SPUR 2022:
Hausdorff Dimension and Projections

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with Professor Guth and Shengwen Gan

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Chapter 1

Pre-Project

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1.1.1 May 23-24

Hi Larry and Shengwen! To catch you up to speed Shengwen, last summer when I was doing research with Larry and Yuqiu, I kept a track of notes over LaTeX in order to keep communications over, and this really helped keep track of ideas and problems I was working on. You can find this set of notes here if you are interested. I figured I would start the same for the SPUR project, at least while working remotely in California!

So, last I spoke with Shengwen we discussed readings to start looking at for the project. The first recommendation is *Fourier Analysis and Hausdorff Dimension* by Mattila (2015). This is available online over SpringerLink, so I have started on the specific suggested readings (SS2.2, 2.5, and Chapter 4). My goal is to finish reading these sections by the end of tomorrow (May 25).

Below are my notes so far on these sections and questions that came up. I also read through Chapter 1 and §2.1 to understand the bigger picture of the text a bit more, as well as get used to some of the notation.

Chapter 1: Introduction

- Measures with compact support in $A \subset \mathbb{R}^n$ (Borel measurable) is denoted $\mathcal{M}(A)$.
- Frostman's Lemma states the Hausdorff dimension of A ($\dim A$) is

$$\dim A = \sup\left\{s \in \mathbb{R} \mid \exists \mu \in \mathcal{M}(A) : \mu(B(x,r)) \le r^s \ \forall x \in \mathbb{R}^n,\ r > 0\right\}.$$

- The s-energy of μ is

$$I_s(\mu) := \iint |x-y|^{-s} \, \mathrm{d}\mu x \mathrm{d}\mu y.$$

  With this definition in hand, we will later show that

$$\dim A = \sup\{s \in \mathbb{R} \mid \exists \mu \in \mathcal{M}(A) : I_s(\mu) < \infty\}.$$

- The Riesz kernel, $k_s(x) = |x|^{-s}$, gives us that

$$I_s(\mu) = \int k_s * \mu \, \mathrm{d}\mu.$$

  We will define the convolution between functions and measures soon. For $0 < s < n$, $\hat{k_s} = \gamma(n,s) k_{n-s}$ where

$\gamma(n,s)$ is a positive constant. Hence, Parseval's Identity gives us that

$$I_s(\mu) = \int \hat{k_s}|\hat{\mu}|^2 = \gamma(n,s)\int |x|^{s-n}|\hat{\mu}(x)|\,\mathrm{d}x.$$

Hence,

$$\dim A = \sup\left\{s \in \mathbb{R} \mid \exists \mu \in \mathcal{M}(A) : \int |x|^{s-n}|\hat{\mu}(x)|^2\,\mathrm{d}x < \infty\right\}.$$

- We denote the one-dimensional Lebesgue measure to be $\mathcal{L}^1$.

Question 1. ~~*How should I/How can I start to picture* $\dim A$ *more concretely?*~~

Question 2. ~~*What is the meaning of one-dimensional Lebesgue measure? Is it just a way to say that* $P_e(A)$ *is one-dimensional without reminding the mathematician reading/writing?*~~

Question 3. ~~*Does the s-energy originate from somewhere before the study of Frostman's Lemma etc? By the terminology, it sounds like a term originating in mathematical physics.*~~

Part I: Preliminaries and some simpler applications of the Fourier transform

Chapter 2: Measure theoretic preliminaries

§2.1: Some basic notation

§2.1 mostly gave a lot of notation which will be useful, and thus is listed below:

- $\mathrm{diam}(A) := d(A)$
- $A^\circ := \mathrm{Int}(A)$
- $\mathrm{spt}\, f := \mathrm{supp}(f)$
- χ_A: Characteristic function on A
- $\mathcal{L}^n :=$ Lebesgue measure on $\mathbb{R}^n$
- $\alpha(n) := \mathcal{L}^n(B(0,1))$; $\sigma^{n-1} :=$ the surface measure on $\mathcal{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$
- $\delta_a :=$ the Dirac delta function
- $C(X); C^+(X)$: continuous functions on X and positive continuous functions respectively
- $C_0(X); C_0^+(X)$: continuous compactly supported functions on X and positive continuous compactly supported functions respectively
- $C^k(X); C_0^k(X)$: k-times differentiable functions on X and k-times differentiable compactly supported functions respectfully
- $C^\infty(X); C_0^\infty(X)$: smooth functions on X and smooth compactly supported functions

It still feels generally unclear why we want to distinguish different Lebesgue measures on $\mathbb{R}^n$, though perhaps it is just to be extra clear in proofs/definitions.

§2.2: Borel and Hausdorff measures

§2.2 had much more new information, so I will be a lot more clear in definitions here.

Definition 4 (Borel measure)
A **Borel measure** is a measure, μ, in which Borel sets are measurable and Borel regular i.e. $\forall A \subset X$, $\exists B$ Borel such that $A \subset B$ and $\mu(A) = \mu(B)$. A Borel measure is *locally finite* if compact sets have finite measure.

We define the *support* of a measure as the smallest closed set F such that $\mu(X \setminus F) = 0$. We define the restriction of a measure to a set A by

$$\mu\big|_A(B) := \mu(A \cap B).$$

Question 5. ~~*At least in the online copy I got through the MIT libraries, for some reason the Mattila seems to use a 'L' shape here instead of the 'regular' $\big|$ notation for restriction. Does this have a clear advantage?*~~

Given $f : X \to Y$, we define the *pushforward* of a measure $f_{\#}$ by

$$f_{\#}\mu(B) := \mu(f^{-1}(B)) \; \forall B \subset Y.$$

It is a Borel measure if μ is a Borel measure and f is a Borel function. Equivalently, for all g Borel functions nonnegative on X,

$$\int g \, \mathrm{d}f_{\#}\mu = \int g \circ f \, \mathrm{d}\mu.$$

We say μ is *absolutely continuous* with respct to ν if

$$\nu(A) = 0 \implies \mu(A) = 0.$$

We denote this $\mu << \nu$. Furthermore, μ and ν are *mutually singular* if there exists a borel set $B \subset X$ such that

$$\mu(X \setminus B) = \nu(B) = 0.$$

Notation 6

We have the shorthand

$$g\mu(B) := \int_B g \, \mathrm{d}\mu.$$

Thus, $\mu\big|_A = \chi_A \mu$.

Definition 7 (Hausdorff measure)

We define a **Hausdorff measure** $\mathcal{H}^s$ for $s \geq 0$ as

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

where, for $0 < \delta \leq \infty$,

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_j \alpha(s) 2^{-s} d(E_j)^s \mid A \subset \bigcup_j E_j, d(E_j) < \delta \right\}.$$

Here, $\alpha(s)$ is a fixed(?) positive number.

If s is an integer, $\alpha(n)$ is the volume of an n-dimensional ball with $\alpha(0) = 1$. Thus, in $\mathbb{R}^n$, $\mathcal{H}^n = \mathcal{L}^n$. If s is not an integer, $\alpha(s)$ is insignificant.

Question 8. *When Mattila says "$\alpha(s)$ is insignificant", does he mean that we can let $\alpha(s)$ be anything? Or is there a smooth interpolation between volumes of n-dimensional balls?*

Definition 9 (Hausdorff dimension)

We define the **Hausdorff dimension** of $A \subset \mathbb{R}^n$ as

$$\dim(A) := \inf\{s \mid \mathcal{H}^s(A) = 0\} = \sup\{s \mid \mathcal{H}^s(A) = \infty\}.$$

Exercise 10. *Show that* $\mathcal{H}^s(A) = 0 \iff \mathcal{H}^s_\infty(A) = 0$.

Given the above exercise (which I am considering trying to show), we get that

$$\dim(A) = \inf\left\{ s \mid \forall \epsilon > 0 \exists E_1, E_2, \dots \subset X : A \subset \bigcup_j E_j \text{ and } \sum_j d(E_j)^s < \epsilon \right\}.$$

Question 11. ~~*I asked this earlier after reading Chapter 1, but how can I picture the Hausdorff dimension? It feels like an unintuitive definition.*~~

If we restricted the E_js to be balls, then we would get the *spherical Hausdorff measure*.

Definition 12 (Orthogonal group)
The **Orthogonal group**, $O(n)$, of $\mathbb{R}^n$ is the set of linear maps $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$g(x) \cdot g(y) = x \cdot y \ \forall x, y \in \mathbb{R}^n.$$

Then, σ^{n-1} is defined up to a constant (under multiplication) by

$$\sigma^{n-1}(g(A)) = \sigma^{n-1}(A) \ \forall A \subset \mathbb{S}^{n-1}, g \in O(n).$$

Question 13. *I can see why this show be true; we want the surface measure on* $\mathbb{S}^{n-1}$ *to be the same up to isometry (which is essentially what this is saying), but I don't see why this results in a unique (up to constant) measure.*

In any case, this implies that there exists a unique Haar probability measure θ_n. In other words, θ_n is the unique Borel measure on $O(n)$ such that $\theta_n(O(n)) = 1$ and

$$\theta_n(\{h \circ g \mid h \in A\}) = \theta_n(\{g \circ h \mid h \in A\}) = \theta_n(A) \ \forall A \subset O(n), g \in O(n).$$

Question 14. *Perhaps this is unimportant, but I am slightly interested in why this unique Haar measure is known to exist. Is it essentially the Riesz representation theorem?*

Thus,

$$\theta_n(\{g \in O(n) \mid g(x) \in A\}) = \frac{\sigma^{n-1}(A)}{\sigma^{n-1}(\mathbb{S}^{n-1})} \ \forall A \subset \mathbb{S}^{n-1}.$$

§2.4: Weak Convergence

I did read through this relatively closely, but as far as I can tell, for now the most important parts will be the definitions of convolution.

Definition 15 (Convolution)
Suppose that f, g are functions, and μ, ν are Borel measures. Then,

$$f * g(x) = \int f(x-y)g(y)\,\mathrm{d}y$$
$$f * \mu(x) = \int f(x-y)\,\mathrm{d}\mu y$$
$$\int \varphi\,\mathrm{d}(\mu * \nu) = \iint \varphi(x+y)\,\mathrm{d}\mu x \mathrm{d}\nu y \quad \forall \varphi \in C_0^+(\mathbb{R}^n).$$

§2.5: Energy-integrals and Frostman's Lemma

Finding lower bounds for Hausdorff measures and dimension can be a really tricky problem. Frostman's Lemma transforms this problem to finding measures with good upper bounds for measures of balls.

Question 16. ~~*Why is this new problem any easier to solve?*~~

In this book, Mattila only explores Frostman's Lemma for compact sets, as any Borel set A with $\mathcal{H}^s(A) > 0$ contains a compact set C with $0 < \mathcal{H}^s(C) < \infty$.

Theorem 17 (Frostman's Lemma)
Let $0 \leq s \leq n$. For $A \subset \mathbb{R}^n$ Borel, $\mathcal{H}^s(A) > 0$ if and only if $\exists \mu \in \mathcal{M}(A)$ such that

$$\mu(B(x,r)) \leq r^s \ \forall x \in \mathbb{R}^n, r > 0. \tag{1.1}$$

In particular, $\dim A = \sup\{s \in \mathbb{R} \mid (1.1) \text{ holds}\}$.

A measure satisfying (1) is called a **Frostman measure**. I need to reread through the proof of Frostman's Lemma more closely before I fully get it. The general idea follows a similar proof method to that used in 18.118: constructing a sequence of measures that weakly converges to a measure with the properties we want. However, the proof also used dyadic cubes, which I currently lack the intuition for (fully). Perhaps it would be worth meeting and talking about the proof? I can try and find another proof of Frostman's Lemma in the meantime and see if I understand it better. After I understand Frostman's Lemma better, then perhaps the rest of §2.5 will make overall more sense. In the meantime, I do have some questions:

Question 18. ~~*What is* $d(\operatorname{spt}\mu)$*? As far as I can tell, its the minimum* r *value such that* $\mu(A) = 0$ *for* $A \in (B(0,r))^c$*.*~~

Question 19. *Using Frostman's Lemma, how can one picture* $\mathcal{H}^s(A)$*? Perhaps this will help illuminate how to visualize Hausdorff measures and dimension.*

More updates to come tomorrow as I read through Chapter 4 and explore other proofs of Frostman's Lemma.

1.1.2 May 25

My plan for today was to read through Chapter 4 and explore other proofs of Frostman's Lemma. I start by reading through Chapter 4. Notes included below.

Chapter 4: Hausdorff dimension of projections and distance sets

§4.1: Projections

We define the projection P_e for $e \in \mathbb{S}^{n-1}$, $n \geq 2$,

$$P_e : \mathbb{R}^n \to \mathbb{R}, \quad P_e(x) = e \cdot x.$$

Then, it follows that

$$\dim P_e(A) \leq \dim A \ \forall A \subset \mathbb{R}^n.$$

Exercise 20. *Show* P_e *is Lipschitz, and show that if* $f : \mathbb{R}^n \to \mathbb{R}$ *is Lipschitz, then*

$$\dim f(A) \leq \dim A \ \forall A \subset \mathbb{R}^n.$$

Theorem 21
Let $A \subset \mathbb{R}^n$ is Borel and let $s = \dim A$. If $s \leq 1$ then

$$\dim P_e(A) = s \ \sigma^{n-1} \text{ almost all } e \in \mathbb{S}^{n-1}.$$

If $s > 1$, then

$$\mathcal{L}^1(P_e(A)) > 0 \ \sigma^{n-1} \text{ almost all } e \in \mathbb{S}^{n-1}.$$

In the proof that follows, I understand the calculation of $\hat{\mu_e}$, but mildly find the results that follow confusing. Perhaps it would be worth writing this proof out carefully using the sections they reference in this part of the proof. The main part I found confusing was $\gamma(t, \ell)$ in calculating the integrals, though this seems to just be some notation I am missing.

Ultimately, what they are showing is that for $0 < t < s$, the t-energy of the pullback of the measure (i.e. t energy of the projection) is finite. Hence, $s = \sup\{s \in \mathbb{R} \mid \exists \mu \in \mathcal{M}(A) : I_s(\mu) < \infty\}$. I don't see why we need to take a sequence of t_i's though such that $t_i \to s$ to finish this proof however.

The second part of the proof shows that μ_e is absolutely continuous with respect to $\mathcal{L}^1$, so given $\mu_e \in \mathcal{M}(P_e(A))$, $\mu_e(A) > 0 \implies \mathcal{L}^1(P_e(A)) > 0$.

Theorem 22
Let $A \subset \mathbb{R}^n$ be Borel, $\dim A > 2$. Then, $P_e(A)$ has non-empty interior for σ^{n-1} almost all $e \in \mathcal{S}^{n-1}$.

It seems unclear to me that this *should* be the case. Perhaps ultimately working on Hausdorff dimension calculations a bit more will help me gain a better intuition behind this theorem. Nonetheless, reading through this proof helped highlight a general approach to these problems; to use that $\dim A > 2$ in some way, take $2 < s < \dim A$ to imply theorems about the projection.

Remark 23. *I particularly like how Mattila structures some of the statements in this book; it feels particularly personal. "I do not know any proof without Fourier transforms for this theorem", and "For a proof of the previous theorem without Fourier transforms, see Mattila [1995]". It provides a lot of interesting insight into theorem proofs that would take me forever to figure out.*

I am mildly interested in how Besicovitch sets can be used to show the bound in the above theorem is sharp, though this is much further into this book.

Theorem 24
Let $A \subset \mathbb{R}^n$ be Lebesgue measurable and let $\mu \in \mathcal{M}(A)$ with $\mu(A) = 1$ and $I_1(\mu) < \infty$. Then,

$$\int \mathcal{L}^1(P_e(A)) \, \mathrm{d}\sigma^{n-1} e \geq \frac{\gamma(n,1)\sigma^{n-1}(\mathbb{S}^{n-1})^2}{2I_1(\mu)}.$$

Again, the proof of this theorem feels relatively straightforward (at least in calculation), but I don't get intuitively why this lower bound makes sense to consider. Is there geometric intuition behind this lower bound?

Question 25. *Is this lower bound sharp?*

§4.2: Distance sets

Definition 26
The **distance set** of $A \subset \mathbb{R}^n$ is
$$D(A) = \{|x - y| \mid x, y \in A\} \subset [0, \infty).$$

It was interesting to read about the conjectures that are still open in this section. It was also useful to note the use of splitting integrals into distinct intervals to make general estimates.

Example 27
For $n \geq 2$, $0 < s < n/2$, there exists a compact set $C \subset \mathbb{R}^n$ with $\dim C = s$ and $\dim D(C) \leq 2s/n$.

The proof makes a more general version of a Cantor set in $\mathbb{R}^n$ with dimension s, which is interesting but begs the question

Question 28. ~~*What is the Hausdorff dimension of the Cantor set? Can you make a Cantor set of any dimension in $[0, n/2)$? The above proof seems to imply this is the case.*~~

§4.3: Dimension of Borel rings

This section was interesting to read through (if nothing else because it was easy to follow the general structure of the proof. I am interested in doing some more investigation into Borel rings themselves.

Question 29. *Is a Borel ring just a Borel set that is also an algebraic subring? Why does this ring make sense to consider if so? Is there an example or intuition behind why to study this sort of structure?*

1.1.3 May 26

Larry got back to me this yesterday (5/25) regarding Question 1, suggesting that I work through some specific examples of Hausdorff dimension on sets. Specifically, he recommended the following:

Exercise 30. *Calculate the Hausdorff dimension of*

1. $\mathbb{R}^n$ *(he says it should be n which makes sense)*
2. *the Cantor set (see Question 28)*
3. $\mathbb{Q}$ *(I believe this should be 0 as it is countable).*

I started to work through Exercise 30 below, before a meeting with Shengwen scheduled for this Friday 4pm EST.

Remark 31. *I later found out that my general approach to this exercise, while useful for gaining intuition into Hausdorff dimension, is not correct. Specifically, I constructed covers that led to the results I was looking for, but I need to calculate the Hausdorff measure for all open covers (or at the very least calculate when it is/isn't infinite or zero.*

Let's first consider the example of $\mathbb{R}$ to start. Consider intervals E_j of diameter $\frac{1}{n}$ covering $\mathbb{R}$. We know that this can cover $\mathbb{R}$, as
$$\sum_j \operatorname{diam}(E_j) = \sum_j \frac{1}{n} \quad \text{diverges.}$$

We can be more explicit in this construction placing the intervals $\frac{1}{2n}$ apart, making it a countable cover, but is (at least not clearly) unnecessary. Then, notice that

$$\sum_j \alpha(s)2^{-s}d(E_j)^s = C_s \sum_j \frac{1}{n^s} = \begin{cases} \text{diverges}, & s \leq 1 \\ \text{converges}, & s < 1 \end{cases}$$

as it is the classic p-series infinite series. Hence,

$$\dim(\mathbb{R}) = \sup\{s \in \mathbb{R} \mid \mathcal{H}^s(\mathbb{R}) = \infty\} = 1.$$

For $\mathbb{R}^n$ in general, you can similarly consider balls of diameter $(1/n)^{\frac{1}{n}}$.

Furthermore, the dimension of $\mathbb{Q}$ *is* 0 as it is countable, which follows as if we write $\mathbb{Q} = \{q_i \mid i \in \mathbb{N}\}$,

$$0 \leq \dim(\mathbb{Q}) \leq \sum_i \dim(q_i) = 0$$

as the Hausdorff dimension of a point is 0.

I wasn't able to completely finish the Cantor set example suggested, but I imagine it has to do with powers of 2 and 3. This intuition from the fact that the Cantor set can be written as the intersection of sets with 2^n intervals of length 3^{-n} and the Cantor set is compact. Hence, given an open cover, there exists a finite subcover, and we can find a refinement $\{E_j\}$ of the cover with 2^n intervals of diameter 3^{-n}. Hence, we should get something along the lines of

$$C_s \sum_{j=1}^{2n} \operatorname{diam}(E_j)^s = C_s \sum_{j=1}^{2^n} \frac{1}{3^{ns}} = C_s \left(\frac{2}{3^s}\right)^n.$$

Hence, the supremum of the s that makes infinite is an upper bound for the Hausdorff dimension of the Cantor set. However, I need to figure out 1) what s does this, and 2) this doesn't actually give us the Hausdorff dimension.

Therefore, I imagine there has to be a better approach to these problems that do not involve constructing a specific open cover/refinements. I will discuss this with Shengwen tomorrow.

In the response from Larry, he also answered Question 3. He stated the following:

> The s-energy is somewhat inspired by potential energy in physics. The potential energy from two masses m_1 and m_2 located at x_1 and x_2 is $-Gm_1m_2|x_1 - x_2|^{-1}$. Here G is the gravitational constant appearing in the formula for the gravitational force. The potential energy of a continuous mass distribution with density $\mu(x)$ is then given by an energy type integral.

I have heard a lot of different mathematical terms used in physics/chemistry recently in 18.118 and analysis in general. This includes potential energy, entropy, and Gibb's free energy. Is there a mathematical physics class at MIT that explores these various intuitive/useful concepts in a measure theoretic sense? It would be really cool to learn more about these various ideas in detail but I don't know where I would find out more about this. Or, perhaps it would be interesting to write an expository paper on various concepts like this. However, of course, this will have to wait until after SPUR.

1.1.4 May 27-29

Today was the day! I met with Shengwen to discuss the readings he suggested, work on Exercise 30, and answer questions I asked in these notes. I will first type out the details regarding the exercise, and then respond to questions asked throughout the notes so far.

Remark 32. *As is often the case in mathematics, there are numerous questions I have asked here that I doubt will be answered, either for lack of time, motivation, or use to this specific project. Nonetheless, I find it is useful to have a track of questions that have come up, and thus I am writing them down here!*

So, initially when working on Exercise 30, I started by consider $\mathbb{R}$ instead of $\mathbb{R}^n$. However, Shengwen suggested we start off *even smaller*, considering the following example:

Example 33
Show that the $\dim([0, 1]) = 1$.

Proof: To do so, we can show that for all $s < 1$,

$$\mathcal{H}^s([0, 1]) = \infty.$$

We use the following properties of $[0, 1]$:

$$\begin{aligned}
[0, 1] &= [0, 1/2] \cup [1/2, 1] \\
[0, 1] &= 2[0, 1/2] \\
[1/2, 1] &= [0, 1/2] + 1/2.
\end{aligned}$$

Hence, by translation invariance of the outer measure $\mathcal{H}^s$ and scaling arguments, we get that

$$\begin{aligned}
\mathcal{H}^s([0, 1]) &= \mathcal{H}^s([0, 1/2]) + \mathcal{H}^s([1/2, 1]) \\
&= 2\mathcal{H}^s([0, 1/2]) \\
&= \frac{2}{2^s} H^s([0, 1/2]).
\end{aligned}$$

If $s < 1$, then $\frac{2}{2^s} > 1 \implies \mathcal{H}^s([0, 1]) = 0$ or ∞. I claim that $\mathcal{H}^s([0, 1]) \neq 0$. To see this, note that for $t < s$, $\mathcal{H}^t(A) \leq \mathcal{H}^s(A)$ for all Borel measurable sets A. This is as the measure is monotonically decreasing with respect to s. Furthermore, $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbb{R}^n$ (as stated on page 14 of the Mattila). Therefore, $\mathcal{H}^1([0, 1]) = 1$, so $\mathcal{H}^s([0, 1]) \neq 0$ for $s < 1$.

Thus, for all $s < 1$, $\mathcal{H}^s([0, 1]) = \infty$. Note again by monotonicity, it also follows that for all $s \geq 1$, $\mathcal{H}^s([0, 1]) < \infty$. Therefore,

$$1 = \sup\{s \in \mathbb{R} \mid \mathcal{H}^s([0, 1]) = \infty\} = \dim([0, 1]).$$

□

Example 34
Similarly, show that $\dim([0, 1]^n) = n$.

Proof: For all $s < n$, we similarly get that

$$\mathcal{H}^s([0, 1]^n) = 2^n \mathcal{H}^s([0, 1/2]) = \frac{2^n}{2^s} \mathcal{H}^s([0, 1]) \implies \mathcal{H}^s([0, 1]^n) = 0 \quad \text{or} \quad = \infty.$$

A similar argument implies that $\mathcal{H}^s([0, 1]^n) \neq 0$, which finishes the result. □

Question 35. *How does one show that $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbb{R}^n$? This seems like it may be an interesting proof to read if we have additional time.*

So, how can we use these previous examples to finish the exercise? Shengwen noted the following theorem:

Theorem 36

Suppose that $A = \bigcup_i A_i$ (countable). Then,

$$\dim A = \sup_i \dim A_i.$$

Proof: One direction is immediately clear, but explained below anyways. Given $A_i \subset A$ for all i, and $\mathcal{H}^s$ is a measure, it follows that $\mathcal{H}^s(A_i) \leq \mathcal{H}^s(A)$. Hence,

$$\mathcal{H}^s(A_i) = \infty \implies \mathcal{H}^s(A) = \infty \; \forall i.$$

The other direction is a bit less immediate, but was briefly discussed by Shengwen.

Suppose that $a = \dim A$. Then, for all $\epsilon > 0$,

$$\sum_i \mathcal{H}^{a-\epsilon}(A_i) = \mathcal{H}^{a-\epsilon}(A) = \infty.$$

Hence, for all $\epsilon > 0$, there exists an i such that $\mathcal{H}^{a-\epsilon}(A_i) = \infty$. Therefore, for all $\epsilon > 0$, there exists an i such that

$$\dim A_i \geq a - \epsilon = \dim A - \epsilon.$$

Thus, $\sup_i(\dim A_i) \geq a = \dim A$. ■

Hence, Example 33 and Theorem 36 implies that $\dim \mathbb{R} = 1$, and similarly Example 34 and the Theorem implies $\dim \mathbb{R}^n = n$.

Notice as well that Theorem 36 implies that $\dim \mathbb{Q} = 0$ as the Hausdorff dimension of a single point is 0 and the rationals are countable.

We then went on to discuss the Cantor set exercise, but first discussed Minkowski dimension to start and approach the problem.

Notation 37

Let $\delta > 0$. Then,

$$C_\delta(X) = \min \#\{\delta - \text{balls to cover } X\}.$$

If $\dim X = k$, then $C_\delta(X) \sim \delta^{-k}$. Then, we can define the upper and lower Minkowski dimension:

Definition 38

The **upper Minkowski dimension** is

$$\overline{\dim}_M(X) = \limsup_{\delta \to 0} \frac{\log C_\delta(X)}{\log \delta^{-1}}.$$

Similarly, the **lower Minkowski dimension** is

$$\underline{\dim}_M(X) = \liminf_{\delta \to 0} \frac{\log C_\delta(X)}{\log \delta^{-1}}.$$

Remark 39. *This is remarkably similar to the topological entropy discussed in 18.118. I would be a bit interested in understanding why this is, but I digress.*

Definition 40 (Minkowski dimension)
If $\overline{\dim}_M(X) = \underline{\dim}_M(X)$, then we define the **Minkowski dimension** of X to be

$$\dim_M(X) = \overline{\dim}_M(X) = \underline{\dim}_M(X).$$

Example 41
Calculate $\dim_M(\mathbb{Q} \cap [0,1])$, if it exists.

Proof: First, notice that $C_\delta(\mathbb{Q} \cap [0,1]) = \delta^{-1}$, the length of the interval divided by the volume of a δ-ball in $\mathbb{R}$.

Then,

$$\overline{\dim}_M(\mathbb{Q} \cap [0,1]) = \underline{\dim}_M(\mathbb{Q} \cap [0,1]) = \lim_{\delta \to 0} \frac{\log \delta^{-1}}{\log \delta^{-1}} = 1.$$

So, $\dim_M(\mathbb{Q} \cap [0,1]) = 1$. □

In this way, the Minkowski dimension is *worse* than Hausdorff dimension, as it makes relatively small sets (such as $\mathbb{Q}$ which is countable) have dimension 1. However, it can be very useful to find upper bounds to Hausdorff dimension to start approaching a problem.

Exercise 42. *Show that* $\underline{\dim}_M X \geq \dim X$*.*

Remark 43. *See §2.3 of Mattila for more details.*

Using the above exercise (shown in Mattila [1995]), let's start approaching Exercise 30 by finding an upper bound to $\dim \mathcal{C}$ where $\mathcal{C}$ is the middle-third Cantor set.

Example 44
Calculate $\dim_M(\mathcal{C})$.

Proof: Let $\delta_k = 3^{-k}$, $k \in \mathbb{N}_0$ (as we are taking the limit as $\delta \to 0$, we can choose the subsequence of δs we wish for our convenience). Then, the number of δ_k-balls covering $\mathcal{C}$ (excluding the endpoints possibly) is 2^k. Then,

$$\dim_M(\mathcal{C}) = \limsup_{k \to \infty} \frac{\log(C_{\delta_k}(\mathcal{C}))}{\log(\delta_k^{-1})} = \limsup_{k \to \infty} \frac{\log 2^k}{\log 3^k} = \frac{\log 2}{\log 3}.$$

□

This gives a useful (and relatively easy to calculate) way to approach the problem of finding the Hausdorff dimension of $\mathcal{C}$. However, when it came down to calculating the dimension of $\mathcal{C}$, we actually used a similar scaling argument as we did for $\mathbb{R}^n$.

Example 45
Find $\dim \mathcal{C}$.

Proof: Let $\mathcal{C}_l = \mathcal{C} \cap [0, 1/3]$ and similarly let $\mathcal{C}_r = \mathcal{C} \cap [2/3, 1]$. Then,

$$\mathcal{H}^s(\mathcal{C}) = 2\mathcal{H}^s(\mathcal{C}_l) = \frac{2}{3^s}\mathcal{H}^s(\mathcal{C}).$$

Hence, if $s < \frac{\log 2}{\log 3}$, then $\mathcal{H}^s(\mathcal{C}) = \infty$. Thus, $\dim \mathcal{C} = \frac{\log 2}{\log 3}$. □

After spending most of the meeting discussing the exercise suggested by Larry (Exercise 30), Shengwen and I went through the questions I had asked so far in the notes to rapid fire respond to some of them. This discussion is included below:

- **Question 1**/**Question 11** was answered using Exercise 30, which we spent today discussing.
- **Question 2**: In particular, I was confused about the difference between L^p and $\mathcal{L}^n$. The difference here is L^p is the space of L^p functions, where as $\mathcal{L}^n$ is the measure. So, it should be clear by context which we are discussing (which is further helped by the use of mathcal).
- **Question 5**: This is mostly just an older notation, but does mean the same thing as the restriction notation I am used to. For this reason, I will continue to use '$|$' to denote restrictions.
- **Question 8**: Shengwen seemed ultimately uncertain with regards to this question, but noted that it is really *insignificant* as we won't need to worry about what the constant $\alpha(s)$ is in general, and it is just there to normalize volumes. That being said, I wanted to see if you had any answer to this Larry.
- **Question 13** and **Question 14**: We ultimately decided to hold off on answering theses questions for now. Shengwen seemed to have an idea about it, but for now it seems unimportant.
- **Question 16**: If anything, this question has been answered through working through the Hausdorff dimension exercise today.
- **Question 18**: $d\,\mathrm{spt}\,\mu$ is the diameter of the support of μ, so my idea of what it meant in this question was accurate! It felt like out-of-nowhere notation, but now it is clear where it comes from.

This is all we discussed in this meeting. I spent the following few days going through the notes, book, and other resources I could find online to flush out these notes so I really internalized what we discussed. The big key detail I felt like I was missing out on before this meeting was how we could use properties of $\mathcal{H}^s$ being an *outer measure* to our advantage.

My current plan moving forward into the next week is to reread Chapter 4, now that I have a better intuition behind Hausdorff dimension. It may also be helpful to read through some sections in Chapter 3. Shengwen also suggested a paper by Fässler and Orponen (*On Restricted Familes of Projections in* $\mathbb{R}^3$), which I plan to look over to better understand the proof of Frostman's lemma.

1.2 May 30-June 05

1.2.1 May 30

Howdy Larry! The introduction into this week's notes is relatively the same as the previous paragraph, but wanted to include it for completeness. I have a few key goals for this week:

- Reread Chapter 4 more closely. Read sections of Chapter 3 that come up to fully digest the proofs here.
- Read Fässler and Orponen's *On Restricted Familes of Projections in* $\mathbb{R}^3$, specifically Lemma 3.13 to better understand the proof of Frostman's lemma.
- Learn more about the high-low frequency method that we would like to use to reprove Marstrand's projection theorem!

As always, if there is anything in the notes so far that you would like to comment on please don't hesitate to send an email! I really found the exercise (Exercise 30) you suggested last week really helpful. I hope your vacation is going well!
