, Larry, Yuqiu, and I will try to meet next week to discuss material. Larry has an idea for a problem regarding Chapters 2, 3, and 4. I also had another idea for a project, which Larry succinctly summarized:

Remark 17. *I could consider projections of $\mathbb{R}^n$ to k-dimensional planes. "The most general theorem of this type is called the Brascamp-Lieb inequality. There was a lot of work on it in the 60s and 70s (and more recently too). We can talk about it as the summer goes on. It builds on the paper that Iosevich mentions by Loomis-Whitney. Learning the full Loomis-Whitney theorem might also be a good project."*

More to follow.

June 30

Before working on **Exercise 3.3**, I first wanted to check one part of Euler's formula ($n - e + f = 2$):

Question 18. *How would we rigorously show that if there does not exist a vertex that is connected to only one edge that there must exist a loop?*

To show this, we use a proof by contradiction. Assume that there does not exist a loop in a graph G, and that each vertex of G is connected to at least two distinct edges. Then, there must exist a longest path in this graph $P = \{v_1, v_2, \ldots, v_n\}$ of n distinct vertices. Now consider v_n: v_n must be connected to two distinct vertices by assumption. Thus, there must exist an adjacent edge that connects to a vertex v_α. If $v_\alpha = v_i$ for $1 \leq i \leq n-1$, then there exists a loop which is a contradiction. If $v_\alpha \neq v_i$ for $1 \leq i \leq n-1$, then the path $P' = \{v_1, v_2, \ldots, v_n, v_\alpha\}$ is a longer path than P. This is also a contradiction. Thus, there must exist a loop in the graph G if G is planar, and every vertex is connected to at least two distinct edges. $\square$

Exercise 3.3: I started to consider Exercise 3.3 today. I started by trying the dimensional analysis concept described on page 18:

$$inches^4 = ((inches^3)^4)^\alpha \implies \alpha = \frac{1}{3}.$$

Note that the 4 comes from the fact that there are 4 3-D projections of a 4D object. Instead of using π for projections, for now I will use τ. I am hoping that this helps cut down on confusion down the road when we transition to 2D projections. Hence, let

$$\tau_1(x) = (x_2, x_3, x_4), \quad \tau_2(x) = (x_1, x_3, x_4), \quad \tau_3(x) = (x_1, x_2, x_4), \quad \text{and} \quad \tau_4(x) = (x_1, x_2, x_3).$$

Then,

$$\begin{aligned}
N = \#S_N &= \sum_x \chi_{S_N}(x) \\
&\leq \sum_x \chi_{\tau_1(S_N)}(x_2, x_3, x_4)\chi_{\tau_2(S_N)}(x_1, x_3, x_4)\chi_{\tau_3(S_N)}(x_1, x_2, x_4)\chi_{\tau_4(S_N)}(x_1, x_2, x_3) \\
&= \sum_{x_1, x_2, x_3} \chi_{\tau_4(S_N)}(x_1, x_2, x_3) \sum_{x_4} \chi_{\tau_1(S_N)}(x_2, x_3, x_4)\chi_{\tau_2(S_N)}(x_1, x_3, x_4)\chi_{\tau_3(S_N)}(x_1, x_2, x_4) \\
&\leq \left(\sum_{x_1, x_2, x_3} \chi^3_{\tau_4(S_N)}(x_1, x_2, x_3)) \right)^{\frac{1}{3}} \\
&\qquad \cdot \left(\sum_{x_1, x_2, x_3} \left(\sum_{x_4} \chi_{\tau_1(S_N)}(x_2, x_3, x_4)\chi_{\tau_2(S_N)}(x_1, x_3, x_4)\chi_{\tau_3(S_N)}(x_1, x_2, x_4) \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} \\
&= \sqrt[3]{\#\tau_4(S_N)} \cdot \left(\sum_{x_1, x_2, x_3} \left(\sum_{x_4} \chi_{\tau_1(S_N)}(x_2, x_3, x_4)\chi_{\tau_2(S_N)}(x_1, x_3, x_4)\chi_{\tau_3(S_N)}(x_1, x_2, x_4) \right)^{\frac{3}{2}} \right)^{\frac{2}{3}}.
\end{aligned}$$

From here I would want to attempt estimation through interpolation, but am having trouble doing so. In the example where we did this in Chapter 3, we had 5 terms instead of 3, allowing us to pick out terms with independent variables. This however is impossible to do in both of the cases of F_1 and F_2.

Let's say, however, that we were able to complete this and show that

$$N \leq \sqrt[3]{\#\tau_1(S_N)} \cdot \sqrt[3]{\#\tau_2(S_N)} \cdot \sqrt[3]{\#\tau_3(S_N)} \cdot \sqrt[3]{\#\tau_4(S_N)}. \tag{1.2}$$

Then, applying equation (2.2) [on Page 15], we get

$$\begin{aligned}
\#S_N = N &\leq \sqrt[6]{\#\pi_{2,3}\#\pi_{2,4}\#\pi_{3,4}} \cdot \sqrt[6]{\#\pi_{1,3}\#\pi_{1,4}\#\pi_{3,4}} \cdot \sqrt[6]{\#\pi_{1,2}\#\pi_{1,4}\#\pi_{2,4}} \cdot \sqrt[6]{\#\pi_{1,2}\#\pi_{1,3}\#\pi_{2,3}} \\
&= \sqrt[3]{\#\pi_{1,2}(S_N)} \cdot \sqrt[3]{\#\pi_{1,3}(S_N)} \cdot \sqrt[3]{\#\pi_{1,4}(S_N)} \cdot \sqrt[3]{\#\pi_{2,3}(S_N)} \cdot \sqrt[3]{\#\pi_{2,4}(S_N)} \cdot \sqrt[3]{\#\pi_{3,4}(S_N)} \\
&= \left(\prod_{i<j; 1 \leq i,j \leq 4} \#\pi_{i,j}(S_N) \right)^{\frac{1}{3}}.
\end{aligned}$$

This uses the fact that $\pi_1(\tau_1(x)) = (x_3, x_4) \implies \#\pi_1(\#\tau_1(S_N)) = \#\pi_{3,4}(S_N)$. What is left to show is equation (2).

1.5 July 01-07

July 01

Hi Larry! Sorry for the semi-late update, it took me a quick second to write them up. This meeting went very well! I was able to present the proofs of Chapter 4 and Chapter 5. After this, we began to talk about **Exercise 3.3** for a bit, and I described what I had done so far to approach the problem. We both agreed that the way I had started to approach the problem so far (shown above) makes the most natural sense, even if we don't have enough terms for the second sum. We didn't finish this problem, but it was helpful to talk through what I was thinking when I worked on the problem, and the issues I was running into with it. Maybe there is another approach that works better here, though this one follows the process used in the other proofs of the same form in the book.

Afterwards, we started discussing Chapters 7 and 8, and things used in this chapter. In Chapter 8, we discussed the last equation on page 61 in particular. I was confused where the 1 comes up in this problem (it comes from the p_0). Yuqiu and I are also confused about why it's L and not $\frac{L}{2}$. One of my goals for this week is to go back through this material and figure out if there is a typo somewhere, and if not, why it isn't $\frac{L}{2}$. Finally, we talked about the bigger picture– what these proofs actually showed (which I was a bit confused about (though now I am not). These proofs show (by contradiction), the $\#B$ must be bigger than $\frac{q^{\frac{d+2}{2}}}{8}$. The ultimate goal is to keep improving this exponent until we get Cq^d. Though in hindsight this feels pretty straight forward, it was useful to take a step back and take a look at the bigger picture.

All things considered, this week was straightforward in terms of our meeting. My goals for next week is as follows:

1. Read through Chapters 2, 3, and 4 more closely. We hope that this will best prepare me for meeting with Larry next Friday to talk about his problem (which is based on these 3 chapters).
2. Read and work through Chapters 9 and 10. We will talk about these chapters next Thursday.
3. Work through Exercises 6.3, 6.4, and hopefully 3.3.

Excited to meet you next week!!

July 02-03

Over these two days I read through Chapters 2, 3, and 4 again so that these would be fresh in my mind for meeting with Larry this Friday. Nothing too much I noted during this preliminary read, though I am interested in **Exercise 4.7** and how that problem might go.

July 04-05

Over these days I read through Chapters 9 and 10, trying to work on the examples before reading the solution (i.e., trying to prove the value of the infinite sums before reading how they are derived). I do have a few questions and concepts that I hope to talk about on Thursday with Yuqiu.

Firstly, I started on an informal exercise on page 67: generalizing the binomial theorem to trinomials and more. The trinomial problem I was able to solve. Applying the binomial theorem twice,

$$\begin{aligned}(a+b+c)^n &= (a+(b+c))^n \\ &= \sum_{k=0}^{n} C(k,n)a^k(b+c)^{n-k} \\ &= \sum_{k=0}^{n} C(k,n)a^k\left(\sum_{m=0}^{n-k} C(m,n-k)b^m c^{n-k-m}\right) \\ &= \sum_{k=0}^{n}\sum_{m=0}^{n-k} C(k,n)C(m,n-k)a^k b^m c^{n-k-m} \\ &= \sum_{k=0}^{n}\sum_{m=0}^{n-k} \frac{n!}{k!m!(n-k-m)!}a^k b^m c^{n-k-m}.\end{aligned}$$

Doing the same process for a 4-degree polynomial, we get

$$(a+b+c+d)^n = \sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{l=0}^{n-k-m} \frac{n!}{k!m!l!(n-k-m-l)!}a^k b^m c^l d^{n-k-m-l}.$$

If I *had* to guess, I would imagine that we could generalize this to the following formula:

$$\left(\sum_{i=1}^{N} a_i\right)^n = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\cdots\sum_{k_{N-1}=0}^{n-\sum_{i=1}^{N-2}k_i} \frac{n!}{\left(\prod_{i=1}^{N-1}k_i!\right)\left(n-\sum_{i=1}^{N-1}k_i\right)!}\cdot\left(\prod_{i=1}^{N-1}a_i^{k_i}\right)\cdot a_N^{n-\sum_{i=1}^{N-1}k_i}.$$

Right now I wonder if there is a better way to write this series of $\sum$s. Furthermore, if this formula is in fact correct, then I can use induction to prove it. I will wait until Thursday to discuss these ideas with Yuqiu.

Question 19. *How does one formally define expected value?*

Question 20. *How can one intuitively calculate the expected value? I.e., the "'expected' number of flips needed to get heads is 2!" on page 69. I don't quite get how I would intuitively reach this conclusion (even though Iosevich outlines the reasoning).*

Question 21. *I believe equation (9.22) can be proven by induction, but my main question is if one can use logic to inductively prove it (as opposed to brute forcing it with DeMorgan's Laws). I am also confused as to what the equation actually* ***is****. For instance, is the first sum $\sum \#(A_{i1}\cap A_{i2}) = \sum_{i_1\neq i_2; i_1<i_2}\#(A_{i1}\cap A_{i2})$? Should the last term simply be $(-1)^{n+1}\#\bigcap_{i=1}^{n}A_i$ supposed to $A_{i1}\cap A_{i2}\cap\cdots\cap A_{in}$?*

July 06-07

On this day I read through Chapter 8 again to try and figure out if the equation on page 61 should be

$$\#B \geq 1 + \mathbf{\frac{1}{2}} L(q-1) \geq \frac{q^{\frac{d-1}{2}}}{4}.$$

Given that the coefficient of $q^{\frac{d+1}{2}}$ is $\frac{1}{4}$ I believe this should be the case. This is the same concept as in **Question 13** on the notes from June 26-27– we add in this factor of $\frac{1}{2}$ to avoid double counting of the points. $\square$

In reading through this chapter, I believe there is another typo. We should assume that $\#B$ is strictly less than $\frac{q^{\frac{d+1}{2}}}{4}$ and $\frac{q^{\frac{d+2}{2}}}{8}$. Otherwise, we don't reach a contradiction at the end of both proofs, and we would instead obtain that $\#B$ is equal to both of these quantities (which clearly cannot be true as these quantities are not equal for fixed q).

Chapter 2

Project

2.1 July 08-14

July 08

Hi Larry! Excited to meet you tomorrow. I will send these notes after tomorrow's meeting so that I can include notes from that meeting in the weekly update. Yuqiu and I met this week to discuss concepts in Chapter 9, mostly pertaining to the questions that I had throughout this week.

Yuqiu said that my general formula for N degree polynomials to the n power is correct, and noted that we can in fact improve the notation. We can denote

$$\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\cdots\sum_{k_{N-1}=0}^{n-\sum_{i=1}^{N-2}k_i} \quad \text{as} \quad \sum_{k_1+k_2+\cdots+k_N=n}.$$

Hence, $k_N = n - \sum_{i=1}^{N-1} k_1$, and thus the formula simplifies to

$$\left(\sum_{i=1}^{N} a_i\right)^n = \sum_{k_1+k_2+\cdots+k_N=n} \frac{n!}{\left(\prod_{i=1}^{N-1} k_i!\right)\left(n-\sum_{i=1}^{N-1} k_i\right)!}\cdot\left(\prod_{i=1}^{N-1} a_i^{k_i}\right)\cdot a_N^{n-\sum_{i=1}^{N-1}k_i} = \sum_{k_1+k_2+\cdots+k_N=n} n!\cdot\prod_{i=1}^{N}\frac{a_i^{k_i}}{k_i!}.$$

However, using this form to prove the equation by induction is proving a bit notationally difficult. I completed this proof after the meeting: Consider the set $\{a_1,\dots,a_{N+1}\}$. Let

$$b_i = \begin{cases} a_i & 1 \le i \le N-1 \\ a_N + a_{N+1} & i = N \end{cases}.$$

Hence,

$$\begin{aligned}\left(\sum_{i=1}^{N+1} a_i\right)^n &= \left(\sum_{i=1}^{N} b_i\right)^n \\ &= \sum_{k_1+\cdots+k_N=n} n!\cdot\prod_{i=1}^{N}\frac{b_i^{k_i}}{k_i!} \\ &= \sum_{k_1+\cdots+k_N=n} n!\cdot\left(\prod_{i=1}^{N-1}\frac{b_i^{k_i}}{k_i!}\right)\cdot\frac{(a_N+a_{N+1})^{k_N}}{k_N!}.\end{aligned}$$

Applying the binomial theorem,

$$\begin{aligned}
\left(\sum_{i=1}^{N+1} a_i\right)^n &= \sum_{k_1+\cdots+k_N=n} n! \cdot \left(\prod_{i=1}^{N-1} \frac{b_i^{k_i}}{k_i!}\right) \cdot \frac{(a_N+a_{N+1})^{k_N}}{k_N!} \\
&= \sum_{k_1+\cdots+k_N=n} n! \cdot \left(\prod_{i=1}^{N-1} \frac{b_i^{k_i}}{k_i!}\right) \cdot \frac{1}{k_N!} \cdot \sum_{k_{N+1}=0}^{k_N} C(k_{N+1},k_N) a_N^{k_{N+1}} a_{N+1}^{k_N-k_{N+1}} \\
&= \sum_{k_1+\cdots+k_N=n} \sum_{k_{N+1}=0}^{k_N} n! \cdot \left(\prod_{i=1}^{N-1} \frac{a_i^{k_i}}{k_i!}\right) \cdot \frac{1}{k_N!} \cdot C(k_{N+1},k_N) a_N^{k_{N+1}} a_{N+1}^{k_N-k_{N+1}} \\
&= \sum_{k'_1+\cdots+k'_N+k'_{N+1}=n} n! \cdot \left(\prod_{i=1}^{N-1} \frac{a_i^{k_i}}{k_i!}\right) \cdot \frac{1}{k_N!} \cdot \frac{k_N!}{k'_{N+1}!(k'_N-k'_{N+1})!} \cdot a_N^{k'_{N+1}} a_{N+1}^{k'_N-k'_{N+1}}.
\end{aligned}$$

Since $k'_N = n - \sum_{i=1}^{N-1} k_i$ and $k'_{N+1} = n - \sum_{i=1}^{N} k_i$, it follows that $k'_N - k'_{N+1} = k_N$. Hence,

$$\begin{aligned}
\left(\sum_{i=1}^{N+1} a_i\right)^n &= \sum_{k'_1+\cdots+k'_N+k'_{N+1}=n} n! \cdot \left(\prod_{i=1}^{N-1} \frac{a_i^{k'_i}}{k'_i!}\right) \cdot \frac{1}{k_N!} \cdot \frac{k_N!}{k'_{N+1}!(k'_N-k'_{N+1})!} \cdot a_N^{k'_{N+1}} a_{N+1}^{k'_N-k'_{N+1}} \\
&= \sum_{k'_1+\cdots+k'_N+k'_{N+1}=n} n! \cdot \left(\prod_{i=1}^{N-1} \frac{a_i^{k'_i}}{k'_i!}\right) \cdot \frac{a_N^{k'_N}}{k'_N!} \cdot \frac{a_{N+1}^{k'_{N+1}}}{k'_{N+1}!} \\
&= \sum_{k'_1+\cdots+k'_{N+1}=n} n! \cdot \prod_{i=1}^{N} \frac{a_i^{k'_i}}{k'_i!} \\
&= \sum_{k_1+\cdots+k_{N+1}=n} n! \cdot \prod_{i=1}^{N} \frac{a_i^{k_i}}{k_i!}.
\end{aligned}$$

□

I am slightly worried that somewhere here I have primes in the wrong spots, but I believe this is the essence of the rigorous proof.

After this, Yuqiu and I discussed how one formally defines expected value. This conversation included topics like weighted sums, random variables, and probability spaces. We also discussed how one could intuitively calculate the expected value, as is often done in the book to hypothesize what sums should equal. I am still left slightly confused, but for now I am not going to worry about this as I haven't taken a class on probability yet (and this will not be the main topic of the project, which is *projected* to be on Chapters 2-4).

Yuqiu and I also discussed the equation at the top of page 84. We came to the conclusion that the equation should be:

$$\#\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i \#A_i - \sum_{i_1\neq i_2; 1\leq i_1<i_2\leq n} \#(A_{i_1}\cap A_{i_2}) + \cdots + (-1)^{n+1} \cdot \#\left(\bigcap_{i=1}^{n} A_i\right).$$

I tried to think about how I would prove this inductively with rigorous notation, but I feel like the proof on page 84 is already enough.

Between today and tomorrow's meeting, I am going to reread through Chapters 2-4.

July 09

We met today! It was again, super nice to meet you. What follows are the notes from today's meeting– I am very excited to continue working on this project and concept.

We start with a basic concept: if the derivative of a function is 0, then that function is a constant. But what does it mean if the derivative of a function is approximately 0? Is that function approximately a constant? And how would we describe the derivative of a function being *nearly* 0?

Well, consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in C_1$, and $\int_{-\infty}^{\infty} |f'(x)| \, \mathrm{d}x \leq 1$. Then, what can we say about f? Well, using the Fundamental Theorem of Calculus, given $-\infty < a, b < \infty$,

$$\begin{aligned} |f(b) - f(a)| &= \left| \int_a^b f'(x) \, \mathrm{d}x \right| \\ &\leq \int_a^b |f'(x)| \, \mathrm{d}x \\ &\leq \int_{-\infty}^{\infty} |f'(x)| \, \mathrm{d}x \leq 1. \end{aligned}$$

Hence, for all $-\infty < a, b < \infty$, $|f(b) - f(a)| \leq 1$.

How does this concept transfer over to higher dimensions? Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \in C_1$ and $\int_{\mathbb{R}^2} |\nabla f| \, \mathrm{d}\vec{x} \leq 1$. Is it true that $|f(\vec{b}) - f(\vec{a})| \leq 1$ for all $\vec{a}, \vec{b} \in \mathbb{R}^2$? No. Here is a counter example: Consider the function

$$f_\epsilon(\vec{x}) = \begin{cases} 0 & |x| > 2\epsilon \\ 1 & x = 0 \end{cases}$$

and furthermore f_ϵ smoothly interpolates between 0 and 1 for all values of $\vec{x}$ such that $0 < |\vec{x}| < 2\epsilon$. Then, ∇f is supported in the region where $0 < |\vec{x}| < 2\epsilon$, and $|\nabla f| \leq \frac{2}{\epsilon}$. Hence,

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} &= \int_{0 \leq |x| \leq 2\epsilon} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} + \int_{|x| > 2\epsilon} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} \\ &= \int_{0 \leq |x| \leq 2\epsilon} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} \\ &= area(Circle) \cdot \frac{2}{\epsilon} \\ &\leq C\epsilon \xrightarrow{\epsilon \to 0} 0 \end{aligned}$$

and where $C > 0$. However, it is not necessarily the case that $|f(\vec{b}) - f(\vec{a})| \leq 1$ for all $\vec{a}, \vec{b} \in \mathbb{R}^2$.

Hence, from here we want to try and prove something weaker. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \in C_1$ and f is compactly supported (i.e. that there exists some rectangle such that for every (x, y) outside of this rectangle, $f(x, y) = 0$). Furthermore, suppose that

$$\int_{\mathbb{R}^2} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} \leq 1.$$

Now given this, we want to estimate the $area(\{\vec{x} \mid f(\vec{x}) > \lambda\})$ in terms of λ.

Question 22. *How will estimating this value help us better understand this problem? I.e., is there a bigger problem that we are actually trying to solve that this will be a key step in?*

Question 23. *Larry mentioned that long term, a question we could look at is the integral $\int_{\mathbb{R}^2} |f|^p \, \mathrm{d}\vec{x}$, but let's not get ahead of ourselves.*

In any case, my first idea in the meeting was to consider the set $U_\lambda = \{\vec{x} \mid |f(\vec{x})| > \lambda\}$, and split the integral of ∇f over $\mathbb{R}^2$ into two integrals over U_λ and U_λ^c. However, this immediately led into some more issues/questions. We have bounds on the integral of ∇f, but how do we relate this to just f? Hence, we went back to the drawing board, in which I suggested using the FTC in one variable as follows:

$$1 \geq \int_{\mathbb{R}^2} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} = \int_{\mathbb{R}} \int_{\mathbb{R}} \sqrt{(\partial_x f)^2 + (\partial_y f)^2} \, \mathrm{d}x \, \mathrm{d}y.$$

Therefore, $\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_x f| \,\mathrm{d}x\,\mathrm{d}y \leq 1$ and $\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_y f| \,\mathrm{d}x\,\mathrm{d}y \leq 1$. Let's consider the first of these integrals. Using what we had shown earlier for single variable functions (with the FTC), we can obtain the following: For all $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned} |f(x_1,y) - f(x_2,y)| &= \left| \int_{x_1}^{x_2} \partial_x f(x,y)\,\mathrm{d}x \right| \\ &\leq \int_{x_1}^{x_2} |\partial_x f(x,y)|\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}} |\partial_x f(x,y)|\,\mathrm{d}x. \end{aligned}$$

Integrating both sides of this inequality, we get that for all $x_1, x_2 \in \mathbb{R}$ (which may depend on y)

$$\int_{\mathbb{R}} |f(x_1,y) - f(x_2,y)|\,\mathrm{d}y \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y \leq 1. \tag{2.1}$$

Replacing x with y, we can similarly get that for all $y_1, y_2 \in \mathbb{R}$ (which may depend on x),

$$\int_{\mathbb{R}} |f(x,y_1) - f(x,y_2)|\,\mathrm{d}x \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_y f(x,y)|\,\mathrm{d}y\,\mathrm{d}x \leq 1. \tag{2.2}$$

Then we went back to my previous idea of splitting an integral over $\mathbb{R}^2$ into an integral over U_λ and U_λ^c– but instead of looking at the integral of ∇f, consider the integral of f:

$$\int_{\mathbb{R}^2} |f(\vec{x})|\,\mathrm{d}\vec{x} = \int_{U_\lambda} |f(\vec{x})|\,\mathrm{d}\vec{x} + \int_{U_\lambda^c} |f(\vec{x})|\,\mathrm{d}\vec{x} \geq \int_{U_\lambda} |f|\,\mathrm{d}\vec{x} \geq \lambda \cdot area(U_\lambda).$$

Hence, if we can get an upper bound on the left hand side in terms of λ, we can get an inequality for $area(U_\lambda)$.

Question 24. *Can we get this upper bound using equations (2.1) and (2.2)?*

Question 25. *Also, how do projections come into this? Given that this project should utilize concepts from Chapters 2-4.*

As for question 25, perhaps we can state the following: for $\Omega \subset \mathbb{R}^2$,

$$area(\Omega) \leq m(\pi_1(\Omega)) \cdot m(\pi_2(\Omega)). \tag{2.3}$$

One could prove this in a similar manner to Exercise 2.3, though it arguably makes more sense pictorially. We surround Ω by the smallest possible rectangle(s) such that Ω is completely in said rectangles. Though, I have the same question that I had for 2.3– must Ω be convex? I imagine not, based on the pictoral reasoning, but I am uncertain. In any case, we have the following equation:

$$area(U_\lambda) \leq m(\pi_1(U_\lambda)) \cdot m(\pi_2(U_\lambda)). \tag{2.4}$$

Here I changed $\pi_1(U_\lambda)$ to $m(\pi_1(U_\lambda)$, where m is the measure function.

There was only a bit more we discussed in today's meeting. Let $y \in \pi_1(U_\lambda)$. Now we are going to use equation (2.1). Since $y \in \pi_1(U_\lambda)$, there exists an $x_2 \in \mathbb{R}$ such that $x_2 = \sup\{x \in \pi_2(U_\lambda) \mid f(x,y) = \lambda\}$. Furthermore, since f is compactly supported, there exists an $x_1 < x_2 \in \mathbb{R}$ such that $f(x_1, y) = 0$. Hence, using these values of x_1 and

x_2 for each $y \in \pi_1(U_\lambda)$ in equation (2.1), we get that

$$\begin{aligned}
1 &\geq \int_{\pi_1(U_\lambda)} |f(x_1, y) - f(x_2, y)| \, \mathrm{d}y \\
&\geq \int_{\pi_1(U_\lambda)} |\lambda| \, \mathrm{d}y \\
&= |\lambda| \cdot m(\pi_1(U_\lambda)) \implies m(\pi_1(U_\lambda)) \leq \frac{1}{|\lambda|}.
\end{aligned}$$

This leaves me slightly concerned though, as when we met we had come to a different conclusion, namely

$$\int_{\mathbb{R}} |\partial_x f(x, y)| \, \mathrm{d}x \geq \lambda.$$

[At least this is what I have in my notes.]

For now, assuming that I didn't make a mistake previously, we can similarly state that

$$1 \geq |\lambda| \cdot m(\pi_2(U_\lambda)) \implies m(\pi_2(U_\lambda)) \leq \frac{1}{|\lambda|}.$$

Question 26. *Could we conclude from here, using equation (2.4), that*

$$area(U_\lambda) \leq m(\pi_1(U_\lambda)) \cdot m(\pi_2(U_\lambda)) \leq \frac{1}{\lambda^2}?$$

Question 27. *Even if this is a solution, can we reach a stronger upper bound by finding an upper bound on $\int_{\mathbb{R}^2} |f(\vec{x})| \, \mathrm{d}\vec{x}$? Would finding an upper bound on this integral help us begin to look at **Question 23**?*

July 10

My first goal for today is to rigorously prove equation (2.4):

$$\begin{aligned}
area(\Omega) &= \int_{\mathbb{R}^2} \chi_\Omega(\vec{x}) \, \mathrm{d}\vec{x} \\
&\leq \iint \chi_{\pi_1(\Omega)}(x_2) \chi_{\pi_2(\Omega)}(x_1) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&= \int_{\mathbb{R}} \chi_{\pi_1(\Omega)}(x_2) \, \mathrm{d}x_2 \cdot \int_{\mathbb{R}} \chi_{\pi_2(\Omega)}(x_1) \, \mathrm{d}x_1 \\
&= m(\pi_1(\Omega)) \cdot m(\pi_2(\Omega)).
\end{aligned}$$

The question still remains– does Ω need to be convex though? If so, is U_λ convex?

Now I want to figure out (presuming I am right about Question 27) what we can say about $area(U_\lambda)$ if

$$\int_{\mathbb{R}^2} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} \leq \alpha$$

where $\alpha \geq 0$. Going through the same process as before with αs instead of 1s, we can conclude that

$$area(U_\lambda) \leq \frac{\alpha^2}{\lambda^2}.$$

While this certainly gives us a way to calculate an upper bound for $area(U_\lambda)$, I wonder if there is a way to find an upper bound on $\int_{\mathbb{R}^2} |f| \, \mathrm{d}x$ that could lend itself nicely to Question 23, and that could possibly allow us to have a weaker condition than f being compactly supported.

July 11-12

On July 11th I worked on trying to prove equation (2.3). I did so by drawing a lot of pictures and trying to visualize $m(\pi_1(\Omega)) \cdot m(\pi_2(\Omega))$ as a rectangle, but I was having trouble figuring out the exact process for proving the equation. However, on July 12th I think I finished the problem.

Consider the sets $X = \pi_2(\Omega)$ and $Y = \pi_1(\Omega)$, and consider the Cartesian product $X \times Y$. It is clear that $\Omega \subset X \times Y$, as by definition

$$X \times Y = \{(x, y) \mid x \in \pi_2(\Omega) \text{ and } y \in \pi_1(\Omega)\}.$$

Therefore,

$$area(\Omega) = \int_{\mathbb{R}^2} \chi_\Omega(x, y) \,\mathrm{d}x \,\mathrm{d}y \leq \int_{\mathbb{R}^2} \chi_{X \times Y}(x, y) \,\mathrm{d}x \,\mathrm{d}y = m(\pi_1(\Omega)) \cdot m(\pi_2(\Omega)).$$

I feel like this should be sufficient, whether or not Ω is convex. However, this argument doesn't easily lend itself to the same equation for 3-dimensions, as here we want that

$$vol(\Omega) \leq \sqrt{area(\pi_1(\Omega))} \cdot \sqrt{area(\pi_2(\Omega))} \cdot \sqrt{area(\pi_1(\Omega))}$$

as opposed to (the true statement)

$$vol(\Omega) \leq area(\pi_1(\Omega)) \cdot area(\pi_2(\Omega)) \cdot area(\pi_1(\Omega)).$$

July 13

I started thinking about how we could possibly prove the last statement from July 12. I was initially frustrated with the fact that we couldn't immediately translate the previous method to this equation, but now I think we might be able to. Consider the following:

Let $X = \{x \mid (x, y, z) \in \Omega\}$, and similarly define Y and Z. Then, it is clear that

$$vol(\Omega) \leq m(X) \cdot m(Y) \cdot m(Z)$$

as this is like surrounding every disjoint path-connected subset of Ω with it's own rectangular prism (similar to surrounding every disjoint path-connected subset of Ω in the 2-D case with it's own little rectangle). Then,

$$vol(\Omega) \leq \sqrt{m(x) \cdot m(Y)} \cdot \sqrt{m(X) \cdot m(Z)} \cdot \sqrt{m(Y) \cdot m(Z)}.$$

Hence, using the 2-D analogous equation, we get

$$vol(\Omega) \leq \sqrt{m(x) \cdot m(Y)} \cdot \sqrt{m(X) \cdot m(Z)} \cdot \sqrt{m(Y) \cdot m(Z)} \geq \sqrt{area(\pi_1(\Omega))} \cdot \sqrt{area(\pi_2(\Omega))} \cdot \sqrt{area(\pi_3(\Omega))}.$$

So far this has all the *parts* of the inequality we want, but is just slightly off. This makes me hopeful that this concept is close to the right approach.

I had another question in what I have done so far:

Question 28. *Is it true that*

$$\int_{\mathbb{R}^2} |\nabla f(\vec{x})| \,\mathrm{d}\vec{x} \leq 1 \implies \int_{\mathbb{R}^2} |\partial_x f(\vec{x})| \,\mathrm{d}\vec{x} \leq 1?$$

We said this is true in the meeting, and I thought this might follow the triangle inequality. However, I do not think this is the case:

$$\int_{\mathbb{R}^2} |\partial_x f(\vec{x})| \,\mathrm{d}\vec{x} + \int_{\mathbb{R}^2} |\partial_y f(\vec{x})| \,\mathrm{d}\vec{x} = \int_{\mathbb{R}^2} |\partial_x f(\vec{x})| + |\partial_y f(\vec{x})| \,\mathrm{d}\vec{x}.$$

Applying the triangle inequality,

$$\int_{\mathbb{R}^2} |\partial_x f(\vec{x})| \, \mathrm{d}\vec{x} + \int_{\mathbb{R}^2} |\partial_y f(\vec{x})| \, \mathrm{d}\vec{x} \geq \int_{\mathbb{R}^2} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} \leq 1.$$

This makes me slightly worried that what we had discussed in the meeting might have been incorrect.

July 14

Using what I have found so far (which I am not certain is fully correct), I was wondering how we might apply this to $|f(x_1, y_1) - f(x_2, y_2)|$ for any two points in $\mathbb{R}^2$.

Let $f(x_1, y_1) = \lambda$, and let $|f(x_1, y_1) - f(x_2, y_2)| < \alpha$. My goal here is to try and find the "volume" of elements in $\mathbb{R}^2$ such that this inequality is satisfied. It must be true that $(x_2, y_2) \in U_i$ for $i \in [\lambda - \alpha, \lambda + \alpha]$. Hence consider the following:

$$\begin{aligned}
\int_{\lambda-\alpha}^{\lambda+\alpha} area(U_\beta) \, \mathrm{d}\beta &\leq \int_{\lambda-\alpha}^{\lambda+\alpha} \frac{1}{\beta^2} \, \mathrm{d}\beta \\
&= \frac{1}{\lambda - \alpha} - \frac{1}{\lambda + \alpha} \\
&= \frac{2\alpha}{\lambda^2 - \alpha^2}.
\end{aligned}$$

I am uncertain how helpful this currently is.

2.2 July 15-21

July 15-16

Hi Larry! I hope the past week has treated you well. It has been very fun working on this problem! For our purposes, I am going to make the notes from today both an update and summary of key results for this problem, from throughout the notes so far.

Firstly, we discussed a bit of confusion regarding the notation $|\nabla f|$, which deals with my confusion from July 13th. I thought we meant that

$$|\nabla f| = |\partial_x f + \partial_y f| \, .$$

However, Yuqiu said that we take $|\nabla f|$ to be

$$|\nabla f| = \sqrt{(\partial_x f)^2 + (\partial_y f)^2}. \tag{2.5}$$

Hence, now it makes sense why

$$\int_{\mathbb{R}^2} |\partial_x f(\vec{x})| \leq 1 \quad \text{and} \quad \int_{\mathbb{R}^2} |\partial_y f(\vec{x})| \, \mathrm{d}\vec{x} \leq 1.$$

We then discussed how I proved $area(\Omega) \leq m(\pi_1(\Omega)) \cdot m(\pi_2(\Omega))$ for all $\Omega \subset \mathbb{R}^2$:

Consider the sets $X = \pi_2(\Omega)$ and $Y = \pi_1(\Omega)$, and consider the Cartesian product $X \times Y$. It is clear that $\Omega \subset X \times Y$, as by definition

$$X \times Y = \{(x, y) \mid x \in \pi_2(\Omega) \text{ and } y \in \pi_1(\Omega)\}.$$

Therefore,

$$area(\Omega) = \int_{\mathbb{R}^2} \chi_\Omega(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathbb{R}^2} \chi_{X\times Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = m(\pi_1(\Omega)) \cdot m(\pi_2(\Omega)).$$

I asked Yuqiu if there might be a similar way to prove the inequality for the volume, as I wasn't quite certain if the proof I had done earlier this summer was sufficient. I was worried that the requirement of convexity had snuck its way into the proof somehow. Thus, we went back to that proof and went through it line by line:

Let Ω be a subset of $\mathbb{R}^3$, and let $A_i = \sqrt{area(\pi_i(\Omega))}$. Then,

$$\begin{aligned}
vol(\Omega) &= \int_{\mathbb{R}^3} \chi_\Omega(x)\,\mathrm{d}x \\
&\leq \iiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\chi_{\pi_3(\Omega)}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&= \iint \chi_{\pi_3(\Omega)}(x_1,x_2)\left(\int \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\,\mathrm{d}x_3\right)\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\leq \left(\iint \chi^2_{\pi_3(\Omega)}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}\left(\iint\left(\int \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\,\mathrm{d}x_3\right)^2\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \\
&\leq A_3^{1/2}\left(\iiiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\chi_{\pi_1(\Omega)}(x_2,x_3')\chi_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_3\,\mathrm{d}x_3'\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \\
&\leq A_3^{1/2}\left(\iiiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_3\,\mathrm{d}x_3'\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \\
&= A_3^{1/2}\left(\iint \chi^2_{\pi_1(\Omega)}(x_2,x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3\right)^{1/2}\left(\iint \chi^2_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_1\,\mathrm{d}x_3'\right)^{1/2} = A_1^{1/2}A_2^{1/2}A_3^{1/2}.
\end{aligned}$$

There was a few typos here and there (i.e., double integrals where there should be one integral, and vice versa), but we ultimately came to the conclusion that this proof is sufficient for all subsets of $\mathbb{R}^3$.

Question 29. *Is there a way to prove that*

$$vol(\Omega) \leq \sqrt{area(\pi_1(\Omega))} \cdot \sqrt{area(\pi_2(\Omega))} \cdot \sqrt{area(\pi_3(\Omega))}$$

geometrically? In a similar way to how I proved the 2-D case using $X \times Y$.

After this, we started to discuss the actual project some more. I raised a concern that to get the equation $area(U_\lambda) \leq \frac{1}{\lambda^2}$, we have to use the fact that $\lambda \neq 0$. It seems unlikely that we will not be able to reach an inequality for U_0, but we dismissed this for now. I also asked if f must be bounded– as the equation $area(U_\lambda) \leq \frac{1}{\lambda^2}$ implies that f is unbounded. Yuqiu rephrased this question as: how **high** can f be from 0?

Yuqiu also pointed out something interesting: In this problem we have assumed that f is compactly supported, but *not every function is like this.* In fact, functions of the form $f(x,y) = c \in \{\mathbb{R} - 0\}$ have $\nabla f = 0$, however these functions are not compactly supported.

Yuqiu then said that $area(U_\lambda) \leq \frac{1}{\lambda^2}$ suggests a nice connection to $L^2(\mathbb{R}^2)$.

Conjecture 30

Maybe we can say something along the lines of:

$$area(U_\lambda) \leq \frac{1}{\lambda^2} \iff \left(\int_{\mathbb{R}^2} |f(\vec{x})|^2\,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} \leq 1.$$

One of these directions is more clear than the other certainly:

$$\begin{aligned} 1 &\geq \int_{\mathbb{R}^2} |f(\vec{x})|^2 \,\mathrm{d}\vec{x} \\ &\geq \int_{U_\lambda} |f(\vec{x})|^2 \,\mathrm{d}\vec{x} \\ &\geq \int_{U_\lambda} |\lambda|^2 \,\mathrm{d}\vec{x}. \end{aligned}$$

Hence,

$$area(U_\lambda) = \int_{U_\lambda} 1 \,\mathrm{d}\vec{x} \leq \frac{1}{\lambda^2}.$$

Yuqiu said that this is known as Chebyshev's inequality, which comes up in probability.

In trying to prove the other direction, Yuqiu suggested introducing new notation to be able and split the integral up into different U_λs:

$$V_\lambda := \{\vec{x} \mid 2\lambda > f(\vec{x}) \geq \lambda\}.$$

Note that $V_\lambda \subset U_\lambda$. Then, we can consider the following:

$$\begin{aligned} \left(\int_{\mathbb{R}^2} |f(\vec{x})| \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} &= \left(\sum_{k\in\mathbb{N}} \int_{V_{2^k}} |f(\vec{x})|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k\in\mathbb{N}} \left(2^{k+1}\right)^2 \cdot area(V_{2^k})\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k\in\mathbb{N}} 2^{2(k+1)} \cdot \frac{1}{2^{2k}}\right)^{\frac{1}{2}} \\ &= 2\left(\sum_{k\in\mathbb{N}} 1\right)^{\frac{1}{2}}. \end{aligned}$$

This sadly diverges, but if we allow f to be bounded, then this becomes a finite sum. Let $f(\vec{x}) \leq Z \in \mathbb{R}$ for all $\vec{x} \in \mathbb{R}^2$. Then,

$$\begin{aligned} \left(\int_{\mathbb{R}^2} |f(\vec{x})|^2 \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} &\leq 2\left(\sum_{k=1}^{\lceil \log_2(Z) \rceil} 1\right)^{\frac{1}{2}} \\ &= 2 \cdot \sqrt{\lceil \log_2(Z) \rceil}. \end{aligned}$$

This sadly diverges if $Z \to \infty$, which means (at least with the approximations done thus far), f must be bounded. However, Yuqiu and I quickly realized some issues that come up here.

Issues start to come up if $\lambda \leq 0$ (in regards to U_λ), and especially when $\lambda = 0$. This is sadly something that we can't just dismiss by putting a condition on f, as it cannot be the case that there exists an ϵ such that $f(\vec{x}) > \epsilon > 0$ for all $\vec{x}$ *and* have it be the case that f is compactly supported.

So, we are left with a few questions:

Question 31. *How can we estimate $area(U_0)$? Is there a way around this? How can we incorporate negative values for $f(\vec{x})$ into the sum to try and estimate $||f||_{L^2(\mathbb{R}^2)}$?*

Yuqiu suggested revisiting the inequalities derived thus far to work on making them sharper. Namely, he suggested that we could look into making this series of inequalities sharper:

$$\begin{aligned}
1 &\geq \int_{\mathbb{R}} |f(x_1, y) - f(x_2, y)| \, \mathrm{d}y \\
&\geq \int_{\pi_1(U_\lambda)} |f(x_1, y) - f(x_2, y)| \, \mathrm{d}y \\
&\geq \int_{\pi_1(U_\lambda)} |\lambda| \, \mathrm{d}y \\
&= |\lambda| \cdot m(\pi_1(U_\lambda)) \implies m(\pi_1(U_\lambda)) \leq \frac{1}{|\lambda|}.
\end{aligned}$$

Here we go from $\mathbb{R}$ to $\pi_1(U_\lambda)$, which throws out a lot of elements in $\mathbb{R}$. To make this sharper, we could consider the following:

$$\begin{aligned}
1 &\geq \int_{\mathbb{R}} |f(x_1, y) - f(x_2, y)| \, \mathrm{d}y \\
&\geq \sum_{k=1}^{N} \int_{\pi_1(V_{2^k})} |f(x_1, y) - f(x_2, y)| \, \mathrm{d}y \\
&\geq \sum_{k=1}^{N} 2^k \cdot m(\pi_1(V_{2^k}))
\end{aligned}$$

where $N \in \mathbb{N}$. The hope is to possibly use this inequality and another for π_2, to derive a new inequality for $area(U_\lambda)$. We also hope that it may be possible to utilize the CS-inequality to do so. This concept hasn't been fully explored yet.

Before we left, I wanted to consider to see if how we defined V_λ affected the final result. My hope is that changing this definition could lead to a convergent sum in the case where f is unbounded.

Let

$$W_{\alpha,\lambda} = \{\vec{x} \mid \alpha\lambda > f(\vec{x}) \geq \lambda\}.$$

Then,

$$\begin{aligned}
\left(\int_{\mathbb{R}^2} |f(\vec{x})|^2 \, \mathrm{d}\vec{x} \right)^{\frac{1}{2}} &= \left(\sum_{k \in \mathbb{N}} \int_{W_{\alpha,\alpha^k}} |f(\vec{x})|^2 \, \mathrm{d}\vec{x} \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k \in \mathbb{N}} \alpha^{2(k+1)} \cdot \frac{1}{\alpha^{2k}} \right)^{\frac{1}{2}}.
\end{aligned}$$

Letting f be bounded by Z again, we get

$$\begin{aligned}
&\leq \left(\sum_{k=1}^{\lceil \log_\alpha(Z) \rceil} \alpha^2 \right)^{\frac{1}{2}} \\
&\leq \alpha \cdot \sqrt{\lceil \log_\alpha(Z) \rceil}.
\end{aligned}$$

I am currently uncertain if this is a better upper bound then the one derived before where $\alpha = 2$, but in either case it still seems to require f to be bounded, as letting $Z \to \infty$ should result in this upper bound go to ∞ as well (if, very slowly). This also does not address the issue of negative values for f.

A lot of progress has been made over the last week! I am excited to continue working on the questions at hand here. My goal is to try make a better estimate for $area(U_\lambda)$ that addresses the issues where $\lambda \approx 0$. I am also hopeful that long term this process will lead to a way to estimate $||f||_{L^p(\mathbb{R}^2)}$ for $p \geq 1$.

July 17

Today I worked on trying to find a bound on the $area(U_0)$. Based on the approximation found so far (namely, $area(U_\lambda) \leq \frac{1}{\lambda^2}$ where $\lambda \neq 0$), it seems to be the case that $area(U_0)$ might just be infinite. This make sense based off of our conditional statements for f: we assume f to be compactly supported, which means there must exist an infinitely large subset of $\mathbb{R}^2$ such that $f(\vec{x}) = 0$. However, consider this:

If $\lambda_1 \leq \lambda_2$, then $U_{\lambda_2} \subset U_{\lambda_1}$ by construction. Thus, $area(U_\lambda) \leq area(U_{\lambda_2})$. Let $\lambda_2 = 0$ and $\lambda_1 = \alpha < 0$. Hence,

$$area(U_0) \leq area(U_\alpha) \leq \frac{1}{\alpha^2} \xrightarrow{\alpha \to -\infty} 0.$$

This seems to be a contradiction with what we have assumed in the problem– that $area(U_0)$ must be infinite. However, perhaps we can redefine U_λ to address a few of our concerns so far. Maybe, for the moment, we can assume that f is strictly non-negative, and then later on redefine our sets U_λ to deal with the cases where f is negative. One idea might be to consider the set

$$A_\lambda = \{\vec{x} \mid |f(\vec{x})| \geq \lambda\}.$$

Thus, $area(A_\lambda) = \infty$ if $\lambda \leq 0$, which satisfies our understanding of f being compactly supported, that wasn't seen by our previous approximation for $area(U_\lambda)$. However, then another question arises– even though this deals with negative values for λ, what is the new approximation for $area(A_\lambda)$?

July 21

In talking to Yuqiu, we believe that the inequality

$$area(U_\lambda) \leq \frac{1}{\lambda^2}$$

should be incorrect if $\lambda < 0$ for the reasons on July 17th, but we aren't certain what goes wrong in the proof. Thus, I spent today going back over the proof of the above inequality to see where, if anything, something should go wrong.

To derive this inequality, we stated that $m(\pi_i(U_\lambda)) \leq \frac{1}{\lambda}$. However, this certainly shouldn't be true if $\lambda < 0$, as f is compactly supported and thus the subset of $\mathbb{R}^2$ where $f(\vec{x}) > \lambda$ is infinite. This would imply that the $m(\pi_i(U_\lambda))$ should also be infinite. It seems reasonable that this should be the case if $\lambda > 0$, but clearly there is some misconception in those few lines. As of right now, I do not particularly see what breaks in my line of reasoning. For now, I am going to go back over the proof so far to try and figure this out. If I still cannot see what is going wrong, I plan to try going through the proof using the notation A_λ to see if this error fixes itself in this method.

Next, I started to consider estimating $m(\pi_i(U_\lambda))$ in terms of λ. Yuqiu and I had started to discuss this earlier, though I realize there was an issue with our reasoning. We stated that

$$\begin{aligned} 1 &\geq \int_{\mathbb{R}} |f(x_1, y) - f(x_2, y)| \, \mathrm{d}y \\ &\geq \sum_{k=0}^{N} \int_{V_{2^k}} |f(x_1, y) - f(x_2, y)| \, \mathrm{d}y. \end{aligned}$$

These two lines are certainly correct, and the hope is to use them to estimate the $area(U_\lambda)$. However, from here we have a few options. Before, we chose x_1 amd x_2 such that this difference was always λ. Here, we have a few more options:

Option 1: Assume that f is unbounded. Then, there exists x_1 and x_2 such that $f(x_1, y) = 2^k$ and $f(x_2, y) = 2^{k+1}$. At the very least, we want to assume that there exists $\vec{x}$ such that $f(\vec{x}) \geq 2^{N+1}$ for this difference to work. Then, we have

$$\begin{aligned}
1 &\geq \int_{\mathbb{R}} |f(x_1, y) - f(x_2, y)| \,\mathrm{d}y \\
&\geq \sum_{k=0}^{N} \int_{\pi_1(V_{2^k})} |f(x_1, y) - f(x_2, y)| \,\mathrm{d}y \\
&= \sum_{k=0}^{N} \int_{\pi_1(V_{2^k})} |2^{k+1} - 2^k| \,\mathrm{d}y \\
&= \sum_{k=0}^{N} |2^{k+1} - 2^k| \cdot m(\pi_1(V_{2^k})).
\end{aligned}$$

Option 2: Here, we do not need to assume that f is unbounded. Choose x_1 and x_2 such that $f(x_1, y) = 0$ and $f(x_2, y) = 2^k$. Note that if there does not exist an x_2 such that $f(x_2, y) = 2^k$, then $V_{2^k} = \emptyset$, and thus this wouldn't effect our sum in the end. Hence, we have

$$\begin{aligned}
1 &\geq \int_{\mathbb{R}} |f(x_1, y) - f(x_2, y)| \,\mathrm{d}y \\
&\geq \sum_{k=0}^{N} \int_{\pi_1(V_{2^k})} |f(x_1, y) - f(x_2, y)| \,\mathrm{d}y \\
&= \sum_{k=0}^{N} \int_{\pi_1(V_{2^k})} |2^k| \,\mathrm{d}y \\
&= \sum_{k=0}^{N} |2^k| \cdot m(\pi_1(V_{2^k})).
\end{aligned}$$

Perhaps we can use these inequalities together to solve for $\sum m(\pi_1(V_{2^k}))$. Note that

$$\sum_{k=\alpha}^{\infty} m(V_{2^k}) = m(U_{2^\alpha})$$

as this is the union of all elements in $\mathbb{R}^2$ that have value at least value 2^α when plugged into f. This is how I hope to go from an estimate of $\sum m(\pi_1(V_{2^k}))$ to $m(\pi_1(U_\lambda))$.

2.3 July 22-29

July 22

Good evening Larry! I hope the week has treated you well, we have made quite a bit of progress I think on this project.

First we started to discuss different possible ways to define U_λ and V_λ. I think the following definitions might be useful to deal with the cases where f is negative: (let $\lambda \neq 0$)

$$U_\lambda := \begin{cases} \{\vec{x} \mid f(\vec{x}) \geq \lambda\} & \lambda > 0 \\ \{\vec{x} \mid f(\vec{x}) \leq \lambda\} & \lambda < 0 \end{cases} \quad \text{and} \quad V_\lambda := \begin{cases} \{\vec{x} \mid 2\lambda > f(\vec{x}) \geq \lambda\} & \lambda > 0 \\ \{\vec{x} \mid 2\lambda < f(\vec{x}) \leq \lambda\} & \lambda < 0 \end{cases}.$$

This is particularly nice as $area(U_\lambda) \leq \frac{1}{\lambda^2}$ for all $\lambda \neq 0$ (at least conceptually this makes more sense for $\lambda < 0$). Then, we can start considering splitting up the integral we did last Thursday, but with negative values for λ: Let $N = \lceil \log_2(\sup_{\vec{x}\in\mathbb{R}^2} |f(\vec{x})|) \rceil$. Then,

$$\begin{aligned}
\left(\int_{\mathbb{R}^2} |f(\vec{x})| \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} &= \left(\sum_{k=-N}^{N} \int_{V_{2^k}} |f(\vec{x})|^2 \,\mathrm{d}x + \int_{V_{-2^k}} |f(\vec{x})|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \left(2 \cdot \sum_{k=-N}^{N} \left(2^{k+1}\right)^2 \cdot area(V_{2^k})\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=0}^{N} 16\right)^{\frac{1}{2}} \\
&= 4 \cdot \sqrt{\left\lceil \log_2\left(\sup_{\vec{x}\in\mathbb{R}^2} |f(\vec{x})|\right)\right\rceil}.
\end{aligned}$$

Except, there's an issue here, and it isn't $\lambda = 0$. Note that we don't particularly need to worry about $\lambda = 0$ as $\int_{U_0} |f(\vec{x})|^2 \,\mathrm{d}\vec{x} = 0$. However, the issue now includes numbers *close* to 0, as these aren't all included in $V_{2^{-N}}$. Hence, the first line above should not be equality but $\geq$. For now, Yuqiu and I dismissed this problem to discuss another that is somewhat related:

Question 32. *Is $area(U_\lambda) \leq 1/\lambda^2$ approximately sharp? Can you find some examples of functions f where U_λ is close to that upper bound for almost all λ?*

Larry proposed this question last week and I have been playing around with possible pictures to consider. I wanted to use the fact that $\frac{1}{\lambda^2}$ is the area of a square with side-length $\frac{1}{\lambda}$, and the fact that if $\lambda_1 \leq \lambda_2$, $U_{\lambda_2} \subset U_{\lambda_1}$. To describe the example, I first start off with a definition:

Consider the following function: $f(x, y) = \lambda$ if (x, y) is on the square of side-length $\frac{1}{\lambda}$ centered at the origin. Thus, $area(U_\lambda) = \frac{1}{\lambda^2}$ for all $\lambda \neq 0$. For the time being, I will call squares of side length k centered at the origin, a square of radius $\frac{1}{2k}$.

However, f is neither continuous (discontinuous at the origin) nor compactly supported. Hence, Yuqiu and I discussed how we can "fix" this example. In the region between the squares of radius R and $2R$, let f smoothly interpolate to 0, and let f be 0 outside of the square of radius $2R$. This lets f be compactly supported, and we can make R arbitrarily large (though finite), implying that $area(U_\lambda) \leq \frac{1}{\lambda^2}$ will be approximately sharp, or exactly sharp, almost everywhere when R is sufficiently large. Addressing the discontinuity at the origin, in the region of the square of radius $\frac{1}{R}$ (related to the R in fixing compactly supported), let $f(\vec{x})$ smoothly interpolate to the value R.

After making these changes to f, it seems to be the case that this works as a sufficient counterexample to show that $area(U_\lambda) \leq \frac{1}{\lambda^2}$ is approximately sharp.

In discussing this question, I also realized that f *must* be bounded. This follows directly from f being both continuous and compactly supported (which I hadn't quite internalized up until this point). After this, Yuqiu and I started to discuss why we look at $||f||_{L^2(\mathbb{R}^2)}$ instead of L^3 or L^p for that matter.

First we considered our Main Problem in a slightly different way. We want to find an upperbound for $||f||_{L^p}$ **only given** $||\nabla f||_{L^1} \leq 1$. We can rephrase this question as follows: does there exist $C > 0$ such that

$$F(f, p, \alpha) = \frac{||f||_{L^p}}{||\nabla f||_{L^1}^{\alpha}} \leq C?$$

We can show that we can only find such a C if $p = 2$ and $\alpha = 1$. Assume, for the sake of contradiction that there exists such a $C > 0$ for all f, α, and p (with $f \in C^1$). Then, let $\beta > 0$ and consider the following:

It should be the case that $F(\beta f, p, \alpha) = \frac{||\beta f||_p}{||\nabla \beta f||_1^\alpha} \leq C$. However, $\frac{||\beta f||_p}{||\nabla \beta f||_1^\alpha} = \frac{\beta}{\beta^\alpha} \frac{||f||_p}{||f||_1^\alpha} \leq \frac{\beta}{\beta^\alpha} \cdot C$. Hence, $\alpha = 1$, as if $\alpha > 1$, letting $\beta \to 0$ implies there doesn't exist a bound for $F(\beta f, p, \alpha)$, and if $\alpha < 1$ then letting $\beta \to 0$ implies $C = 0$ (clearly not the case).

Hence, we can now let $\alpha = 1$, and we can do a similar rescaling argument to show that $p = 2$. Let $g(x, y) = f(\beta x, \beta y)$. Hence, we get the following:

$$\begin{aligned}||g||_{L^p(\mathbb{R}^2)} &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} (g(x,y))^p \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} (f(\beta x,\beta y))^p \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}}.\end{aligned}$$

Letting $u = \beta x$ and $v = \beta y$,

$$\begin{aligned}&= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\beta^2}\cdot (f(u,v))^p \,\mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{p}} \\ &= \frac{1}{\beta^{\frac{2}{p}}}||f||_{L^p}.\end{aligned}$$

Similarly,

$$\begin{aligned}||\nabla g|| &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sqrt{(\partial_x g(x,y))^2 + (\partial_y g(x,y))^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sqrt{(\partial_x f(\beta x,\beta y))^2 + (\partial_y f(\beta x,\beta y))^2}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Letting $u = \beta x$ and $v = \beta y$,

$$\begin{aligned}&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\beta^2}\cdot\sqrt{(\beta\partial_u f(u,v))^2 + (\beta\partial_v f(u,v))^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\beta}\cdot\sqrt{(\partial_u f(u,v))^2 + (\partial_v f(u,v))^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \frac{1}{\beta}\cdot||\nabla f||_{L^1}.\end{aligned}$$

Therefore,

$$\frac{||g||_{L^p}}{||\nabla g||_{L^1}} = \beta^{1-\frac{2}{p}}\frac{||f||_{L^p}}{||\nabla f||_{L^1}} \leq \beta^{1-\frac{2}{p}}\cdot C.$$

By a similar argument for α, this directly implies p can only be 2. Hence, if there exists a C such that $F(f, p, \alpha) \leq C$, then $p = 2$ and $\alpha = 1$.

This actually answers **Question 23**; we can only possibly find an upperbound for $||f||_{L^p}$ if $p = 2$. Now the question is: given $p = 2$, does there exist such an upper bound?

This thought process lends itself to a few new questions:

Question 33. *Can this argument be used to start to generalize this problem to functions with more variables? Is it always the case that there is only one such scaling invariant L^p space (i.e., will there always only exist one such p and α)? What if we were instead given $||\nabla f||_{L^q}$?*

Furthermore, the rescaling argument allows us to reframe our problem.

Problem 34

Let $f(x, y) \in C^1$ be a compactly supported function, such that $|f(x, y)| \leq 1$ for all $(x, y) \in \mathbb{R}^2$ and $f(x, y) = 0$ for $(x, y) \notin [-1, 1] \times [-1, 1]$. Given $||\nabla f||_{L^1} \leq 1$, find an upper bound for $||f||_{L^2}$.

This is equivalent to our previous problem, as previously we assumed f was compactly supported and bounded. Therefore, we can rescale f to have upper bound 1, and to be 0 outside of a given rectangle (in this case, $[-1,1]\times[-1,1]$).

My goals for this upcoming week:

1. Go through the argument we have started to go through, with these additional constrictions on f. We still have the issue of $\int_{U_\epsilon} |f|$ for ϵ relatively small, but I hope that these additional constrictions can illuminate a way to approach this issue.
2. I also would like to start a summary of the work done so far in it's own small self-contained document.

Have a good week!

July 24

Today I started working on making a summary of the project work done so far, but I did want to note something. The function created this last Thursday as an example of U_λ being sharp, can equivalently be written as

$$f = \frac{1}{\max\{|x|,|y|\}}.$$

2.4 July 30-August 05

July 30

Hi Larry! We made a lot of good progress this week; in fact, I think we might be done with this problem with up to some small nuanced parts of the problem (i.e., notation and summation indexing). For today's purposes, I will put a summary of what we talked about during this meeting.

First, we started discussing the rescaling arguments for functions of more variables. Let $f:\mathbb{R}^n\to\mathbb{R}$, and let $\beta>0$. Assume that there exists a fixed $C>0$, p, and α such that for all $f\in C^1$ with f compactly supported such that $\frac{||f||_p}{||\nabla f||_1^\alpha} < C$.

To see that $\alpha = 1$, consider the function $g(\vec{x}) = \beta f(\vec{x})$. Then,

$$\frac{||g||_p}{||\nabla g||_1^\alpha} = \frac{||\beta f||_p}{||\nabla \beta f||_1^\alpha} = \frac{\beta}{\beta^\alpha}\cdot\frac{||f||_p}{||\nabla f||_1^\alpha} \le \beta^{1-\alpha}\cdot C.$$

Just as we showed in last week's argument, this implies that $\alpha = 1$.

Now we will show that $p = \frac{n}{n-1}$ (note that this agrees with the case where $n=2$ where $p=2$). Consider the function $g(\vec{x}) = f(\beta\vec{x})$. Then,

$$\begin{aligned}||g||_p &= \left(\int_{\mathbb{R}^n} |g(\vec{x})|^p \,\mathrm{d}\vec{x}\right)^{\frac{1}{p}}\\ &= \left(\int_{\mathbb{R}^n} |f(\beta\vec{x})|^p \,\mathrm{d}\vec{x}\right)^{\frac{1}{p}}.\end{aligned}$$

Let $u_i = \beta x_i$, such that $\mathrm{d}\vec{u} = \beta^n \,\mathrm{d}\vec{x}$. We will use this substitution in the next part of the problem too. Hence,

$$
\begin{aligned}
&= \left(\int_{\mathbb{R}^n} \frac{1}{\beta^n} \cdot |f(\vec{u})|^p \,\mathrm{d}\vec{u}\right)^{\frac{1}{p}} \\
&= \beta^{-\frac{n}{p}} \cdot ||f||_p.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
||\nabla g||_1 &= \int_{\mathbb{R}^n} \sqrt{\sum_{i=1}^n \partial_{x_i}^2 (g(\vec{x}))} \,\mathrm{d}\vec{x} \\
&= \int_{\mathbb{R}^n} \sqrt{\sum_{i=1}^n \partial_{x_i}^2 (f(\beta\vec{x}))} \,\mathrm{d}\vec{x} \\
&= \frac{1}{\beta^n} \cdot \int_{\mathbb{R}^n} \sqrt{\beta^2 \cdot \sum_{i=1}^n \partial_{u_i}^2 (f(\vec{u}))} \,\mathrm{d}\vec{u} \\
&= \beta^{1-n} \cdot ||\nabla f||_1.
\end{aligned}
$$

Therefore,

$$
\frac{||g||_p}{||\nabla g||_1} = \beta^{n-\frac{n}{p}-1} \frac{||f||}{||\nabla f||} \leq \beta^{n-\frac{n}{p}-1} \cdot C.
$$

Hence,

$$
n - \frac{n}{p} - 1 = 0 \implies p = \frac{n}{n-1}.
$$

Therefore, *if* there exists a $C > 0$, such that for all compactly supported and continuous $f : \mathbb{R}^n \to \mathbb{R}$, $\frac{||f||_p}{||\nabla f||^\alpha} < C$, then $p = \frac{n}{n-1}$ and $\alpha = 1$.

The numbers here feel oddly related to the Loomis-Whitney inequality. It might be interesting to looking into this inequality next if we are indeed done with this problem.

Next, I discussed how I had tried to redefine V_λ such that the splitting of the integral over $\mathbb{R}^2$ led to a better/stronger inequality than the one we already have. However, no matter how I defined V_λ, I reached the conclusion that

$$
\left(\int_{\mathbb{R}} \int_{\mathbb{R}} |f(\vec{x})|^2 \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} \leq \left(\sum_{k\in\mathbb{N}} 1\right)^{\frac{1}{2}}.
$$

Hence, Yuqiu suggested that we can't just play around with notation/sets to get an inequality that we want. Rather, we should look back on our problem in totality and try and find places where we lose sharpness and work on those.

Specifically, we started to reconsider the following (which we had done earlier):

$$
\begin{aligned}
1 &\geq \int_{\mathbb{R}^2} |\partial_x f(\vec{x})| \,\mathrm{d}\vec{x} \\
&\geq \int_{\mathbb{R}} \int_{x_1}^{x_2} |\partial_x f(\vec{x})| \,\mathrm{d}\vec{x} \\
&= \int_{\mathbb{R}} |f(x_1, y) - f(x_2, y)| \,\mathrm{d}y.
\end{aligned}
$$

This last line uses the FTC. From here, we would go into splitting up $\mathbb{R}$ into $\pi_1(V_{2^k})$. However, as Yuqiu pointed out, we throw out a lot of $\mathbb{R}$ when we go from that to $[x_1, x_2]$. So, instead Yuqiu suggested *starting* by splitting $\mathbb{R}^2$ into V_{2^k}, and then applying the FTC, as follows.

Firstly, instead of assuming that $||\nabla f|| \leq 1$, for now we will just keep it as $||\nabla f||$. Then, we have

$$||\nabla f||_1 \geq \int_{\mathbb{R}^2} |\partial_x f(\vec{x})\, \mathrm{d}\vec{x}.$$

Choose N sufficiently large such that $2^{-N} \leq \sup_{\vec{x}\in\mathbb{R}^2} f(\vec{x}) \leq 1$. Then,

$$\begin{aligned}
&\geq \sum_{k=0}^{N} \int_{V_{2^{-k}}} |\partial_x f(x,y)|\, \mathrm{d}x\, \mathrm{d}y \\
&= \sum_{k=0}^{N} \int_{\pi_1(V_{2^{-k}})} \int_{\{\vec{x}|\vec{x}\in V_{2^{-k}}\}} |\partial_x f(x,y)|\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}$$

Choose $x_{1,k}$ and $x_{2,k}$ such that $f(x_{1,k}, y) = 2^{-k-1}$ and $f(x_{2,k}, y) = 2^{-k}$. We know that we can choose such values as $2^{-N} \leq \sup f(\vec{x})$. There is one such caveat with this that I will address later in this summary. Anyways, then

$$\begin{aligned}
&\geq \sum_{k=0}^{N} \int_{\pi_1(V_{2^{-k}})} \int_{x_{1,k}}^{x_{2,k}} |\partial_x f(x,y)|\, \mathrm{d}x\, \mathrm{d}y \\
&= \sum_{k=0}^{N} \int_{\pi_1(V_{2^{-k}})} |f(x_{2,k}, y) - f(x_{1,k}, y)|\, \mathrm{d}y \\
&= \sum_{k=0}^{N} \int_{\pi_1(V_{2^{-k}})} |2^{-k} - 2^{-k-1}|\, \mathrm{d}y \\
&= \sum_{k=0}^{N} |2^{-k} - 2^{-k-1}| \cdot m(\pi_1(V_{2^{-k}})) \\
&= \sum_{k=0}^{N} 2^{-k} \cdot \left|1 - \frac{1}{2}\right| \cdot m(\pi_1(V_{2^{-k}}))
\end{aligned}$$

Therefore, we have the two inequalities

$$||\nabla f||_1 \geq \frac{1}{2} \sum_{k=0}^{N} 2^{-k} \cdot m(\pi_1(V_{2^{-k}})).$$

$$||\nabla f||_1 \geq \frac{1}{2} \sum_{k=0}^{N} 2^{-k} \cdot m(\pi_2(V_{2^{-k}})).$$

Therefore,

$$||\nabla f||_1^2 \geq \frac{1}{4} \sum_{k=0}^{N} 2^{-2k} \cdot m(\pi_1(V_{2^{-k}})) \cdot m(\pi_2(V_{2^{-k}})) \geq \frac{1}{4} \sum_{k=0}^{N} 2^{-2k} \cdot area(V_{2^{-k}})$$

using equation (2.4).

Hence,

$$||\nabla f||_1 \geq \frac{1}{2} \left(\sum_{k=0}^{N} 2^{-2k} \cdot area(V_{2^{-k}}) \right)^{\frac{1}{2}}.$$

We can similarly estimate $||f||_2$:

$$\begin{aligned}||f||_2 &= \left(\int_{\mathbb{R}^2} |f(\vec{x})|^2 \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} \\ &= \left(\int_{-1}^{1}\int_{-1}^{1} |f(\vec{x})|^2 \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} \\ &= \left(\sum_{k\in\mathbb{N}} \int_{V_{2^{-k}}} |f(\vec{x})|^2 \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k\in\mathbb{N}} 2^{-2k}\cdot area(V_{2^{-k}})\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\frac{||f||_2}{||\nabla f||_1} \leq \frac{\left(\sum_{k\in\mathbb{N}} 2^{-2k}\cdot area(V_{2^{-k}})\right)^{\frac{1}{2}}}{\frac{1}{2}\left(\sum_{k=0}^{N} 2^{-2k}\cdot area(V_{2^{-k}})\right)^{\frac{1}{2}}} \leq 2\cdot\left(\frac{\sum_{k=0}^{N} 2^{-2k}\cdot area(V_{2^{-k}})}{\sum_{k=0}^{N} 2^{-2k}\cdot area(V_{2^{-k}})}\right)^{\frac{1}{2}} = 2. \tag{2.6}$$

From here, I am mostly concerned with indexing of these sums, but this feels fairly close. Below are my goals for this week:

1. ~~Consider the rescaling argument if given $||\nabla f||_q$ instead of $||\nabla f||_1$.~~
2. ~~Go back through this argument, and double check every line for errors in indexing and making the argument more refined.~~
3. Yuqiu also pointed one more thing out: $x_{1,k}$ and $x_{2,k}$ must be connected by a line segment. Yuqiu says that this should be the case, so my goal is to prove this small portion of the proof.
4. ~~I want to add these new topics to the project's official notes (which will be sent in today's email)~~.

There are quite a few questions and places for this project to go from here. I think the Loomis-Whitney inequality may be neat to explore, but for now will work on the items above. This is very exciting!

August 4

Okay! Mostly done with notes but there is One Main Question left which Yuqiu suggests should have an answer to it. I will phrase the question.

Right now we have that

$$\begin{aligned}||\nabla f||_1 &\geq \sum_{k=N}^{\infty} \int_{V_{2^{-k}}} |\partial_x f(\vec{x})| \,\mathrm{d}\vec{x} \\ &= \sum_{k=N}^{\infty} \int_{\pi_1(V_{2^{-k}}} \int_{\pi_2^{-1}(V_{2^{-k}}} |\partial_x f(x,y)| \,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

and then we choose $x_{1,k}$ and $x_{2,k}$ such that etc etc. However, the question is: does there exist an $x_{1,k}$ and $x_{2,k}$ such that $[x_{1,k}, x_{2,k}] \subset \pi_2^{-1}(V_{2^{-k}})$ with the properties that $f(x_{1,k}, y) = 2^{-k-1}$ and $f(x_{2,k}, y) = 2^{-k}$? Or similarly, swap the values of $x_{1,k}$ and $x_{2,k}$. Question: Do these ys need to be the same? I do not believe so conceptually, but also the $\mathrm{d}y$ in the integral makes me suspicious of this. Based on our choice of N, we certainly know that there exists $\vec{x}_1$ and $\vec{x}_2$ such that $f(\vec{x}_1) = 2^{-k-1}$ and $f(\vec{x}_2) = 2^{-k}$, but how do we know that there is a specific interval where they are connected in $\pi_2(V_{2^{-k}})$? Does this follow from the continuity of $\pi_1 f$ and $\pi_2 f$?

Chapter 3

The Project In All of It's Glory

3.1 The Cauchy-Schwarz Inequality

This document was made to summarize all the key parts of this UROP project. This includes theorems and exercises from the book *A View from the Top* by Alex Iosevich, and general notes and pending questions from the UROP. For the first five weeks of this UROP, from June 03-July 07, Yuqiu and I read through Chapters 1-8 of Iosevich's text, and from then on we started considering questions brought up Section 2. The first place we can and will start off in these notes however is with the CS inequality (as the title of this Section implies).

3.1.1 The Inequality Itself

We will prove this inequality in a few ways, both of which are outlined in Iosevich's text, either directly or in an exercise.

Let $a, b \in \mathbb{R}$. Then,

$$\begin{aligned}(a-b)^2 &\geq 0\\ a^2 - 2ab + b^2 &\geq 0\\ \implies ab &\leq \frac{a^2+b^2}{2}.\end{aligned}$$

Hence, consider the finite sums

$$A_N = \sum_{k=1}^{N} a_k \quad \text{and} \quad B_N = \sum_{k=1}^{N} b_k$$

where $a_i, b_i \in \mathbb{R}$ for all $1 \leq i \leq N$. To simplify terms long term, let

$$X_N = \left(\sum_{k=1}^{N} a_k^2\right)^{\frac{1}{2}} \quad \text{and} \quad Y_N = \left(\sum_{k=1}^{N} b_k^2\right)^{\frac{1}{2}}.$$

Note that X_N and Y_N are just constants! Then, we get the following:

$$\sum_{k=1}^{N} a_k b_k = X_N Y_N \sum_{k=1}^{N} \frac{a_n}{X_N} \cdot \frac{b_k}{Y_N}.$$

Using the fact that $ab \leq \frac{a^2+b^2}{2}$ for all $a, b \in \mathbb{R}$,

$$
\begin{aligned}
\#S_N = \sum_{k=1}^{N} a_k b_k &\leq X_N Y_N \cdot \left(\sum_{k=1}^{N} \frac{1}{2} \cdot \left(\frac{a_k}{X_N}\right)^2 + \frac{1}{2} \cdot \left(\frac{b_k}{Y_N}\right)^2 \right) \\
&= \frac{X_N Y_N}{2X_N^2} \left(\sum_{k=1}^{N} a_k^2 \right) + \frac{X_N Y_N}{2Y_N^2} \left(\sum_{k=1}^{N} b_k^2 \right) \\
&= \frac{X_N Y_N}{2X_N^2} \left(X_N^2 \right) + \frac{X_N Y_N}{2Y_N^2} \left(Y_N^2 \right) \\
&= X_N Y_N \\
&= \left(\sum_{k=1}^{N} a_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} b_k^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

Theorem 35 (The Cauchy-Schwarz Inequality)
Therefore, we have

$$
\sum_{k=1}^{N} a_k b_k \leq \left(\sum_{k=1}^{N} a_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} b_k^2 \right)^{\frac{1}{2}}. \tag{3.1}
$$

For another way to prove this inequality, consider the standard $\mathbb{R}^n$ Hermitian inner product $\langle a - tb, a - tb \rangle$ for $t \in [0, 1]$. We will minimize this inner product using calculus:

$$
\begin{aligned}
\langle a - tb, a - tb \rangle &= (a_1 - tb_1)^2 + (a_2 - tb_2)^2 + \cdots + (a_N - tb_N)^2 \\
&= \sum_{k=1}^{N} a_k^2 - 2t \sum_{k=1}^{N} a_k b_k + t^2 \sum_{k=1}^{M} b_k^2 \\
&= \|a\|^2 + t^2 \|b\|^2 - 2t \langle a, b \rangle.
\end{aligned}
$$

It is evident that the critical point of this equation is located at $t = \frac{\langle a, b \rangle}{||b||^2}$. It is furthermore clear that this is where the minimum of the equation is. Therefore, we get the following minimum value:

$$
\left\langle a - \frac{\langle a, b \rangle}{||b||^2} b, a - \frac{\langle a, b \rangle}{||b||^2} b \right\rangle = \|a\|^2 - \frac{\langle a, b \rangle^2}{\|b\|^2}.
$$

Multiplying by $\|b\|^2$ on both sides, we can conclude

$$
\begin{aligned}
0 &\leq \left\| \|b\|^2 a - \langle a, b \rangle b \right\|^2 \\
&= \left\langle \|b\|^2 a - \langle a, b \rangle b, \|b\|^2 a - \langle a, b \rangle b \right\rangle \\
&= \|a\|^2 \|b\|^2 - \langle a, b \rangle^2 \\
\implies &\langle a, b \rangle \leq \|a\| \cdot \|b\|.
\end{aligned}
$$

□

In Chapter 2, Iosevich begins to outline more of what we will be utilizing for this project: projections in $\mathbb{R}^2$ and $\mathbb{R}^3$ (which could extend over to $\mathbb{R}^d$). He does so to begin using the CS inequality in a cool way. For both $\mathbb{R}^2$ and $\mathbb{R}^3$, I will write out the proof that Iosevich uses for the discrete case, and then prove the same inequalities for the "continuous case".

3.1.2 2D Projections

Let S_N be a set of N points in $\mathbb{R}^2$. Furthermore, let $\pi_1(x_1, x_2) = x_2$ and $\pi_2(x_1, x_2) = x_1$ where $(x_1, x_2) \in \mathbb{R}^2$. Then,

$$\begin{aligned}\sum_{x_1,x_2} \chi_{S_N}(x_1, x_2) &\leq \sum_{x_1,x_2} \chi_{\pi_1(S_N)}(x_2) \cdot \chi_{\pi_2(S_N)}(x_1) \\ &= \sum_{x_1} \chi_{\pi_2(S_N)}(x_1) \cdot \sum_{x_2} \chi_{\pi_1(S_N)}(x_2) \\ &= \#\pi_1(S_N) \cdot \#\pi_2(S_N).\end{aligned}$$

From here you can state that $N^{\frac{1}{2}} \leq \max_{i=1,2} \#\pi_i(S_N)$, but this isn't as important for this project.

Now, instead of considering N points in $\mathbb{R}^2$, lets try to make this more general. Let $\Omega \subset \mathbb{R}^2$, can we say something similar for this case? Well, the answer ends up being yes, using similar logic:

$$\begin{aligned}\text{area}(\Omega) &= \int_{\mathbb{R}^2} \chi_\Omega(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\ &\leq \int_{\mathbb{R}^2} \chi_{\pi_1(\Omega)}(x_2) \cdot \chi_{\pi_2(\Omega)}(x_1)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\ &= \int_{\mathbb{R}} \chi_{\pi_2(\Omega)}(x_1)\, \mathrm{d}x_1 \cdot \int_{\mathbb{R}} \chi_{\pi_1(\Omega)}(x_2)\, \mathrm{d}x_2 \\ &= \mathrm{m}(\pi_1(\Omega)) \cdot \mathrm{m}(\pi_2(\Omega))\end{aligned}$$

where m is the measure (i.e. the length of the subset of $\mathbb{R}$). We get a very useful equation for this project: for all $\Omega \subset \mathbb{R}^2$,

$$\text{area}(\Omega) \leq \mathrm{m}(\pi_1(\Omega)) \cdot \mathrm{m}(\pi_2(\Omega)). \tag{3.2}$$

We can prove this one more way: geometrically. Let $X = \pi_2(\Omega)$ and $Y = \pi_1(\Omega)$ and consider

$$X \times Y := \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

Then, it is clear that $\Omega \subset X \times Y$, and then $\text{area}(\Omega) \leq \text{area}(X \times Y)$. Hence,

$$\text{area}(\Omega) \leq \text{area}(X \times Y) = \mathrm{m}(X) \cdot \mathrm{m}(Y) = \mathrm{m}(\pi_1(\Omega)) \cdot \mathrm{m}(\pi_2(\Omega)).$$

3.1.3 3D Projections

The 3D projections case is relatively the same, except this time it actually utilizes the CS inequality (and thus, why it is in this section). Note that this part of the notes is not pivotal to the project itself in Section 2, and can be skipped. This time, let $\pi_1(x_1, x_2, x_3) = (x_2, x_3)$ and so on and so forth for π_2 and π_3. Hence, for the discrete case, we get

$$\begin{aligned}\#S_N = \sum_x \chi_{S_N}(x) &\leq \sum_{x_1,x_2,x_3} \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3)\chi_{\pi_3(S_N)}(x_1, x_2) \\ &= \sum_{x_1,x_2} \chi_{\pi_3(S_N)}(x_1, x_2) \left(\sum_{x_3} \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3) \right).\end{aligned}$$

Applying the CS inequality,

$$\leq \left(\sum_{x_1,x_2} \chi^2_{\pi_3(S_N)}(x_1,x_2)\right)^{\frac{1}{2}} \cdot \left(\sum_{x_1,x_2}\left(\sum_{x_3} \chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3)\right)^2\right)^{\frac{1}{2}}.$$

Between this line and the next, we use the fact that $\chi^2(\vec{x}) = \chi(\vec{x})$ as $\chi(\vec{x})$ either equals 0 or 1 and $0^2 = 0$ and $1^2 = 1$.

$$\begin{aligned}
&= \left(\sum_{x_1,x_2} \chi_{\pi_3(S_N)}(x_1,x_2)\right)^{\frac{1}{2}} \cdot \left(\sum_{x_1,x_2}\left(\sum_{x_3} \chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3)\right)^2\right)^{\frac{1}{2}} \\
&= \#\pi_3(S_N) \cdot \left(\sum_{x_1,x_2}\left(\sum_{x_3} \chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3)\right)^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Now don't be intimidated by the large amount of variables and letters in this next line. All we are doing is adding in a new variable when we square the inside sum. Then, the goal from here is to split up the sum by *separating* these many variables.

$$\begin{aligned}
&= \#\pi_3(S_N) \cdot \left(\sum_{x_1,x_2}\sum_{x_3,x_3'} \chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3)\chi_{\pi_1(S_N)}(x_2,x_3')\chi_{\pi_2(S_N)}(x_1,x_3')\right)^{\frac{1}{2}} \\
&\leq \#\pi_3(S_N) \cdot \left(\sum_{x_1,x_2}\sum_{x_3,x_3'} \chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3')\right)^{\frac{1}{2}} \\
&= \#\pi_3(S_N) \cdot \left(\sum_{x_2,x_3} \chi_{\pi_1(S_N)}(x_2,x_3) \cdot \sum_{x_1,x_3'} \chi_{\pi_2(S_N)}(x_1,x_3')\right)^{\frac{1}{2}} \\
&= \sqrt{\#\pi_1(S_N)} \cdot \sqrt{\#\pi_2(S_N)} \cdot \sqrt{\#\pi_3(S_N)}.
\end{aligned}$$

Again, from here, we could get that $N^{\frac{2}{3}} \leq \max_{i=1,2,3} \#\pi_i(S_N)$. Now we will show the continuous case, which works out exactly the same but with integrals!

Let $\Omega \subset \mathbb{R}^3$, and let $A_i = \sqrt{\mathrm{area}(\pi_i(\Omega))}$. Then,

$$\begin{aligned}
\mathrm{vol}(\Omega) &= \int_{\mathbb{R}^3} \chi_\Omega(x)\,\mathrm{d}x \\
&\leq \iiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\chi_{\pi_3(\Omega)}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&= \iint \chi_{\pi_3(\Omega)}(x_1,x_2)\left(\int \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\,\mathrm{d}x_3\right)\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}$$

Note that we can swap these integrals like this by Fubini's Theorem! By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\leq \left(\iint \chi^2_{\pi_3(\Omega)}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}\left(\iint\left(\int \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\,\mathrm{d}x_3\right)^2\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}\\
&\leq A_3^{1/2}\left(\iiiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\chi_{\pi_1(\Omega)}(x_2,x_3')\chi_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_3\,\mathrm{d}x_3'\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}\\
&\leq A_3^{1/2}\left(\iiiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_3\,\mathrm{d}x_3'\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}\\
&= A_3^{1/2}\left(\iint \chi^2_{\pi_1(\Omega)}(x_2,x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3\right)^{1/2}\left(\iint \chi^2_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_1\,\mathrm{d}x_3'\right)^{1/2} = A_1^{1/2}A_2^{1/2}A_3^{1/2}.
\end{aligned}
$$

Hence, we achieve the nice inequality:

$$
\mathrm{vol}(\Omega) \leq \sqrt{\mathrm{area}(\pi_1(\Omega))}\cdot\sqrt{\mathrm{area}(\pi_2(\Omega))}\cdot\sqrt{\mathrm{area}(\pi_3(\Omega))}.
$$

$\square$

Question 36. *Is there a nice geometric argument similar to the 2D case that we can use to prove this inequality?*

3.2 Sobolev's Inequality

We start with a basic concept: if the derivative of a function is 0, then that function is a constant. But what does it mean if the derivative of a function is approximately 0? Is that function approximately a constant? And how would we describe the derivative of a function being *nearly* 0?

Well, consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in C_0^1(\mathbb{R})$, and $\int_{-\infty}^{\infty} |f'(x)| \,\mathrm{d}x \leq 1$. What can we say about f? Well, using the Fundamental Theorem of Calculus, given $-\infty < a, b < \infty$,

$$\begin{aligned} |f(b) - f(a)| &= \left| \int_a^b f'(x) \,\mathrm{d}x \right| \\ &\leq \int_a^b |f'(x)| \,\mathrm{d}x \\ &\leq \int_{-\infty}^{\infty} |f'(x)| \,\mathrm{d}x \leq 1. \end{aligned}$$

Hence, for all $-\infty < a, b < \infty$, $|f(b) - f(a)| \leq 1$.

How does this concept transfer over to higher dimensions? Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \in C_0^1(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} |\nabla f(\vec{x})| \,\mathrm{d}\vec{x} \leq 1$. Is it true that $|f(\vec{b}) - f(\vec{a})| \leq 1$ for all $\vec{a}, \vec{b} \in \mathbb{R}^2$? No. Here is a counter example: consider the function

$$f_\epsilon(\vec{x}) = \begin{cases} 0 & |\vec{x}| > 2\epsilon \\ 1 & \vec{x} = 0 \end{cases}$$

and let f_ϵ smoothly interpolate between 0 and 1 for all values of $\vec{x}$ such that $0 < |\vec{x}| < 2\epsilon$. Then, ∇f is supported in the region where $0 < |\vec{x}| < 2\epsilon$, and $|\nabla f| \leq \frac{2}{\epsilon}$. Hence,

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla f(\vec{x})| \,\mathrm{d}\vec{x} &= \int_{0 \leq |x| \leq 2\epsilon} |\nabla f(\vec{x})| \,\mathrm{d}\vec{x} + \int_{|x| > 2\epsilon} |\nabla f(\vec{x})| \,\mathrm{d}\vec{x} \\ &= \int_{0 \leq |x| \leq 2\epsilon} |\nabla f(\vec{x})| \,\mathrm{d}\vec{x} \\ &= \text{area}(Circle) \cdot \frac{2}{\epsilon} \\ &\leq C\epsilon \xrightarrow{\epsilon \to 0} 0 \end{aligned}$$

and where $C > 0$. However, it is not necessarily the case that $|f(\vec{b}) - f(\vec{a})| \leq 1$ for all $\vec{a}, \vec{b} \in \mathbb{R}^2$. In other words, though the "derivative" tends to 0, the difference $|f(\vec{b}) - f(\vec{a})|$ does not.

Hence, from here we want to try and prove something weaker. Suppose that $f \in C_0^1(\mathbb{R}^2)$.

Remark 37. *Recall that a function is compactly supported if there exists some finitely sized rectangle such that for every (x, y) outside of this rectangle in $\mathbb{R}^2$, $f(x, y) = 0$. For our purposes, I will be picturing a square centered at the origin instead of just "some rectangle", though this is logically equivalent.*

Note 38

For now, I will denote $L^p(\mathbb{R}^2)$ as simply p, i.e.,

$$\|f\|_{L^p(\mathbb{R}^2)} := \|f\|_p = \left(\int_{\mathbb{R}^2} |f(\vec{x})|^p \,\mathrm{d}\vec{x} \right)^{\frac{1}{p}}.$$

I will use the same notation when dealing with $L^p(\mathbb{R}^n)$, though this should be clear from the context.

Now the question is: can we find an upper bound for $\|f\|_p$ with respect to $\|\nabla f\|_1$?

3.2.1 Rescaling Arguments

We can instead ask a broader question than this which will prove helpful: can we find an upperbound for $\|f\|_p$ with respect to $\|\nabla f\|_1$? To start answering this question, we can ask: for which p and α does there exist a $C > 0$ such that

$$F(f, p, \alpha) := \frac{\|f\|_p}{\|\nabla f\|_1^\alpha} \leq C$$

for all $f \in C_0^1(\mathbb{R}^2)$?

For this, we can use *rescaling arguments* to prove that $p = 2$ and $\alpha = 1$. Assume, for the sake of contradiction that there exists such a $C > 0$ for all f, α, and p (with $f \in C_0^1(\mathbb{R}^2)$). Then, let $0 < \beta \in \mathbb{R}$ and consider the following:

It should be the case that

$$F(\beta f, p, \alpha) = \frac{\|\beta f\|_p}{\|\nabla \beta f\|_1^\alpha} = \frac{\beta}{\beta^\alpha} \frac{\|f\|_p}{\|f\|_1^\alpha} \leq \frac{\beta}{\beta^\alpha} \cdot C.$$

Hence, $\alpha = 1$, as if $\alpha > 1$, letting $\beta \to 0$ implies there doesn't exist a finite bound for $F(\beta f, p, \alpha)$, and if $\alpha < 1$ then letting $\beta \to 0$ implies $C = 0$ (clearly not the case by considering any nontrivial example).

Hence, we can now let $\alpha = 1$, and we can do a similar rescaling argument to show that $p = 2$. Let $g(x, y) = f(\beta x, \beta y)$. Then,

$$\begin{aligned}
\|g\|_p &= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} (g(x, y))^p \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} (f(\beta x, \beta y))^p \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}}.
\end{aligned}$$

Letting $u = \beta x$ and $v = \beta y$,

$$\begin{aligned}
&= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\beta^2} \cdot (f(u, v))^p \, \mathrm{d}u \, \mathrm{d}v \right)^{\frac{1}{p}} \\
&= \frac{1}{\beta^{\frac{2}{p}}} \|f\|_p.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\nabla g\|_1 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \sqrt{(\partial_x g(x, y))^2 + (\partial_y g(x, y))^2} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \sqrt{(\partial_x f(\beta x, \beta y))^2 + (\partial_y f(\beta x, \beta y))^2} \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

Letting $u = \beta x$ and $v = \beta y$,

$$\begin{aligned}
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\beta^2} \cdot \sqrt{(\beta \partial_u f(u, v))^2 + (\beta \partial_v f(u, v))^2} \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\beta} \cdot \sqrt{(\partial_u f(u, v))^2 + (\partial_v f(u, v))^2} \, \mathrm{d}u \, \mathrm{d}v \\
&= \frac{1}{\beta} \cdot \|\nabla f\|_1.
\end{aligned}$$

Therefore,

$$\frac{\|g\|_p}{\|\nabla g\|_1} = \beta^{1 - \frac{2}{p}} \frac{\|f\|_p}{\|\nabla f\|_1} \leq \beta^{1 - \frac{2}{p}} \cdot C.$$

By a similar argument for α, this directly implies p can only be 2. Hence, if there exists a C such that $F(f, p, \alpha) \leq C$, then $p = 2$ and $\alpha = 1$. Now the question is: given $p = 2$ and $\alpha = 1$, does there exist such an upper bound?

This rescaling argument allows us to reframe our main goal of this project.

Remark 39. *Note that as we assume $f \in C_0^1(\mathbb{R}^2)$, f must be bounded.*

Problem 40

Let $f(x, y) \in C_0^1(\mathbb{R}^2)$ such that $|f(x, y)| \leq 1$ for all $(x, y) \in \mathbb{R}^2$ and $f(x, y) = 0$ for $(x, y) \notin [-1, 1] \times [-1, 1]$. Let there exist an $\vec{x}$ such that $|f(\vec{x})| = 1$. Show there exists an upper bound (C) for $\frac{\|f\|_2}{\|\nabla f\|_1}$.

This is equivalent to our previous problem, as previously we assumed $f \in C_0^1(\mathbb{R}^2)$ was compactly supported, and thus bounded. Therefore, we can rescale f to have upper bound 1, and to be 0 outside of a given square centered at the origin (in this case, $[-1, 1] \times [-1, 1]$). However, this part is just semantics and will simply be a corollary when we solve Problem 6, so we will revisit in the Concluding Remarks.

Question 41. *Can a rescaling argument be used to start to generalize this problem to functions with more variables? Is it always the case that there is only one such scaling invariant L^p space (i.e., will there always only exist one such p and α)? What if we were instead given $\|\nabla f\|_q$ instead of $\|\nabla f\|_1$?*

Let's answer these questions! We will answer the most general version of this question.

Higher Dimensions, with $\|\nabla \mathbf{f}\|_\mathbf{q}$: Let $f : \mathbb{R}^n \to \mathbb{R}$, and let $\beta > 0$. Assume that there exists a fixed $C > 0$, p, and α such that for all $f \in C_0^1(\mathbb{R}^2)$ such that $\frac{\|f\|_p}{\|\nabla f\|_q^\alpha} < C$. We will find values for both p and α.

Consider $g(\vec{x}) = \beta f(\vec{x})$. Then,

$$\frac{\|g\|_p}{\|\nabla g\|_q^\alpha} = \frac{\|\beta f\|_p}{\|\nabla \beta f\|_q^\alpha} = \beta^{1-\alpha} \cdot \frac{\|f\|_p}{\|\nabla f\|_q^\alpha} \leq \beta^{1-\alpha} \cdot C \implies \alpha = 1.$$

Of course, the more interesting part of this problem will be what p is. Let $g(\vec{x}) = f(\beta \vec{x})$. From the last part, we have that $\|g\|_p = \beta^{-\frac{n}{p}} \cdot \|f\|_p$. Now we consider $\|\nabla g\|_q$.

$$\begin{aligned}
\|\nabla g\|_q &= \left(\int_{\mathbb{R}^n} \left(\sqrt{\sum_{i=1}^n \partial_{x_i}^2 g(\vec{x})} \right)^q \mathrm{d}\vec{x} \right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}^n} \left(\sqrt{\sum_{i=1}^n \partial_{x_i}^2 f(\beta \vec{x})} \right)^q \mathrm{d}\vec{x} \right)^{\frac{1}{q}}.
\end{aligned}$$

Let $u_i = \beta x_i$. Hence, $d\vec{u} = \beta^n \,\mathrm{d}\vec{x}$, and $\partial_{x_i}^2 f(\beta \vec{x}) = \beta^2 \partial_{u_i}^2 f(\vec{u})$. Therefore,

$$\begin{aligned}
&= \left(\frac{1}{\beta^n} \cdot \int_{\mathbb{R}^n} \left(\sqrt{\sum_{i=1}^n \beta^2 \partial_{u_i}^2 f(\vec{u})} \right)^q \mathrm{d}\vec{u} \right)^{\frac{1}{q}} \\
&= \beta^{1-\frac{n}{q}} \cdot \left(\int_{\mathbb{R}^n} \left(\sqrt{\sum_{i=1}^n \partial_{u_i}^2 f(\vec{u})} \right)^q \mathrm{d}\vec{u} \right)^{\frac{1}{q}} \\
&= \beta^{1-\frac{n}{q}} \cdot \|\nabla f\|_q.
\end{aligned}$$

Hence,

$$\frac{\|g\|_p}{\|\nabla g\|_q} = \beta^{\frac{n}{q} - \frac{n}{p} - 1} \cdot \frac{\|f\|_1}{\|\nabla f\|_q} \leq \beta^{\frac{n}{q} - \frac{n}{p} - 1} \cdot C.$$

Therefore, $p = \frac{nq}{n-q}$. This agrees with what we have shown previously with $q = 1$ and $n = 2$.

Question 42. *What can we do if given both $\|\nabla f\|_{q_1}$ and $\|\nabla f\|_{q_2}$ for $q_1 \neq q_2$? What happens if $q = n$?*

3.2.2 Bounding $\|f\|_2$

Now we actually want to try and bound the ratio

$$\frac{\|f\|_2}{\|\nabla f\|_1} = \frac{\left(\int_{\mathbb{R}^2} |f(\vec{x})|^2 \, \mathrm{d}\vec{x}\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^2} \sqrt{\partial_x^2(f(x,y)) + \partial_y^2(f(x,y))} \, \mathrm{d}x \, \mathrm{d}y}.$$

To do so, we can try to find an upper bound for $\|f\|_2$ and find a lower bound to $\|\nabla f\|_1$. In this subsection, we will try to find a bound for $\|f\|_2$.

First, we can split $\mathbb{R}^2$ into disjoint subsets in which we know the values of f. Consider the sets

$$V_{2^{-k}} := \{\vec{x} \mid 2^{-k} \geq |f(\vec{x})| > 2^{-k-1}\}.$$

Since $|f(\vec{x})| \leq 1$, we have the following:

$$\begin{aligned}
\|f\|_2 &= \left(\int_{\mathbb{R}^2} |f(\vec{x})|^2 \, \mathrm{d}\vec{x}\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=0}^{\infty} \int_{V_{2^{-k}}} |f(\vec{x})|^2 \, \mathrm{d}\vec{x}\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence, taking the maximum over each set, we get

$$\leq \left(\sum_{k=0}^{\infty} 2^{-2k} \cdot \mathrm{area}(V_{2^{-k}})\right)^{\frac{1}{2}}.$$

Can we find an upper bound to the area part of this inequality? Would that result in some nice inequality? Note that in the end finding an upper bound to the area won't actually pan out. If the reader prefers, they can skip ahead to the next subsection.

We can consider similarly defined sets such that V_λ is a subset of them, and thus has less area than them. Consider the sets

$$U_\lambda := \begin{cases} \{\vec{x} \mid f(\vec{x}) \geq \lambda\} & \lambda > 0 \\ \{\vec{x} \mid f(\vec{x}) \leq \lambda\} & \lambda < 0 \end{cases}.$$

Note the following:

$$\begin{aligned}
\|\nabla f\|_1 &= \int_{\mathbb{R}^2} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} \\
&= \int_{\mathbb{R}^2} \sqrt{(\partial_x f(\vec{x}))^2 + (\partial_y f(\vec{x}))^2} \, \mathrm{d}\vec{x} \\
&\geq \int_{\mathbb{R}^2} \sqrt{(\partial_x f(\vec{x}))^2} \, \mathrm{d}\vec{x} \\
\|\nabla f\|_1 &\geq \int_{\mathbb{R}^2} |\partial_x f(\vec{x})| \, \mathrm{d}\vec{x}.
\end{aligned}$$

Similarly,

$$\|\nabla f\|_1 \geq \int_{\mathbb{R}^2} |\partial_y f(\vec{x})| \, \mathrm{d}\vec{x}.$$

Let's consider the first of these two inequalities. Using the method we used for single variable functions (with the Fundamental Theorem of Calculus), we can obtain the following: For all $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned}|f(x_1,y)-f(x_2,y)| &= \left|\int_{x_1}^{x_2} \partial_x f(x,y)\,\mathrm{d}x\right| \\ &\leq \int_{x_1}^{x_2} |\partial_x f(x,y)|\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}} |\partial_x f(x,y)|\,\mathrm{d}x.\end{aligned}$$

Integrating both sides of this inequality, we get that for all $x_1, x_2 \in \mathbb{R}$ (which may depend on y)

$$\int_{\mathbb{R}} |f(x_1,y)-f(x_2,y)|\,\mathrm{d}y \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y \leq \|\nabla f\|_1. \tag{3.3}$$

Replacing x with y, we can similarly get that for all $y_1, y_2 \in \mathbb{R}$ (which may depend on x),

$$\int_{\mathbb{R}} |f(x,y_1)-f(x,y_2)|\,\mathrm{d}x \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_y f(x,y)|\,\mathrm{d}y\,\mathrm{d}x \leq \|\nabla f\|_1. \tag{3.4}$$

Furthermore, as $U_\lambda \subset \mathbb{R}^2$, we have that

$$\mathrm{area}(U_\lambda) \leq \mathrm{m}(\pi_1(U_\lambda)) \cdot \mathrm{m}(\pi_2(U_\lambda)). \tag{3.5}$$

Let $y \in \pi_1(U_\lambda)$. Now we are going to use equation (4). Since $y \in \pi_1(U_\lambda)$, there exists an $x_2 \in \mathbb{R}$ such that $x_2 = \sup\{x \in \pi_2(U_\lambda) \mid f(x,y) = \lambda\}$. Furthermore, since f is compactly supported, there exists an $x_1 < x_2 \in \mathbb{R}$ such that $f(x_1,y) = 0$. Hence, using these values of x_1 and x_2 for each $y \in \pi_1(U_\lambda)$ in equation (2.1), we get that

$$\begin{aligned}\|\nabla f\|_1 &\geq \int_{\pi_1(U_\lambda)} |f(x_1,y)-f(x_2,y)|\,\mathrm{d}y \\ &\geq \int_{\pi_1(U_\lambda)} |\lambda|\,\mathrm{d}y \\ &= |\lambda| \cdot \mathrm{m}(\pi_1(U_\lambda)) \implies \mathrm{m}(\pi_1(U_\lambda)) \leq \frac{\|\nabla f\|_1}{|\lambda|}.\end{aligned}$$

Similarly,

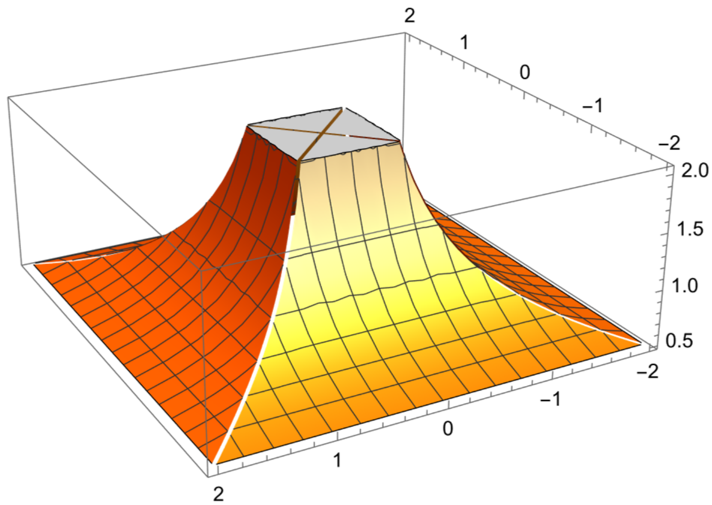
$$1 \geq |\lambda| \cdot \mathrm{m}(\pi_2(U_\lambda)) \implies \mathrm{m}(\pi_2(U_\lambda)) \leq \frac{\|\nabla f\|_1}{|\lambda|}.$$

Hence, applying equation (6), we have that

$$\mathrm{area}(U_\lambda) \leq \frac{\|\nabla f\|_1^2}{\lambda^2}. \tag{3.6}$$

Question 43. *Is this inequality approximately sharp?*

For simplicity, let $\|\nabla f\|_1 = 1$ to visualize this example, though note this would easily translate over $\|\nabla f\| = c$ for any nonnegative constant c. Consider the following example: $f(x,y) = \frac{1}{\max\{|x|,|y|\}}$.



Thus, U_λ is a square of side-length $\frac{1}{\lambda}$, and thus $\text{area}(U_\lambda) = \frac{1}{\lambda^2}$ for all $\lambda \neq 0$. However, f is neither continuous (discontinuous at the origin) nor compactly supported. How we can "fix" this example? In the region where $R \leq \max\{|x|, |y|\} \leq 2R$, let f smoothly interpolate to 0, and let f be 0 if $\max\{|x|, |y|\} \geq 2R$. This lets f be compactly supported, and we can make R arbitrarily large (though finite), implying that $\text{area}(U_\lambda) \leq \frac{1}{\lambda^2}$ will be approximately sharp, or exactly sharp, almost everywhere when R is sufficiently large. Addressing the discontinuity at the origin, in the region of the region where $0 \leq \max\{|x|, |y|\} \leq \frac{1}{R}$ (related to the R in fixing compactly supported), let $f(\vec{x})$ smoothly interpolate to the value R.

After making these changes to f, it seems to be the case that this works as a sufficient counterexample to show that $\text{area}(U_\lambda) \leq \frac{\|\nabla f\|_1^2}{\lambda^2}$ is approximately sharp.

So, given this inequality is approximately sharp, can we derive some sort of connection between $\text{area}(U_\lambda) \leq \frac{\|\nabla f\|_1^2}{\lambda^2}$ and $\|f\|_2$?

Conjecture 44

Perhaps we can say something along the lines of

$$\left(\int_{\mathbb{R}^2} |f(\vec{x})|^2 \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} \leq \|\nabla f\|_1^2 \iff \forall \lambda \neq 0,\ \text{area}(U_\lambda) \leq \frac{\|\nabla f\|_1^2}{\lambda^2}.$$

Well we can certainly prove the forward direction.

$$\begin{aligned}\|\nabla f\|_1^2 &\geq \int_{\mathbb{R}^2} |f(\vec{x})|^2 \,\mathrm{d}\vec{x} \\ &\geq \int_{U_\lambda} |f(\vec{x})|^2 \,\mathrm{d}\vec{x} \\ &\geq \int_{U_\lambda} |\lambda|^2 \,\mathrm{d}\vec{x}.\end{aligned}$$

Hence,

$$\text{area}(U_\lambda) = \int_{U_\lambda} 1 \,\mathrm{d}\vec{x} \leq \frac{\|\nabla f\|_1^2}{\lambda^2}.$$

This is known as Chebyshev's inequality, which comes up in probability.

But in trying to prove the other direction, how can we use $\text{area}(U_\lambda) \leq \frac{\|\nabla f\|_1^2}{\lambda^2}$ to approximate this integral? Well, this goes back to the initial part of this section: finding an upper bound for $\|f\|_2$. We could certainly try and plug in the area inequality. However, this will give us a divergent series, as shown below:

$$\begin{aligned}\|f\|_2 &\le \left(\sum_{k\in\mathbb{N}} 2^{-2(k-1)} \cdot \operatorname{area}(V_{2^{-(k-1)}})\right)^{\frac{1}{2}} \\ &\le \left(\sum_{k\in\mathbb{N}} 2^{-2(k-1)} \cdot \operatorname{area}(U_{2^{-(k-1)}})\right)^{\frac{1}{2}} \\ &\le \left(\sum_{k\in\mathbb{N}} 2^{-2(k-1)} \cdot \frac{\|\nabla f\|_1^2}{2^{-2(k-1)}}\right)^{\frac{1}{2}} \\ &= \left(\sum_{k\in\mathbb{N}} \|\nabla f\|_1^2\right)^{\frac{1}{2}}.\end{aligned}$$

Hence, unless $\|\nabla f\|_1 = 0$ (which implies f is constant and thus $\|f\|_2 = 0$ as f is compactly supported), we get a divergent series. Thus, this approach of finding an upper bound for the area doesn't quite pan out. Therefore, we hope to (and will) instead find a lower bound for $\|\nabla f\|_1$ that depends on the areas of V_λ and hope that things cancel out in the end (they will).

3.2.3 Bounding $\|\nabla f\|_1$

Instead, we can try to find a lower bound for $\|\nabla f\|_1$ without assuming $\|\nabla f\|_1 \le 1$. In trying to find an upper bound to the ratio $\frac{\|f\|_2}{\|\nabla f\|_1}$, we don't assume anything about the value of $\|\nabla f\|_1$, but the inequality $\|\nabla f\|_1 \le 1$ was crucial in first proving $\operatorname{area}(U_\lambda) \le \frac{1}{\lambda^2}$, which we ultimately can't use [as again, it leads to a divergent series]. Hence, we will *not* be using this inequality, though it was useful to start exploring this problem. We have the following.

$$\begin{aligned}\|\nabla f\|_1 &= \int_{\mathbb{R}^2} \sqrt{\partial_x^2 f(\vec{x}) + \partial_y^2 f(\vec{x})}\,\mathrm{d}\vec{x} \\ &\ge \int_{\mathbb{R}^2} |\partial_x f(\vec{x})|\,\mathrm{d}\vec{x} \\ &\ge \sum_{k\ even} \int_{V_{2^{-k}} \cup V_{2^{-k-1}}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Recall that we assume there exists a $\vec{x}$ such that $|f(\vec{x})| = 1$, and thus we can state the last line. Then,

$$\begin{aligned}\|\nabla f\|_1 &\ge \sum_{k\ even} \int_{V_{2^{-k}} \cup V_{2^{-k-1}}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{k\ even} \int_{\pi_1(V_{2^{-k}} \cup V_{2^{-k-1}})} \int_{\{x \mid (x,y)\in(V_{2^{-k}} \cup V_{2^{-k-1}})\}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Similarly, we can get that

$$\|\nabla f\|_1 \ge \sum_{k\ odd} \int_{\pi_1(V_{2^{-k}} \cup V_{2^{-k-1}})} \int_{\{x \mid (x,y)\in(V_{2^{-k}} \cup V_{2^{-k-1}})\}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y.$$

Adding these two inequalities together, we get that

$$\begin{aligned}\|\nabla f\|_1 &\ge \frac{1}{2}\sum_{k=0}^{\infty} \int_{\pi_1(V_{2^{-k}} \cup V_{2^{-k-1}})} \int_{\{x \mid (x,y)\in(V_{2^{-k}} \cup V_{2^{-k-1}})\}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\ &\ge \frac{1}{2}\sum_{k=0}^{\infty} \int_{\pi_1(V_{2^{-k}})} \int_{\{x \mid (x,y)\in(V_{2^{-k}} \cup V_{2^{-k-1}})\}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We can pick $x_{1,k}$ and $x_{2,k}$ such that $f(x_{1,k}, y) = 2^{-k-2}$ and $f(x_{2,k}, y) = 2^{-k-1}$. I will explain at the end of this proof why we can do this. By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\|\nabla f\|_1 &\geq \frac{1}{2}\sum_{k=0}^{\infty}\int_{\pi_1(V_{2^{-k}})}\int_{\{x|(x,y)\in(V_{2^{-k}}\cup V_{2^{-k-1}})\}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \sum_{k=0}^{\infty}\int_{\pi_1(V_{2^{-k}})}\int_{x_{1,k}}^{x_{2,k}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\int_{\pi_1(V_{2^{-k}})} \frac{1}{4}\cdot 2^{-k}\,\mathrm{d}x\,\mathrm{d}y \\
\|\nabla f\|_1 &\geq \frac{1}{8}\sum_{k=0}^{\infty} 2^{-k}\cdot \mathrm{m}(\pi_1(V_{2^{-k}})).
\end{aligned}$$

We can similarly do this with π_2 instead of π_1. Then we get the inequality

$$\|\nabla f\|_1 \geq \frac{1}{8}\sum_{k=0}^{\infty} 2^{-k}\cdot \mathrm{m}(\pi_2(V_{2^{-k}})).$$

Therefore,

$$\|\nabla f\|_1^2 \geq \frac{1}{16}\sum_{k=0}^{\infty} 2^{-2k}\cdot \mathrm{m}(\pi_1(V_{2^{-k}}))\cdot \mathrm{m}(\pi_2(V_{2^{-k}})) \geq \frac{1}{16}\sum_{k=0}^{\infty} 2^{-2k}\cdot \mathrm{area}(V_{2^{-k}})$$

using equation (2).

There is just that one caveat that we need to prove on. In order to use the Fundamental Theorem of Calculus to find a lower bound for $\|\nabla f\|_1$, we need to show that $x_{1,k}$ and $x_{2,k}$ are connected by a line segment in $V_{2^{-k}} \cup V_{2^{-k-1}}$ when $y \in \pi_1(V_{2^{-k}})$.

This is why we went through the trouble of splitting the sum into even and odd parts– so that this step of the proof worked out the way we wanted to. Let $y \in \pi_1(V_{2^{-k}})$. Since this function is compactly supported, we know that there exists an x such that $f(x, y) = 2^{-k-1}$ and thus $(x, y) \in V_{2^{-k}} \cup V_{2^{-k-1}}$. Let $x_{2,k}$ be the smallest such x that satisfies $f(x_{2,k}, y) = 2^{-k-1}$.

Since f is compactly supported, this implies that for all $x \leq x_{2,k}$, $f(x, y) \leq 2^{-k-1}$, as otherwise we would have a smaller $x_{2,k}$. Hence, there must exist an $x_{1,k}$ in the set

$$\{x \in \mathbb{R} \mid x \leq x_{2,k} \text{ and } (x, y) \in V_{2^{-k-1}}\}$$

with $f(x_{1,k}, y) = 2^{-k-2}$ such that $[x_{1,k}, x_{2,k}] \times y$ is a connected line segment in $V_{2^{-k}} \cup V_{2^{-k-1}}$. Therefore, we were in fact able to apply the Fundamental Theorem of Calculus.

Hence,

$$\|\nabla f\|_1 \geq \frac{1}{4}\left(\sum_{k=0}^{\infty} 2^{-2k}\cdot \mathrm{area}(V_{2^{-k}})\right)^{\frac{1}{2}}.$$

Therefore,

$$\frac{\|f\|_2}{\|\nabla f\|_1} \leq \frac{\left(\sum_{k=0}^{\infty} 2^{-2k}\cdot \mathrm{area}(V_{2^{-k}})\right)^{\frac{1}{2}}}{\frac{1}{4}\left(\sum_{k=0}^{\infty} 2^{-2k}\cdot \mathrm{area}(V_{2^{-k}})\right)^{\frac{1}{2}}} = 4.$$

3.3 Concluding Remarks

So at this point, we have proven the following theorem:

Theorem 45

Let $f(x,y) \in C_0^1(\mathbb{R}^2)$ such that $|f(x,y)| \leq 1$ for all $(x,y) \in \mathbb{R}^2$ and $f(x,y) = 0$ for all $(x,y) \notin [-1,1] \times [-1,1]$. Then,

$$\frac{\|f\|_2}{\|\nabla f\|_1} \leq 4.$$

We can very easily generalize this though for all two-dimensional continuous compactly-supported functions in general:

Corollary 46

Let $f(x,y) \in C_0^1(\mathbb{R}^2)$ such that $|f(x,y)| \leq Z$ for all $(x,y) \in \mathbb{R}^2$ and $f(x,y) = 0$ for all $(x,y) \notin [-R,R] \times [-R,R]$. Then, we have

$$\frac{\|f\|_2}{\|\nabla f\|_1} \leq 4.$$

Proof: Assume Z is nonzero, as then this is trivially true. Consider the function $g(x,y) = \frac{1}{Z} \cdot f(Rx, Ry)$. Hence, $g(x,y) \in C_0^1(\mathbb{R}^2)$, $|g(x,y)| \leq 1$ for all $(x,y) \in \mathbb{R}^2$, and $g(x,y) = 0$ for all $(x,y) \notin [-1,1] \times [-1 \times 1]$. Therefore, based on Theorem 10,

$$\frac{\|g\|_2}{\|\nabla g\|_1} \leq 4.$$

We can use this to find $\frac{\|f\|_2}{\|\nabla f\|_1}$:

$$\begin{aligned}
4 &\geq \frac{\|g\|_2}{\|\nabla g\|_1} \\
&= \frac{\left(\int_{\mathbb{R}^2} |g(\vec{x})|^2 \, \mathrm{d}\vec{x}\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^2} |\nabla g(\vec{x})| \, \mathrm{d}\vec{x}} \\
&= \frac{\left(\int_{\mathbb{R}^2} |\frac{1}{Z} \cdot f(R\vec{x})|^2 \, \mathrm{d}\vec{x}\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^2} |\nabla \frac{1}{Z} \cdot f(R\vec{x})| \, \mathrm{d}\vec{x}} \\
&= \frac{\left(\int_{\mathbb{R}^2} |f(R\vec{x})|^2 \, \mathrm{d}\vec{x}\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^2} |\nabla f(R\vec{x})| \, \mathrm{d}\vec{x}}.
\end{aligned}$$

This actually shows something pretty interesting; the upper bound on $|f|$ actually plays no effect on the ratio $\frac{\|f\|_2}{\|\nabla f\|_1}$, though perhaps this is inherently clear from the rescaling arguments earlier in this chapter. In any case, let $\vec{u} = R\vec{x}$ such that $\mathrm{d}\vec{u} = R^2 \, \mathrm{d}\vec{x}$. Hence,

$$\begin{aligned}
&= \frac{\left(\int_{\mathbb{R}^2} \frac{1}{R^2} \cdot |f(\vec{u})|^2 \, \mathrm{d}\vec{u}\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^2} \frac{R}{R^2} \cdot |\nabla f(\vec{u})| \, \mathrm{d}\vec{u}} \\
&= \frac{\|f\|_2}{\|\nabla f\|_1}.
\end{aligned}$$

□