SPUR 2022:
Hausdorff Dimension and Projections

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with Professor Guth and Shengwen Gan

Contents

Chapter 1

Pre-Project

1.1 May 23-29

1.1.1 May 23-24

Hi Larry and Shengwen! To catch you up to speed Shengwen, last summer when I was doing research with Larry and Yuqiu, I kept a track of notes over LaTeX in order to keep communications over, and this really helped keep track of ideas and problems I was working on. You can find this set of notes here if you are interested. I figured I would start the same for the SPUR project, at least while working remotely in California!

So, last I spoke with Shengwen we discussed readings to start looking at for the project. The first recommendation is Mattila's *Fourier Analysis and Hausdorff Dimension* (2015) **[M]**. This is available online over SpringerLink here, so I have started on the specific suggested readings (SS2.2, 2.5, and Chapter 4). My goal is to finish reading these sections by the end of tomorrow (May 25).

Below are my notes so far on these sections and questions that came up. I also read through Chapter 1 and §2.1 to understand the bigger picture of the text a bit more, as well as get used to some of the notation.

Chapter 1: Introduction

- Measures with compact support in $A \subset \mathbb{R}^n$ (Borel measurable) is denoted $\mathcal{M}(A)$.
- Frostman's Lemma states the Hausdorff dimension of A ($\dim A$) is

$$\dim A = \sup \left\{ s \in \mathbb{R} \mid \exists \mu \in \mathcal{M}(A) : \mu(B(x,r)) \leq r^s \ \forall x \in \mathbb{R}^n,\ r > 0 \right\}.$$

- The s-energy of μ is

$$I_s(\mu) := \iint |x-y|^{-s} \, \mathrm{d}\mu x \mathrm{d}\mu y.$$

  With this definition in hand, we will later show that

$$\dim A = \sup\{ s \in \mathbb{R} \mid \exists \mu \in \mathcal{M}(A) : I_s(\mu) < \infty \}.$$

- The Riesz kernel, $k_s(x) = |x|^{-s}$, gives us that

$$I_s(\mu) = \int k_s * \mu \, \mathrm{d}\mu.$$

  We will define the convolution between functions and measures soon. For $0 < s < n$, $\hat{k_s} = \gamma(n,s) k_{n-s}$ where

$\gamma(n, s)$ is a positive constant. Hence, Parseval's Identity gives us that

$$I_s(\mu) = \int \hat{k_s} |\hat{\mu}|^2 = \gamma(n, s) \int |x|^{s-n} |\hat{\mu}(x)| \, \mathrm{d}x.$$

Hence,

$$\dim A = \sup \left\{ s \in \mathbb{R} \mid \exists \mu \in \mathcal{M}(A) : \int |x|^{s-n} |\hat{\mu}(x)|^2 \, \mathrm{d}x < \infty \right\}.$$

- We denote the one-dimensional Lebesgue measure to be $\mathcal{L}^1$.

Question 1. ~~*How should I/How can I start to picture* $\dim A$ *more concretely?*~~

Question 2. ~~*What is the meaning of one-dimensional Lebesgue measure? Is it just a way to say that* $P_e(A)$ *is one-dimensional without reminding the mathematician reading/writing?*~~

Question 3. ~~*Does the s-energy originate from somewhere before the study of Frostman's Lemma etc? By the terminology, it sounds like a term originating in mathematical physics.*~~

Part I: Preliminaries and some simpler applications of the Fourier transform

Chapter 2: Measure theoretic preliminaries

§2.1: Some basic notation

§2.1 mostly gave a lot of notation which will be useful, and thus is listed below:

- $\mathrm{diam}(A) := d(A)$
- $A^\circ := \mathrm{Int}(A)$
- $\mathrm{spt}\, f := \mathrm{supp}(f)$
- χ_A: Characteristic function on A
- $\mathcal{L}^n :=$ Lebesgue measure on $\mathbb{R}^n$
- $\alpha(n) := \mathcal{L}^n(B(0,1))$; σ^{n-1}:= the surface measure on $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$
- δ_a:= the Dirac delta function
- $C(X); C^+(X)$: continuous functions on X and positive continuous functions respectively
- $C_0(X); C_0^+(X)$: continuous compactly supported functions on X and positive continuous compactly supported functions respectively
- $C^k(X); C_0^k(X)$: k-times differentiable functions on X and k-times differentiable compactly supported functions respectfully
- $C^\infty(X); C_0^\infty(X)$: smooth functions on X and smooth compactly supported functions

It still feels generally unclear why we want to distinguish different Lebesgue measures on $\mathbb{R}^n$, though perhaps it is just to be extra clear in proofs/definitions.

§2.2: Borel and Hausdorff measures

§2.2 had much more new information, so I will be a lot more clear in definitions here.

Definition 4 (Borel measure)

A **Borel measure** is a measure, μ, in which Borel sets are measurable and Borel regular i.e. $\forall A \subset X$, $\exists B$ Borel such that $A \subset B$ and $\mu(A) = \mu(B)$. A Borel measure is *locally finite* if compact sets have finite measure.

We define the *support* of a measure as the smallest closed set F such that $\mu(X \setminus F) = 0$. We define the restriction of a measure to a set A by

$$\mu\big|_A(B) := \mu(A \cap B).$$

Question 5. ~~*At least in the online copy I got through the MIT libraries, for some reason the Mattila seems to use a 'L' shape here instead of the 'regular' $\big|$ notation for restriction. Does this have a clear advantage?*~~

Given $f : X \to Y$, we define the *pushforward* of a measure $f_\#$ by

$$f_\#\mu(B) := \mu(f^{-1}(B)) \; \forall B \subset Y.$$

It is a Borel measure if μ is a Borel measure and f is a Borel function. Equivalently, for all g Borel functions nonnegative on X,

$$\int g \, \mathrm{d}f_\#\mu = \int g \circ f \, \mathrm{d}\mu.$$

We say μ is *absolutely continuous* with respct to ν if

$$\nu(A) = 0 \implies \mu(A) = 0.$$

We denote this $\mu \ll \nu$. Furthermore, μ and ν are *mutually singular* if there exists a borel set $B \subset X$ such that

$$\mu(X \setminus B) = \nu(B) = 0.$$

Notation 6

We have the shorthand

$$g\mu(B) := \int_B g \, \mathrm{d}\mu.$$

Thus, $\mu\big|_A = \chi_A \mu$.

Definition 7 (Hausdorff measure)

We define a **Hausdorff measure** $\mathcal{H}^s$ for $s \geq 0$ as

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

where, for $0 < \delta \leq \infty$,

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_j \alpha(s) 2^{-s} d(E_j)^s \mid A \subset \bigcup_j E_j, d(E_j) < \delta \right\}.$$

Here, $\alpha(s)$ is a fixed positive number.

If s is an integer, $\alpha(n)$ is the volume of an n-dimensional ball with $\alpha(0) = 1$. Thus, in $\mathbb{R}^n$, $\mathcal{H}^n = \mathcal{L}^n$. If s is not an integer, $\alpha(s)$ is insignificant.

Question 8. *When Mattila says "$\alpha(s)$ is insignificant", does he mean that we can let $\alpha(s)$ be anything? Or is there a smooth interpolation between volumes of n-dimensional balls?*

Definition 9 (Hausdorff dimension)

We define the **Hausdorff dimension** of $A \subset \mathbb{R}^n$ as

$$\dim(A) := \inf\{s \mid \mathcal{H}^s(A) = 0\} = \sup\{s \mid \mathcal{H}^s(A) = \infty\}.$$

Exercise 10. *Show that* $\mathcal{H}^s(A) = 0 \iff \mathcal{H}^s_\infty(A) = 0$.

Given the above exercise (which I am considering trying to show), we get that

$$\dim(A) = \inf\left\{ s \mid \forall \epsilon > 0 \exists E_1, E_2, \dots \subset X : A \subset \bigcup_j E_j \text{ and } \sum_j d(E_j)^s < \epsilon \right\}.$$

Question 11. ~~*I asked this earlier after reading Chapter 1, but how can I picture the Hausdorff dimension? It feels like an unintuitive definition.*~~

If we restricted the E_js to be balls, then we would get the *spherical Hausdorff measure*.

Definition 12 (Orthogonal group)
The **Orthogonal group**, $O(n)$, of $\mathbb{R}^n$ is the set of linear maps $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$g(x) \cdot g(y) = x \cdot y \ \forall x, y \in \mathbb{R}^n.$$

Then, σ^{n-1} is defined up to a constant (under multiplication) by

$$\sigma^{n-1}(g(A)) = \sigma^{n-1}(A) \ \forall A \subset \mathbb{S}^{n-1}, g \in O(n).$$

Question 13. *I can see why this show be true; we want the surface measure on* $\mathbb{S}^{n-1}$ *to be the same up to isometry (which is essentially what this is saying), but I don't see why this results in a unique (up to constant) measure.*

In any case, this implies that there exists a unique Haar probability measure θ_n. In other words, θ_n is the unique Borel measure on $O(n)$ such that $\theta_n(O(n)) = 1$ and

$$\theta_n(\{h \circ g \mid h \in A\}) = \theta_n(\{g \circ h \mid h \in A\}) = \theta_n(A) \ \forall A \subset O(n), g \in O(n).$$

Question 14. *Perhaps this is unimportant, but I am slightly interested in why this unique Haar measure is known to exist. Is it essentially the Riesz representation theorem?*

Thus,

$$\theta_n(\{g \in O(n) \mid g(x) \in A\}) = \frac{\sigma^{n-1}(A)}{\sigma^{n-1}(\mathbb{S}^{n-1})} \ \forall A \subset \mathbb{S}^{n-1}.$$

§2.4: Weak Convergence

I did read through this relatively closely, but as far as I can tell, for now the most important parts will be the definitions of convolution.

Definition 15 (Convolution)
Suppose that f, g are functions, and μ, ν are Borel measures. Then,

$$f * g(x) = \int f(x-y) g(y) \, \mathrm{d}y$$
$$f * \mu(x) = \int f(x-y) \, \mathrm{d}\mu y$$
$$\int \varphi \, \mathrm{d}(\mu * \nu) = \iint \varphi(x+y) \, \mathrm{d}\mu x \mathrm{d}\nu y \quad \forall \varphi \in C_0^+(\mathbb{R}^n).$$

§2.5: Energy-integrals and Frostman's Lemma

Finding lower bounds for Hausdorff measures and dimension can be a really tricky problem. Frostman's Lemma transforms this problem to finding measures with good upper bounds for measures of balls.

Question 16. ~~*Why is this new problem any easier to solve?*~~

In this book, Mattila only explores Frostman's Lemma for compact sets, as any Borel set A with $\mathcal{H}^s(A) > 0$ contains a compact set C with $0 < \mathcal{H}^s(C) < \infty$.

Theorem 17 (Frostman's Lemma)
Let $0 \le s \le n$. For $A \subset \mathbb{R}^n$ Borel, $\mathcal{H}^s(A) > 0$ if and only if $\exists \mu \in \mathcal{M}(A)$ such that

$$\mu(B(x,r)) \le r^s \ \forall x \in \mathbb{R}^n, r > 0. \tag{1.1}$$

In particular, $\dim A = \sup\{s \in \mathbb{R} \mid (1.1) \text{ holds}\}$.

A measure satisfying (1) is called a **Frostman measure**. I need to reread through the proof of Frostman's Lemma more closely before I fully get it. The general idea follows a similar proof method to that used in 18.118: constructing a sequence of measures that weakly converges to a measure with the properties we want. However, the proof also used dyadic cubes, which I currently lack the intuition for (fully). Perhaps it would be worth meeting and talking about the proof? I can try and find another proof of Frostman's Lemma in the meantime and see if I understand it better. After I understand Frostman's Lemma better, then perhaps the rest of §2.5 will make overall more sense. In the meantime, I do have some questions:

Question 18. ~~*What is $d(\operatorname{spt}\mu)$? As far as I can tell, its the minimum r value such that $\mu(A) = 0$ for $A \in (B(0,r))^c$.*~~

Question 19. *Using Frostman's Lemma, how can one picture $\mathcal{H}^s(A)$? Perhaps this will help illuminate how to visualize Hausdorff measures and dimension.*

More updates to come tomorrow as I read through Chapter 4 and explore other proofs of Frostman's Lemma.

1.1.2 May 25

My plan for today was to read through Chapter 4 and explore other proofs of Frostman's Lemma. I start by reading through Chapter 4. Notes included below.

Chapter 4: Hausdorff dimension of projections and distance sets

§4.1: Projections

We define the projection P_e for $e \in \mathbb{S}^{n-1}$, $n \ge 2$,

$$P_e : \mathbb{R}^n \to \mathbb{R}, \quad P_e(x) = e \cdot x.$$

Then, it follows that

$$\dim P_e(A) \le \dim A \ \forall A \subset \mathbb{R}^n.$$

Exercise 20. *Show P_e is Lipschitz, and show that if $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then*

$$\dim f(A) \le \dim A \ \forall A \subset \mathbb{R}^n.$$

Theorem 21
Let $A \subset \mathbb{R}^n$ is Borel and let $s = \dim A$. If $s \leq 1$ then

$$\dim P_e(A) = s \ \sigma^{n-1} \text{ almost all } e \in \mathbb{S}^{n-1}.$$

If $s > 1$, then

$$\mathcal{L}^1(P_e(A)) > 0 \ \sigma^{n-1} \text{ almost all } e \in \mathbb{S}^{n-1}.$$

In the proof that follows, I understand the calculation of $\hat{\mu_e}$, but mildly find the results that follow confusing. Perhaps it would be worth writing this proof out carefully using the sections they reference in this part of the proof. The main part I found confusing was $\gamma(t, \ell)$ in calculating the integrals, though this seems to just be some notation I am missing.

Ultimately, what they are showing is that for $0 < t < s$, the t-energy of the pullback of the measure (i.e. t energy of the projection) is finite. Hence, $s = \sup\{s \in \mathbb{R} \mid \exists \mu \in \mathcal{M}(A) : I_s(\mu) < \infty\}$. I don't see why we need to take a sequence of t_i's though such that $t_i \to s$ to finish this proof however.

The second part of the proof shows that μ_e is absolutely continuous with respect to $\mathcal{L}^1$, so given $\mu_e \in \mathcal{M}(P_e(A))$, $\mu_e(A) > 0 \implies \mathcal{L}^1(P_e(A)) > 0$.

Theorem 22
Let $A \subset \mathbb{R}^n$ be Borel, $\dim A > 2$. Then, $P_e(A)$ has non-empty interior for σ^{n-1} almost all $e \in \mathbb{S}^{n-1}$.

It seems unclear to me that this *should* be the case. Perhaps ultimately working on Hausdorff dimension calculations a bit more will help me gain a better intuition behind this theorem. Nonetheless, reading through this proof helped highlight a general approach to these problems; to use that $\dim A > 2$ in some way, take $2 < s < \dim A$ to imply theorems about the projection.

Remark 23. *I particularly like how Mattila structures some of the statements in this book; it feels particularly personal. "I do not know any proof without Fourier transforms for this theorem", and "For a proof of the previous theorem without Fourier transforms, see Mattila [1995]". It provides a lot of interesting insight into theorem proofs that would take me forever to figure out.*

I am mildly interested in how Besicovitch sets can be used to show the bound in the above theorem is sharp, though this is much further into this book.

Theorem 24
Let $A \subset \mathbb{R}^n$ be Lebesgue measurable and let $\mu \in \mathcal{M}(A)$ with $\mu(A) = 1$ and $I_1(\mu) < \infty$. Then,

$$\int \mathcal{L}^1(P_e(A)) \, \mathrm{d}\sigma^{n-1} e \geq \frac{\gamma(n, 1)\sigma^{n-1}(\mathbb{S}^{n-1})^2}{2I_1(\mu)}.$$

Again, the proof of this theorem feels relatively straightforward (at least in calculation), but I don't get intuitively why this lower bound makes sense to consider. Is there geometric intuition behind this lower bound?

Question 25. *Is this lower bound sharp?*

§4.2: Distance sets

Definition 26
The **distance set** of $A \subset \mathbb{R}^n$ is
$$D(A) = \{|x - y| \mid x, y \in A\} \subset [0, \infty).$$

It was interesting to read about the conjectures that are still open in this section. It was also useful to note the use of splitting integrals into distinct intervals to make general estimates.

Example 27
For $n \geq 2$, $0 < s < n/2$, there exists a compact set $C \subset \mathbb{R}^n$ with $\dim C = s$ and $\dim D(C) \leq 2s/n$.

The proof makes a more general version of a Cantor set in $\mathbb{R}^n$ with dimension s, which is interesting but begs the question

Question 28. ~~*What is the Hausdorff dimension of the Cantor set? Can you make a Cantor set of any dimension in* $[0, n/2)$*? The above proof seems to imply this is the case.*~~

§4.3: Dimension of Borel rings

This section was interesting to read through (if nothing else because it was easy to follow the general structure of the proof. I am interested in doing some more investigation into Borel rings themselves.

Question 29. *Is a Borel ring just a Borel set that is also an algebraic subring? Why does this ring make sense to consider if so? Is there an example or intuition behind why to study this sort of structure?*

1.1.3 May 26

Larry got back to me this yesterday (5/25) regarding Question 1, suggesting that I work through some specific examples of Hausdorff dimension on sets. Specifically, he recommended the following:

Exercise 30. *Calculate the Hausdorff dimension of*

1. $\mathbb{R}^n$ *(he says it should be* n *which makes sense)*
2. *the Cantor set (see Question 28)*
3. $\mathbb{Q}$ *(I believe this should be 0 as it is countable).*

I started to work through Exercise 30 below, before a meeting with Shengwen scheduled for this Friday 4pm EST.

Remark 31. *I later found out that my general approach to this exercise, while useful for gaining intuition into Hausdorff dimension, is not correct. Specifically, I constructed covers that led to the results I was looking for, but I need to calculate the Hausdorff measure for all open covers (or at the very least calculate when it is/isn't infinite or zero.*

Let's first consider the example of $\mathbb{R}$ to start. Consider intervals E_j of diameter $\frac{1}{n}$ covering $\mathbb{R}$. We know that this can cover $\mathbb{R}$, as
$$\sum_j \text{diam}(E_j) = \sum_j \frac{1}{n} \quad \text{diverges.}$$

We can be more explicit in this construction placing the intervals $\frac{1}{2n}$ apart, making it a countable cover, but is (at least not clearly) unnecessary. Then, notice that

$$\sum_j \alpha(s)2^{-s}d(E_j)^s = C_s \sum_j \frac{1}{n^s} = \begin{cases} \text{diverges}, & s \leq 1 \\ \text{converges}, & s < 1 \end{cases}$$

as it is the classic p-series infinite series. Hence,

$$\dim(\mathbb{R}) = \sup\{s \in \mathbb{R} \mid \mathcal{H}^s(\mathbb{R}) = \infty\} = 1.$$

For $\mathbb{R}^n$ in general, you can similarly consider balls of diameter $(1/n)^{\frac{1}{n}}$.

Furthermore, the dimension of $\mathbb{Q}$ *is* 0 as it is countable, which follows as if we write $\mathbb{Q} = \{q_i \mid i \in \mathbb{N}\}$,

$$0 \leq \dim(\mathbb{Q}) \leq \sum_i \dim(q_i) = 0$$

as the Hausdorff dimension of a point is 0.

I wasn't able to completely finish the Cantor set example suggested, but I imagine it has to do with powers of 2 and 3. This intuition from the fact that the Cantor set can be written as the intersection of sets with 2^n intervals of length 3^{-n} and the Cantor set is compact. Hence, given an open cover, there exists a finite subcover, and we can find a refinement $\{E_j\}$ of the cover with 2^n intervals of diameter 3^{-n}. Hence, we should get something along the lines of

$$C_s \sum_{j=1}^{2n} \text{diam}(E_j)^s = C_s \sum_{j=1}^{2^n} \frac{1}{3^{ns}} = C_s \left(\frac{2}{3^s}\right)^n.$$

Hence, the supremum of the s that makes infinite is an upper bound for the Hausdorff dimension of the Cantor set. However, I need to figure out 1) what s does this, and 2) this doesn't actually give us the Hausdorff dimension.

Therefore, I imagine there has to be a better approach to these problems that do not involve constructing a specific open cover/refinements. I will discuss this with Shengwen tomorrow.

In the response from Larry, he also answered Question 3. He stated the following:

> The s-energy is somewhat inspired by potential energy in physics. The potential energy from two masses m_1 and m_2 located at x_1 and x_2 is $-Gm_1m_2|x_1 - x_2|^{-1}$. Here G is the gravitational constant appearing in the formula for the gravitational force. The potential energy of a continuous mass distribution with density $\mu(x)$ is then given by an energy type integral.

I have heard a lot of different mathematical terms used in physics/chemistry recently in 18.118 and analysis in general. This includes potential energy, entropy, and Gibb's free energy. Is there a mathematical physics class at MIT that explores these various intuitive/useful concepts in a measure theoretic sense? It would be really cool to learn more about these various ideas in detail but I don't know where I would find out more about this. Or, perhaps it would be interesting to write an expository paper on various concepts like this. However, of course, this will have to wait until after SPUR.

1.1.4 May 27-29

Today was the day! I met with Shengwen to discuss the readings he suggested, work on Exercise 30, and answer questions I asked in these notes. I will first type out the details regarding the exercise, and then respond to questions asked throughout the notes so far.

Remark 32. *As is often the case in mathematics, there are numerous questions I have asked here that I doubt will be answered, either for lack of time, motivation, or use to this specific project. Nonetheless, I find it is useful to have a track of questions that have come up, and thus I am writing them down here!*

So, initially when working on Exercise 30, I started by consider $\mathbb{R}$ instead of $\mathbb{R}^n$. However, Shengwen suggested we start off *even smaller*, considering the following example:

Example 33

Show that the $\dim([0,1]) = 1$.

Proof: To do so, we can show that for all $s < 1$,

$$\mathcal{H}^s([0,1]) = \infty.$$

We use the following properties of $[0,1]$:

$$\begin{aligned}[0,1] &= [0,1/2] \cup [1/2,1] \\ [0,1] &= 2[0,1/2] \\ [1/2,1] &= [0,1/2] + 1/2.\end{aligned}$$

Hence, by translation invariance of the outer measure $\mathcal{H}^s$ and scaling arguments, we get that

$$\begin{aligned}\mathcal{H}^s([0,1]) &= \mathcal{H}^s([0,1/2]) + \mathcal{H}^s([1/2,1]) \\ &= 2\mathcal{H}^s([0,1/2]) \\ &= \frac{2}{2^s} H^s([0,1/2]).\end{aligned}$$

If $s < 1$, then $\frac{2}{2^s} > 1 \implies \mathcal{H}^s([0,1]) = 0$ or ∞. I claim that $\mathcal{H}^s([0,1]) \neq 0$. To see this, note that for $t < s$, $\mathcal{H}^t(A) \leq \mathcal{H}^s(A)$ for all Borel measurable sets A. This is as the measure is monotonically decreasing with respect to s. Furthermore, $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbb{R}^n$ (as stated on page 14 of the Mattila). Therefore, $\mathcal{H}^1([0,1]) = 1$, so $\mathcal{H}^s([0,1]) \neq 0$ for $s < 1$.

Thus, for all $s < 1$, $\mathcal{H}^s([0,1]) = \infty$. Note again by monotonicity, it also follows that for all $s \geq 1$, $\mathcal{H}^s([0,1]) < \infty$. Therefore,

$$1 = \sup\{s \in \mathbb{R} \mid \mathcal{H}^s([0,1]) = \infty\} = \dim([0,1]).$$

□

Example 34

Similarly, show that $\dim([0,1]^n) = n$.

Proof: For all $s < n$, we similarly get that

$$\mathcal{H}^s([0,1]^n) = 2^n \mathcal{H}^s([0,1/2]) = \frac{2^n}{2^s} \mathcal{H}^s([0,1]) \implies \mathcal{H}^s([0,1]^n) = 0 \quad \text{or} \quad = \infty.$$

A similar argument implies that $\mathcal{H}^s([0,1]^n) \neq 0$, which finishes the result. □

Question 35. *How does one show that $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbb{R}^n$? This seems like it may be an interesting proof to read if we have additional time.*

So, how can we use these previous examples to finish the exercise? Shengwen noted the following theorem:

Theorem 36
Suppose that $A = \bigcup_i A_i$ (countable). Then,

$$\dim A = \sup_i \dim A_i.$$

Proof: One direction is immediately clear, but explained below anyways. Given $A_i \subset A$ for all i, and $\mathcal{H}^s$ is a measure, it follows that $\mathcal{H}^s(A_i) \leq \mathcal{H}^s(A)$. Hence,

$$\mathcal{H}^s(A_i) = \infty \implies \mathcal{H}^s(A) = \infty \ \forall i.$$

The other direction is a bit less immediate, but was briefly discussed by Shengwen.

Suppose that $a = \dim A$. Then, for all $\epsilon > 0$,

$$\sum_i \mathcal{H}^{a-\epsilon}(A_i) = \mathcal{H}^{a-\epsilon}(A) = \infty.$$

Hence, for all $\epsilon > 0$, there exists an i such that $\mathcal{H}^{a-\epsilon}(A_i) = \infty$. Therefore, for all $\epsilon > 0$, there exists an i such that

$$\dim A_i \geq a - \epsilon = \dim A - \epsilon.$$

Thus, $\sup_i(\dim A_i) \geq a = \dim A$. ■

Hence, Example 33 and Theorem 36 implies that $\dim \mathbb{R} = 1$, and similarly Example 34 and the Theorem implies $\dim \mathbb{R}^n = n$.

Notice as well that Theorem 36 implies that $\dim \mathbb{Q} = 0$ as the Hausdorff dimension of a single point is 0 and the rationals are countable.

We then went on to discuss the Cantor set exercise, but first discussed Minkowski dimension to start and approach the problem.

Notation 37
Let $\delta > 0$. Then,

$$C_\delta(X) = \min \#\{\delta - \text{balls to cover } X\}.$$

If $\dim X = k$, then $C_\delta(X) \sim \delta^{-k}$. Then, we can define the upper and lower Minkowski dimension:

Definition 38
The **upper Minkowski dimension** is

$$\overline{\dim}_M(X) = \limsup_{\delta \to 0} \frac{\log C_\delta(X)}{\log \delta^{-1}}.$$

Similarly, the **lower Minkowski dimension** is

$$\underline{\dim}_M(X) = \liminf_{\delta \to 0} \frac{\log C_\delta(X)}{\log \delta^{-1}}.$$

Remark 39. *This is remarkably similar to the topological entropy discussed in 18.118. I would be a bit interested in understanding why this is, but I digress.*

Definition 40 (Minkowski dimension)
If $\overline{\dim}_M(X) = \underline{\dim}_M(X)$, then we define the **Minkowski dimension** of X to be

$$\dim_M(X) = \overline{\dim}_M(X) = \underline{\dim}_M(X).$$

Example 41
Calculate $\dim_M(\mathbb{Q} \cap [0,1])$, if it exists.

Proof: First, notice that $C_\delta(\mathbb{Q} \cap [0,1]) = \delta^{-1}$, the length of the interval divided by the volume of a δ-ball in $\mathbb{R}$.

Then,

$$\overline{\dim}_M(\mathbb{Q} \cap [0,1]) = \underline{\dim}_M(\mathbb{Q} \cap [0,1]) = \lim_{\delta \to 0} \frac{\log \delta^{-1}}{\log \delta^{-1}} = 1.$$

So, $\dim_M(\mathbb{Q} \cap [0,1]) = 1$. □

In this way, the Minkowski dimension is *worse* than Hausdorff dimension, as it makes relatively small sets (such as $\mathbb{Q}$ which is countable) have dimension 1. However, it can be very useful to find upper bounds to Hausdorff dimension to start approaching a problem.

Exercise 42. *Show that* $\underline{\dim}_M X \geq \dim X$.

Remark 43. *See §2.3 of Mattila for more details.*

Using the above exercise (shown in Mattila [1995]), let's start approaching Exercise 30 by finding an upper bound to $\dim \mathcal{C}$ where $\mathcal{C}$ is the middle-third Cantor set.

Example 44
Calculate $\dim_M(\mathcal{C})$.

Proof: Let $\delta_k = 3^{-k}$, $k \in \mathbb{N}_0$ (as we are taking the limit as $\delta \to 0$, we can choose the subsequence of δs we wish for our convenience). Then, the number of δ_k-balls covering $\mathcal{C}$ (excluding the endpoints possibly) is 2^k. Then,

$$\dim_M(\mathcal{C}) = \limsup_{k \to \infty} \frac{\log(C_{\delta_k}(\mathcal{C}))}{\log(\delta_k^{-1})} = \limsup_{k \to \infty} \frac{\log 2^k}{\log 3^k} = \frac{\log 2}{\log 3}.$$

□

This gives a useful (and relatively easy to calculate) way to approach the problem of finding the Hausdorff dimension of $\mathcal{C}$. However, when it came down to calculating the dimension of $\mathcal{C}$, we actually used a similar scaling argument as we did for $\mathbb{R}^n$.

Example 45
Find $\dim \mathcal{C}$.

Proof: Let $\mathcal{C}_l = \mathcal{C} \cap [0, 1/3]$ and similarly let $\mathcal{C}_r = \mathcal{C} \cap [2/3, 1]$. Then,

$$\mathcal{H}^s(\mathcal{C}) = 2\mathcal{H}^s(\mathcal{C}_l) = \frac{2}{3^s}\mathcal{H}^s(\mathcal{C}).$$

Hence, if $s < \frac{\log 2}{\log 3}$, then $\mathcal{H}^s(\mathcal{C}) = \infty$. Thus, $\dim \mathcal{C} = \frac{\log 2}{\log 3}$. □

After spending most of the meeting discussing the exercise suggested by Larry (Exercise 30), Shengwen and I went through the questions I had asked so far in the notes to rapid fire respond to some of them. This discussion is included below:

- **Question 1**/**Question 11** was answered using Exercise 30, which we spent today discussing.
- **Question 2**: In particular, I was confused about the difference between L^p and $\mathcal{L}^n$. The difference here is L^p is the space of L^p functions, where as $\mathcal{L}^n$ is the measure. So, it should be clear by context which we are discussing (which is further helped by the use of mathcal).
- **Question 5**: This is mostly just an older notation, but does mean the same thing as the restriction notation I am used to. For this reason, I will continue to use '$|$' to denote restrictions.
- **Question 8**: Shengwen seemed ultimately uncertain with regards to this question, but noted that it is really *insignificant* as we won't need to worry about what the constant $\alpha(s)$ is in general, and it is just there to normalize volumes. That being said, I wanted to see if you had any answer to this Larry.
- **Question 13** and **Question 14**: We ultimately decided to hold off on answering theses questions for now. Shengwen seemed to have an idea about it, but for now it seems unimportant.
- **Question 16**: If anything, this question has been answered through working through the Hausdorff dimension exercise today.
- **Question 18**: $d\,\mathrm{spt}\,\mu$ is the diameter of the support of μ, so my idea of what it meant in this question was accurate! It felt like out-of-nowhere notation, but now it is clear where it comes from.

This is all we discussed in this meeting. I spent the following few days going through the notes, book, and other resources I could find online to flush out these notes so I really internalized what we discussed. The big key detail I felt like I was missing out on before this meeting was how we could use properties of $\mathcal{H}^s$ being an *outer measure* to our advantage.

My current plan moving forward into the next week is to reread Chapter 4, now that I have a better intuition behind Hausdorff dimension. It may also be helpful to read through some sections in Chapter 3. Shengwen also suggested a paper by Fässler and Orponen (*On Restricted Familes of Projections in* $\mathbb{R}^3$) **[FO]**, which I plan to look over to better understand the proof of Frostman's lemma.

1.2 May 30-June 05

1.2.1 May 30-31

Howdy Larry! The introduction into this week's notes is relatively the same as the previous paragraph, but wanted to include it for completeness. I have a few key goals for this week:

- ~~Reread Chapter 4 more closely. Read sections of Chapter 3 that come up to fully digest the proofs here.~~
- ~~Read Fässler and Orponen's *On Restricted Familes of Projections in* $\mathbb{R}^3$, specifically Lemma 3.13 to better understand the proof of Frostman's lemma.~~
- ~~Learn more about the high-low frequency method that we would like to use to reprove Marstrand's projection theorem!~~

As always, if there is anything in the notes so far that you would like to comment on please don't hesitate to send an email! I really found the exercise (Exercise 30) you suggested last week really helpful. I hope your vacation is going well!

Notes on §4.1 Reading **[M]**

Chapter 4: Hausdorff dimension of projections and distance sets

This chapter gives an application of Fourier transforms used for geometric problems involving Hausdorff dimension.

§4.1: Projections

For $e \in \mathbb{S}^{n-1}$, $n \geq 2$, consider the projection $P_e : \mathbb{R}^n \to \mathbb{R}$, where $P_e(x) = e \cdot x$. I still want to do Exercise 20 to fully get why, but this implies that

$$\dim P_e(A) \leq \dim A \ \forall A \subset \mathbb{R}^n.$$

Remark 46. *I get why, intuitively, this is true, but I think it would still be useful for more intuition on Hausdorff dimension.*

Theorem 47

Let $A \subset \mathbb{R}^n$ be Borel, with $s = \dim A$. If $s \leq 1$, then $\dim P_e(A) = s$ for σ^{n-1} almost all $e \in \mathbb{S}^{n-1}$. If $s > 1$, then $\mathcal{L}^1(P_e(A)) > 0$ for σ^{n-1} almost all $e \in \mathbb{S}^{n-1}$.

First, we note some important propeties that we will use in this proof, notational and from Chapter 3. Firstly, recall that $g\mu(B) = \int_B g \mathrm{d}\mu$. Hence,

$$P_{e\#}\mu(B) = \int_B P_{e\#} \mathrm{d}\mu = \int_{P_e(B)} \mathrm{d}\mu = \mu(P_e(B)) \ \forall B \subset \mathbb{R}.$$

Notation 48

We denote $\gamma(n, s)$ to be the positive constant, fixed, such that $\hat{k_s} = \gamma(n, s) k_{n-s}$ as tempered distributions. Specifically, such that

$$\int \hat{k_s} \varphi = \gamma(n, s) \int k_{n-s} \varphi \ \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Note 49 (Polar coordinates)

For $f \in L^1(\mathbb{R}^n)$, we have the following polar coordinate decomposition:

$$\int_{\mathbb{R}^n} f \, \mathrm{d}\mathcal{L}^n = \int_{\mathbb{S}^{n-1}} \int_0^\infty f(rx) r^{n-1} \, \mathrm{d}r \mathrm{d}\sigma^{n-1} x.$$

Theorem 50

For $0 < s < n$,

$$I_s(\mu) = \gamma(n, s) \int |\hat{\mu}(x)|^2 |x|^{s-n} \, \mathrm{d}x.$$

Proof: This proof handwaves over some mild details (covered in the book), but the tldr is

$$I_s(\mu) = \int k_s * \mu \, \mathrm{d}\mu = \int \widehat{k_s * \mu}\overline{\hat{\mu}} = \int \hat{k_s}|\hat{\mu}|^2 = \gamma(n,s)\int |\hat{\mu}(x)|^2|x|^{s-n}\,\mathrm{d}x.$$

■

Proof of Theorem 47: Note, if $\mu \in \mathcal{M}(A)$, let $\mu_e = P_{e\#}\mu \in \mathcal{M}(P_e(A))$. Then, note that

$$\begin{aligned}\hat{\mu_e}(r) &= \int_{-\infty}^{\infty} e^{-2\pi i r\cdot x}\,\mathrm{d}\mu_e x \\ &= \int_{-\infty}^{\infty} e^{-2\pi i r\cdot(e\cdot y)}\,\mathrm{d}\mu y \\ &= \int_{-\infty}^{\infty} e^{-2\pi i (re)\cdot x}\,\mathrm{d}\mu y \\ &= \hat{\mu}(re).\end{aligned}$$

Suppose $0 < s = \dim A \le 1$. Fix $0 < t < s$ and pick $\mu \in \mathcal{M}(A)$ such that $I_t(\mu) < \infty$ (which exists as $t < s$. Then, we have that

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} I_t(\mu_e) d\sigma^{n-1}e &= \gamma(1,t)\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty} |\hat{\mu}_e(r)|^2 r^{t-1}\,\mathrm{d}r\mathrm{d}\sigma^{n-1}e \\ &= 2\gamma(1,t)\int_{\mathbb{S}^{n-1}}\int_{0}^{\infty} |\hat{\mu}(re)|^2 r^{t-1}\,\mathrm{d}r\mathrm{d}\sigma^{n-1}e \\ &= 2\gamma(1,t)\int_{\mathbb{R}^n} |\hat{\mu}(x)|^2|x|^{t-n}\,\mathrm{d}x \\ &= 2\gamma(1,t)\gamma(n,t)^{-1}I_t(\mu) < \infty.\end{aligned}$$

So, $I_t(\mu_e) < \infty$ for σ^{n-1} almost all $e \in \mathbb{S}^{n-1}$. In other words, for all $0 < t < s$, there exists a $\mu_e \in \mathcal{M}(P_e(A))$ such that $I_t(\mu_e) < \infty$ for almost all e. Hence, $\dim P_e(A) = \dim A = s$.

Now suppose that $s > 1$. Hence, there exists a $\mu \in \mathcal{M}(A)$ such that $I_1(\mu) < \infty$. Then, we similarly get that

$$\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty} |\hat{\mu_e}(r)|^2\,\mathrm{d}r\mathrm{d}\sigma^{n-1}e = 2\gamma(n,1)^{-1}I_1(\mu) < \infty.$$

So, $\hat{\mu_e} \in L^2(\mathbb{R})$ for σ^{n-1} almost all $e \in \mathbb{S}^{n-1}$. Thus, $\mu_e \in L^2(\mathbb{R})$ for almost all e. So, μ_e is absolutely continuous with respect to $\mathcal{L}^1$ for almost all e. Therefore, given $\mu_e \in \mathcal{M}(P_e(A))$ (i.e. positive), $\mathcal{L}^1(P_e(A)) > 0$ for almost all e. ■

Question 51. *Why do we need the σ^{n-1} almost all e? Why isn't it μ-almost all e or $\mathcal{L}^1$?*

Question 52. *It feels unclear that μ_e is absolutely continuous with respect to $\mathcal{L}^1$, but I do understand how this finishes the proof.*

Remark 53. *Though I have the above questions, I still feel pretty comfortable with this proof now. Overall, the structure of these proofs is quite interesting.*

Theorem 54

Let $A \subset \mathbb{R}^n$ be a Borel set and $\dim A > 2$. Then, $P_e(A)$ has non-empty interior for σ^{n-1} almost all $e \in \mathbb{S}^{n-1}$.

Rather than re-type out this proof, I will simply note where certain constants arise. Firstly, the 2 comes from changing the integral from $\int_{-\infty}^{\infty}$ to $2\int_0^{\infty}$. Furthermore, the second term in the second line simply comes from 'peeling off' the unit ball in the integral. It feels unclear to me why this disappears in the next mine however.

Question 55. ~~*What is $C(\mu)$? It seems as though we use $C(n)$ notation in equation (3.34), but I have yet to find where this comes from.*~~

In any case, the Schwartz inequality lets us undo the polar coordinates, giving us a constant times the s-energy. In the end, we get that $\hat{\mu_e} \in L^1(\mathbb{R})$ for almost all e, which implies μ_e is continuous for such e. Furthermore, given $\mu_e \in \mathcal{M}(P_e(A))$, $P_e(A)^\circ \neq \emptyset$ for almost all e.

Question 56. *It feels a bit unclear that the interior must then be non-empty, but I think this may follow from continuity. I.e., it can't instantaneously jump to a positive value, but rather increase over an interval which is enough to imply non-empty.*

Theorem 57

Let $A \subset \mathbb{R}^n$ be Lebesgue measurable and let $\mu \in \mathcal{M}(A)$ with $\mu(A) = 1$ and $I_1(\mu) < \infty$. Then,

$$\int \mathcal{L}^1(P_e(A))\, \mathrm{d}\sigma^{n-1} e \geq \frac{\gamma(n,1)\sigma^{n-1}(\mathbb{S}^{n-1})^2}{2I_1(\mu)}.$$

Remark 58. *This lower bound feels unintuitive, though I get how it arises in the proof. Is this inequality sharp? I imagine so as all of the inequalities used are sharp. Is there a geometric picture to this lower bound constant?*

Proof of Theorem 57: There are two parts of the proof here which I think are of note. Firstly, the use of the Schwartz Inequality felt unclear until I noted that

$$1 = P_{e\#}(\mathbb{R})^2 = \left(\int_{P_e(A)} \mu_e \cdot 1\, \mathrm{d}\mathcal{L}^1\right)^2 = \left(\left(\int_{P_e(A)} \mu_e^2\, \mathrm{d}\mathcal{L}^1\right)^{1/2} \left(\int_{P_e(A)} \mu_e^2\, \mathrm{d}\mathcal{L}^1\right)^{1/2}\right)^2 = \mathcal{L}^1(P_e(A)) \int \mu_e^2\, \mathrm{d}\mathcal{L}^1.$$

Secondly, it is unclear to me how the Schwartz inequality is used in the final part of the proof, specifically with the first implication. ■

Remark 59. *Initially I had planned to read through all of Chapter 4 and go through with the rigor I have had so far in this section of the notes. However, given that the main part of the project is on Marstrand's projection theorem, I think it may be best to instead start on the next item on the to-do list.*

1.2.2 June 01-02

My plan for these days is to read through Fässler and Orponen's paper, listed earlier on the to-do list, and recommended by Shengwen. Specifically, he suggested reading Lemma 3.13, the proof of Frostman's Lemma which I wanted to read some more about.

Notes on Lemma 3.13 **[FO]**

Definition 60 ((δ, s)-sets)

Let $\delta, s > 0$, and let $P \subset \mathbb{R}^3$ be a finite δ-separated set. Then, P is a **(δ, s)-set** if

$$|P \cap B(x,r)| \lesssim \left(\frac{r}{\delta}\right)^s, \quad x \in \mathbb{R}^3, r \geq \delta.$$

Definition 61 (Hausdorff content)
We denote the *s-dimensional Hausdorff content* of a set to be $\mathcal{H}^s_\infty$. By definition, the s-dimensional Hausdorff dimension of S is

$$\mathcal{H}^s_\infty(S) := \limsup_{r_i \to 0} \inf \left\{ \sum_i r_i^s \mid \text{ there is a cover of } S \text{ by balls of radii } r_i > 0 \right\}.$$

Here, I say "defined" as we have this same notation used in Definition 7. Then, we have the following lemma:

Lemma 62 (Frostman)
Let $\delta, s > 0$ and let $B \subset \mathbb{R}^3$ be any set with $\mathcal{H}^s_\infty(B) =: \kappa > 0$. Then, there exists a (δ, s)-set $P \subset B$ with cardinality $P \gtrsim \kappa \cdot \delta^{-s}$.

Remark 63. *This is proven in Appendix A of the paper, but I wanted to include it here for completion, and also to work through the details of a proof using dyadic cubes. I also note that this lemma is called "a discrete version of Frostman's Lemma". This makes sense, as the spacing condition of the set in the lemma is similar to the condition of measure in Frostman's lemma.*

Proof of Lemma 62: Assume without loss of generality that $\delta = 2^{-k}$ for some $k \in \mathbb{N}$ and $B \subset [0,1]^3$. Let $\mathcal{D}_k$ be the dyadic cubes in $\mathbb{R}^3$ of side-length 2^{-k}. Consider all of the dyadic cubes $Q^k \in \mathcal{D}_k$ which intersect B, and choose a single point $x \in B \cap Q^k$ for each such Q^k. This gives us a finite set P_0 (finite as there are finitely many options given $B \subset [0,1]^3$). We then modify the set P_0 (which seems to be a usual approach to proofs using dyadic cubes). Consider the cubes in $\mathcal{D}_{k-1}$. If one of these, Q^{k-1} satisfies

$$|P_0 \cap Q^{k-1}| > \left(\frac{d(Q^{k-1})}{\delta} \right)^s,$$

remove points from $P_0 \cap Q^{k-1}$ until the reduced set P_0' satisfies

$$\frac{1}{2} \left(\frac{d(Q^{k-1})}{\delta} \right)^s \leq |P_0' \cap Q^{k-1}| \leq \left(\frac{d(Q^{k-1})}{\delta} \right)^s.$$

Repeat this until for all $Q^{k-1} \in \mathcal{D}_{k-1}$ to obtain P_1. Then, continue this getting P_js (defined more explicitly in the paper, but by the same general logic). Stop the process when the remaining set of points, P, is entirely contained in some dyadic cube $Q \subset [0,1]^3$. We claim that for every $x \in P_0$, there exists a unique maximal dyadic cube $Q_x \subset Q_0$ such that $\ell(Q_x) \geq \delta$ and

$$|P \cap Q_x| \geq \frac{1}{2} \left(\frac{d(Q_x)}{\delta} \right)^s.$$

We only need show there is *at least* one such Q_x as the uniqueness follows from the dyadic structure.

If $x \in P$, then we have the above Q_x for the $Q_x \in \mathcal{D}_k$ containing x. On the other hand, if $x \in P_0 \setminus P$, the point x was 'deleted' from P_0 at some point. So, we define Q_x as the dyadic cube containing x where the 'last deletion of points' occurred. We show this Q_x satisfies the above. For instance, if this happened when defining P_{j-1}, then $Q_{k-j-1} = Q_x$. Thus, $P_j' \cap Q_x = P \cap Q_x$, which satisfies the requirements of Q_x.

Observe the following: the cubes $\{Q_x \mid x \in P_0\}$

1. cover $B \subset [0,1]^3$ as they cover every cube in $\mathcal{D}_k$ containing a point in P_0, which cover B,

2. and they are disjoint, hence partitioning P.

So, we show P is the desired (δ, s)-set. For the cubes $Q \in \mathcal{D}_l$ with $l \leq k$, it follows from the construction of P that

$$|P \cap Q| \leq \left(\frac{d(Q)}{\delta}\right)^s.$$

The general statement for balls $B \subset \mathbb{R}^3$ with $d(B) \geq \delta$ follows by noting any such balls can be covered by dyadic cubes of diameter $\sim d(B)$. ∎

Question 64. *What do we mean by $\sim d(B)$ in the last line? I get the general idea of what this means conceptually, but I wish it was a bit more specific about the side-lengths of the dyadic cubes there.*

I think in general this paper is very interesting to read, but for now will shift my focus to the high-low method as this will be a key point in the SPUR project.

1.2.3 June 03-05

The paper recommended by Shengwen on the high-low method was Guth, Solomon, and Wang's *Incidence Estimates For Well Spaced Tubes* **[GSW]**, found here. I plan to read this paper (with notes below) over the next few days, in time for a meeting with Shegnwen June 6, 4pm EST.

Notes on **[GSW]**: Introduction and Proposition 2.1

Remark 65. *Incidence geometry is about patterns of intersections of lines, while the Kakeya problem is about patters of intersection of tubes.*

In this paper, a number of examples are given depicting the importance of the well spacing of tubes to gain *results* for tubes. Consider for instance the Szemerédi-Trotter theorem (ST). Given a set $\mathscr{L}$ of lines in the plane, define (for $r \geq 2$)

$$P_r(\mathscr{L}) = \{\text{points on at least } r \text{lines}\}.$$

Then,

Theorem 66 (Szemerédi-Trotter)
We have the sharp inequality

$$|P_r(\mathscr{L})| \lesssim r^{-3}|\mathscr{L}|^2 + r^{-1}|\mathscr{L}|.$$

So now, let $\mathbb{T}$ be a set of $\delta \times 1$ tubes i.e. rectangles in $[0,1]^2$, called ***δ*-tubes**. The set of δ-balls intersecting at least r-tubes in $\mathbb{T}$ is infinite, so for tubes we need a different definition of $P_r(\bullet)$.

Definition 67 (r-rich)
We define the set of r-rich δ-balls for a set of tubes $\mathbb{T}$ to be the set of δ-balls with centers in $\delta\mathbb{Z}^2$ intersecting at least r-tubes of $\mathbb{T}$. We denote this set $P_r(\mathbb{T})$.

However, (ST) doesn't yet hold under these assumptions. For instance, consider the set $\mathbb{T}$ of small perturbations (by $\epsilon < \delta$) of one tube. Then, $P_r(\mathbb{T}) \sim \delta^{-1}$ for $r \sim |\mathbb{T}|$.

One may think the issue here is the strong overlap of the tubes.

Definition 68 (Essentially Distinct)
We say two tubes, T_1, T_2, are essentially distinct if

$$|T_1 \cap T_2| \geq \frac{1}{2}|T_1|.$$

However, (ST) also doesn't hold for essentially distinct tubes either. Consider the rectangle $R = r\delta \times 1$, which contains $\sim r^2$ essentially distinct δ-tubes, $\mathbb{T}_R$. Then $P_r(\mathbb{T}_R) \sim r\delta^{-1}$.

Question 69. *I am having a bit of trouble picturing by there are $\sim r^2$ essentially distinct δ-tubes in R, though I get why there are approximately $r\delta^{-1}$ r-rich balls (up to constant) for this set of tubes pictorally.*

So, now we could consider essentially distinct tubes in different directions, but this still won't be enough. Consider the set

$$\mathbb{T} = \{\delta^{-1} \quad \delta-\text{tubes in different directions through the origin in } \delta-\text{separated directions}\}.$$

Then, for $1 \ll r \ll \delta^{-1}$, $|P_r(\mathbb{T})| \sim r^{-2}|\mathbb{T}|$, which is still larger than the RHS of (ST).

Remark 70. *There are $\sim r^2|\mathbb{T}|$ for all $1 \ll r \ll \delta^{-1}$ as this is approximately the number of lattice points in a circle around the origin of radius δ^{-1} that could possibly intersect those many tubes. At the very least, this is how I pictured the above statement.*

So, GSW makes a bigger/stronger assumption on the spacing of the tubes. They claim this is the strongest assumption they can make on the spacing.

Question 71. *Why is this the strongest? If they assumed more, would the problem be obvious? Or is it the strongest they were able to make to figure out a proof? I am interested in why this may be the "strongest", though I don't know how what possible stronger assumptions could be made.*

Fix $W \geq 1$. Then, there are $\sim W^2$ essentially distinct $W^{-1} \times 1$ rectangles in $[0,1]^2$. (I find this mildly confusing, but this should be similar to the r^2 rectangles on the other example.) Then, fix some $\delta < 1/W$ and let $\mathbb{T}$ be a set of W^2 δ-tubes, each of which is contained in one of the $W^{-1} \times 1$ essentially distinct rectangles. Even under this spacing condition, if $r \leq \delta|\mathbb{T}|$, then $P_r(\mathbb{T}) \sim \delta^{-2}$ (similar to before) so ST doesn't hold. However, if $r > \delta|\mathbb{T}|$, we have the following theorem:

Theorem 72 (1.1)
Suppose $1 \leq W \leq \delta^{-1}$. Let $\mathbb{T}$ be a set of $\sim W^2$ δ-tubes in $[0,1]^2$, with at most one δ-tube in each $W^{-1} \times 1$ rectangle. Then, for any $\epsilon > 0$

$$\text{if } r > \max(\delta^{1-\epsilon}|\mathbb{T}|, 1) \implies |P_r(\mathbb{T})| \leq C(\epsilon)\delta^{-\epsilon}r^{-3}|\mathbb{T}|^2.$$

Remark 73. *Here, $C(\epsilon)$ is a positive constant that only depends on ϵ. This is the notation that I was confused about earlier in Question 55.*

We can even improve this result:

Theorem 74 (1.2)
Suppose $1 \le W \le \delta^{-1}$ and $1 \le N_1 \le (W\delta)^{-1}$. Divide the circle into arcs θ of length δ. For each θ, and each $1 \le j \le W$, let $T_{\theta,j} \subset [0,1]^2$ be a δ-tube. Suppose that for each θ, and each $W^{-1} \times 1$ rectangle in direction θ, there are uniformly N_1 tubes $T_{\theta,j}$ in the rectangle. Let $\mathbb{T}$ be the set of all the tubes $T_{\theta,j}$. Then, for any $\epsilon > 0$,

$$\text{if } r > C_1(\epsilon)\delta^{1-\epsilon}|\mathbb{T}| \implies |P_r(\mathbb{T})| \le C_2(\epsilon)\delta^{-\epsilon}W^{-1}r^{-2}|\mathbb{T}|^2.$$

Question 75. *Why does assuming there are $\sim N_1$ tubes in each rectangle make sense to consider geometrically? Or was this theorem considered from "Cordóba's theorem"?*

There are a number of other theorems and proposition that GSW prove in this paper, but I am going focus on digesting the proof of Proposition 2.1 as this is used throughout the paper and most directly used the high-low method.

Proposition 76 (2.1)
Suppose P is a set of unit balls in $[0,D]^n$ and $\mathbb{T}$ is a set of essentially distinct tubes of length D and radius 1 in $[0,D]^n$. Suppose that each ball of P lies in about E tubes of $\mathbb{T}$. Let $S = D^{\epsilon/10n}$ for tiny $\epsilon > 0$. Then, either

Thin case. $|P| \lessapprox_n S^n E^{-2}|\mathbb{T}|D^{n-1}$ or

Thick case. There is a set of finitely overlapping $2S$-balls Q_j (called heavy balls) such that

1. $\cup_j Q_j$ contains a fraction $\gtrapprox_n 1$ of the balls of P,
2. Each Q_j intersects $\gtrapprox_n S^{n-1}E$ tubes of $\mathbb{T}$.

Here, $\lessapprox_n$ means $\le C(\epsilon, n)D^{10n\epsilon^3}$.

Remark 77. *Do you (Larry or Shengwen) have any suggestions on examples of papers/proofs where y'all "thicken the balls and go to the next scale"? Perhaps there is an example in this paper but I would be interested in understanding where this name comes from.*

Remark 78. *In the following proof, $N_S(\bullet)$ is an S-radius ball (i.e. Neighborhood) centered at q.*

Proof of Proposition 76: Define

$$W_S(q) = \#\{T \in \mathbb{T} \mid T \cap N_S(q) \neq \emptyset\}.$$

We choose a subset $P' \subset P$ such that $|P'| \gtrapprox |P|$ and $W_S(q_1) \sim W_S(q_2)$ for any $q_1, q_2 \in P'$. In essence, from what I can tell, we are throwing out all of the points in P where the number of rich S-balls isn't comparable. Relabel P' to P.

For each unit ball q of P, define ψ_q to be a smooth bump function approximation χ_q, in that $\operatorname{supp}\psi_q \subset 2q$ and $\psi_q = 1$ on q. Let $f = \sum_{q\in P}\psi_q$. For each tube $T \in \mathbb{T}$, let ψ_T be a smooth bump function approximating χ_T, and similarly let $g = \sum_{T\in\mathbb{T}}\psi_T$. Let $I(P,\mathbb{T})$ denote the cardinality of the set $\{(q,T) \mid |q \cap T| \ge |q|/2, q \in P, T \in \mathbb{T}\}$.

Remark 79. *Is this basically essentially distinct tubes, but in this case essentially distinct sets (the sets here being q and T)?*

If q intersects T, then $\int \psi_q\psi_T \gtrapprox 1$ so

$$I(P,\mathbb{T}) \lesssim \int fg = \int \hat{f}\bar{\hat{g}}.$$

This is the part where we decompose Fourier space into high-frequency and low-frequency pieces (for the high-low method). Let ρ be a real number that is slightly larger than S^{-1}, i.e. $\rho = D^{\epsilon^3}S^{-1}$. Let η_0 be a smooth bump function equal to 1 on the unit ball, supported in the ball of radius 2. Then, take $\eta(\omega) = \eta_0(\rho^{-1}\omega)$. Then,

$$I(P, \mathbb{T}) \lesssim \int \eta \hat{f}\bar{\hat{g}} + \int (1-\eta)\hat{f}\bar{\hat{g}}.$$

Remark 80. *Firstly, I really like how the η_0 was described simply by using a bump function on the ball of radius 2 and rescaling. Secondly, I want to know more about why the first integral in the above line is the "low-frequency" piece and the second is the "high-frequency"– where does this terminology come from?*

If the high-frequency piece dominates, we will show that the conclusion of the thin case holds, and if the low-frequency piece dominates, then we will show that the conclusion of the thick case holds.

The high frequency case. If the high-frequency term dominates, then we have

$$\begin{aligned} I(P, \mathbb{T}) &\lesssim \int \eta \hat{f}\bar{\hat{g}} + \int (1-\eta)\hat{f}\bar{\hat{g}} \\ &\lesssim \int (1-\eta)\hat{f}\bar{\hat{g}} \\ &\leq \left(\int (1-\eta)|\hat{f}|^2\right)^{1/2} \left(\int (1-\eta)|\hat{g}|^2\right)^{1/2}. \end{aligned}$$

We first note that $\|f\|_{L^2} \sim |P|^{1/2}$, so that term is bounded.

Question 81. *I find this comparison to be a bit confusing.*

So, we now bound the factor involving g, taking advantage of the support of the Fourier transform of ψ_T. Cover the unit sphere $\mathbb{S}^{n-1}$ by $1/D$-caps θ. Then, we call the outer normal direction of the center of θ on $\mathbb{S}^{n-1}$ the direction of θ.

Let $\mathbb{T}_\theta$ be the set of $T \in \mathbb{R}$ in direction θ, and let $\sum_{T\in\mathbb{T}_\theta} \psi_T$. If T is a $1 \times D$ tube in direction θ, then $\hat{\psi_T}$ is rapidly decaying outside of θ^* where θ^* is a $D^{-1} \times 1 \times \cdots \times 1$ slab through the origin perpendicular to the direction of θ.

Remark 82. *I am having a bit of trouble understanding this picture, but Shengwen and I can talk about it Monday.*

Now consider the integral

$$\int (1-\eta)|\hat{g}|^2 = \int (1-\eta(\omega)) \left|\sum_\theta \hat{g}_\theta(\omega)\right|^2 d\omega.$$

If $1\eta(\omega) \neq 0$, then $|\omega| \geq \rho$ (as $\eta(\omega) = 1$ when $|\omega| < \rho$). Furthermore, then $\omega \in D^{\epsilon^3}\theta^*$, for $\lesssim \rho^{-n}D^{n-2+n\epsilon^3}$. So, $\hat{g}_\theta$ is essentially supported in θ^* with rapidly decaying tail. So, outside of $D^{\epsilon^3}\theta^*$, $\hat{g}_\omega| \leq C_N D^{-N}$. So, for any N,

$$\left|\sum_\theta \hat{g}_\theta(\omega)\right|^2 \lesssim \rho^{-n}D^{n-2+n\epsilon^3} \sum_\theta \int |\hat{g}_\theta|^2 = \rho^{-n}D^{n-2+n\epsilon^3} \sum_\theta \int |g_\theta|^2.$$

The term $C_N D^{-N}$ is negligible (representing the rapidly decaying tail). So,

$$\int (1-\eta)|\hat{g}|^2 \lesssim \rho^{-n}D^{n-2+n\epsilon^3} \sum_\theta \int |g_\theta|^2.$$

Given the tubes $T \in \mathbb{T}_\theta$ are essentially distinct, by orthogonality from finite overlap, we have

$$\rho^{-n}D^{n-2} \sum_\theta \int |g_\theta|^2 = \rho^{-n}D^{n-2} \sum_{T\in\mathbb{T}} \int |\psi_T|^2 \sim \rho^{-n}D^{n-1}|\mathbb{T}|.$$

So,

$$E|P| \approx I(P, \mathbb{T}) \lesssim \rho^{-n/2}D^{\frac{n-1+n\epsilon^3}{2}}|P|^{1/2}|\mathbb{T}|^{1/2}.$$

So,

$$|P| \lesssim \rho^{-n} D^{n\epsilon^3} E^{-2} D^{n-1} |\mathbb{T}| \lesssim S^n E^{-2} D^{n-1} |\mathbb{T}|.$$

The low frequency case. If the low-frequency term dominates, then we have

$$I(P, \mathbb{T}) \lesssim \int \eta \hat{f} \bar{\hat{g}} = \int f(g * \eta^\vee) = \sum_{q \in P} \sum_{T \in \mathbb{T}} \int \psi_q (\psi_T * \eta^\vee).$$

Note that $\psi_T * \eta^\vee$ is rapidly decaying outside of the $\rho^{-1} \times D$ tube around T and $|\psi_T * \eta^\vee| \leq \rho^{n-1}$. Since $S = D^{\epsilon^3} \rho^{-1}$, $\psi_T * \eta^\vee$ is negligible outside of a $S \times D$ tube around T. Therefore,

$$\sum_{T \in \mathbb{T}} \int \psi_q (\psi_T * \eta^\vee) \lesssim \rho^{n-1} W_S(q) \lesssim S^{-(n-1)} W_S(q).$$

Given $W_S(q)$ are approximately the same for all $q \in P$ (this is where we use the reduction from the beginning of the proof), we have that

$$W_S(q) \gtrapprox S^{n-1} E.$$

■

Based on the above proposition/proof, I think the main things to focus on during my meeting with Shengwen on Monday are:

- Go through questions on the reading (in these notes)
- Discuss how the Fourier transform effects the support of a function
- Try to understand where the terms "high"- and "low"-frequency come from, whether technical or pedagogical
- Ask how this might apply to reproving Marstrand's projection theorem.

Hopefully these questions can help guide both the discussion with Shengwen, and may lend itself to picking the next thing to read/work on.