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WHAT MAKES A GOOD FRIEND? THE MATHEMATICS OF ROCK CLIMBING*

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Abstract. The shape of the cams used in rock climbing friends is derived using a linear system of differential equations.

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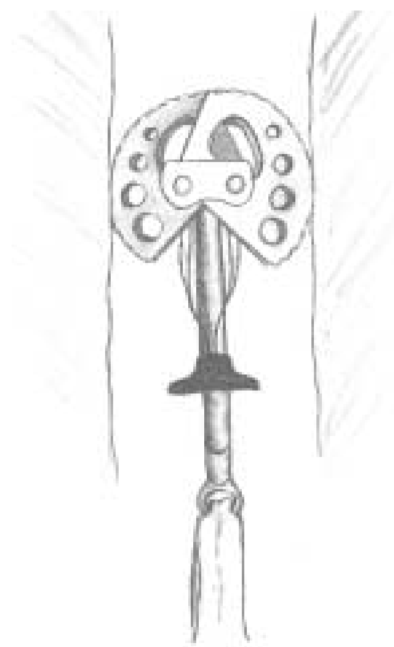
The purpose of this note is to determine the shape of the cams used in the devices rock climbers call *friends,* used to secure ropes to cracks in a rock face. Ray Jardine invented friends in 1973, and their use is now universal. The derivation of the shape of friends is an elementary but charming application of concepts found in any differential equations course.

Each friend is made up of usually four cams, which are attached symmetrically at a pivot point and rotate by a spring and pulley mechanism. The cams have a spiral shape such that rotating the lobes enables the friend to fit into a range of crack sizes.



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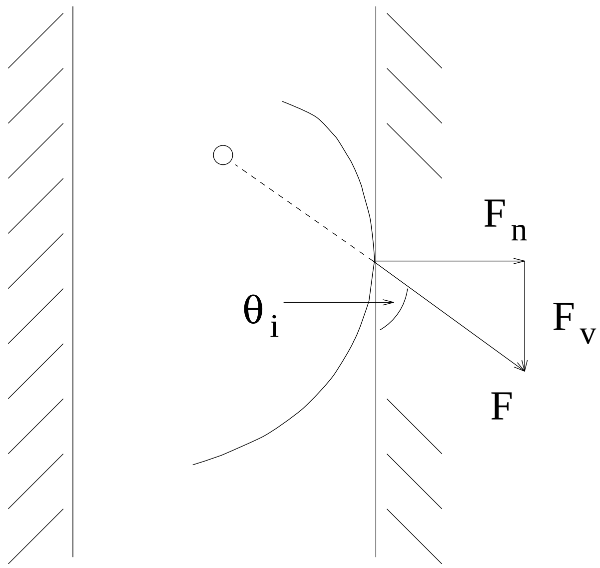

FIG. 1.

1. The derivation. The physical principles of friends are straightforward. A force pulling down on the rope attached to the friend wedges the cams more tightly into the crack. The induced static frictional force is then able to hold the load. In terms of vector forces, the load induces a force F on each cam which acts along the line between the pivot point and the point of contact between the cam and the rock. See Figure 1.

The force F decomposes into a normal force, F_n, and a vertical force parallel to the crack wall, F_v. Let θ_i be the angle of incidence of F against the crack wall. The forces F, F_n, and F_h satisfy the obvious Pythagorean law and the equation

$$F_v = F_n \cot \theta_i. \tag{1.1}$$

As is shown in any elementary physics textbook, there is a crucial relationship between the angle θ_i and the friction force holding the weight. Let μ be the coefficient of static friction of the cam material against the rock. Then the cam will not slip against the rock as long as $F_v < F_n\mu$, or, by (1.1), $F_n \cot \theta_i < F_n\mu$. It follows that a necessary design specification for the cams is that the angle of incidence of the force F satisfy $\cot \theta_i < \mu$.

It is natural then to require that the angle of incidence θ_i always be some constant satisfying $\cot \theta_i < \mu$, no matter what size crack is encountered. This precision in the design establishes confidence that the cam will perform as intended.

We are now ready to formulate the problem of finding the shape of a cam in strictly mathematical terms. Consider a cam lobe set inside a crack. Taking the pivot point as the origin of a coordinate system, the boundary of the cam which contacts the rock can be represented as a parameterized curve $(x(t), y(t))$. At the point of contact, the tangent vector to the curve is parallel with the crack face. Hence there exist functions $K_1(t)$ and $K_2(t)$ such that

$$(x, y) + K_1(x', y') = K_2(y', -x'). \tag{1.2}$$

See Figure 2. (Note that K_1 and K_2 will have the same parity.)

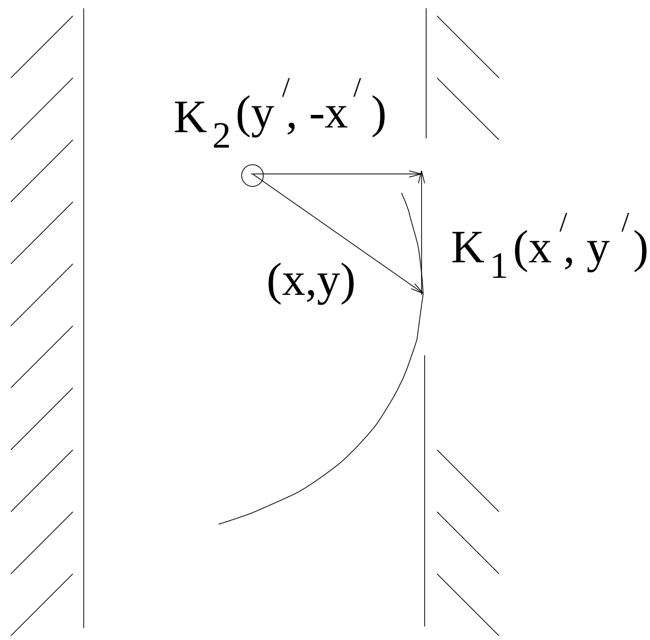

FIG. 2.

Solving for K_1 and K_2, we have

$$K_1 = \frac{-xx' - yy'}{(x')^2 + (y')^2}, \tag{1.3}$$

$$K_2 = \frac{-yx' + xy'}{(x')^2 + (y')^2}. \tag{1.4}$$

Since the force F acts along the line through the pivot point and the point of contact, the triangles in Figures 1 and 2 are similar. Hence we have

$$\cot\theta_i = \frac{|K_1|\,||(x', y')||}{|K_2|\,||(y', -x')||} = \left|\frac{K_1}{K_2}\right| = \frac{K_1}{K_2} = \frac{-xx' - yy'}{-yx' + xy'}. \tag{1.5}$$

Our goal is for θ_i to be constant, and hence $\cot\theta_i$ will also be constant. Simplifying the last equation, and using $C = \cot\theta_i$, the parameterized curve must satisfy

$$x'(-x + Cy) = y'(Cx + y). \tag{1.6}$$

But this equation can be split into a system of linear differential equations! Namely,

$$\frac{dx}{dt} = -Cx - y, \tag{1.7}$$

$$\frac{dy}{dt} = x - Cy. \tag{1.8}$$

Note that we have written the system such that the corresponding solution curves rotate counterclockwise. We leave it as an exercise to the reader that equation (1.8) and this system have the same nontrivial solutions, up to changes of time scale and direction.

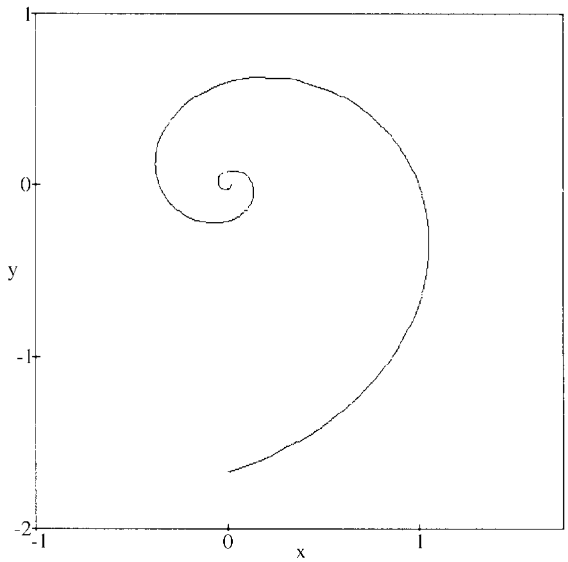

FIG. 3.

A nice looking particular solution to (1.7) and (1.8) is

$$(x(t), y(t)) = e^{-Ct}(\cos t, \sin t), \tag{1.9}$$

whose trace is shown in Figure 3 (with $C = .32$).

In fact, the solutions to the system of differential equations display rotational symmetry: they all exhibit the same shape as any one particular solution. Thus the polar coordinate curve $r = e^{-C\theta}$ can serve as a template for constructing friend cams. The problem of deciding exactly where to cut the curve to make a functional cam is considered next.

2. Cutting the curve. The special symmetry of the design of friends restricts how much of the cam curve given by (1.9) can actually be used in one individual cam. This restriction stems from the pivot point of the friend lying at the exact center of the crack. When the cam is operating at its minimum radius, the total sweep of the cam must be small enough for the entire device to fit in the crack (see Figure 4).

It turns out that the maximum sweep of the cam is a function only of the angle of incidence θ_i described above, which we shall now demonstrate.

Suppose the cam is operating at its minimum radius, and let α, β, l, and w be as shown in Figure 4. We can assume that the boundary of the cam is given by the polar coordinate equation $r(\theta) = Ke^{C\theta}$, where K is some constant, C is as above, and θ is the angle measured *clockwise* from l, with the pivot point again taken as the origin of the coordinate system.

Then the geometry determines the following equations:

$$r(\pi/2 - \theta_i)\cos(\pi/2 - \theta_i) = w/2, \tag{2.1}$$

$$r(\pi/2 + \beta)\sin(\beta) = w/2, \tag{2.2}$$

$$\beta = \alpha - \theta_i. \tag{2.3}$$

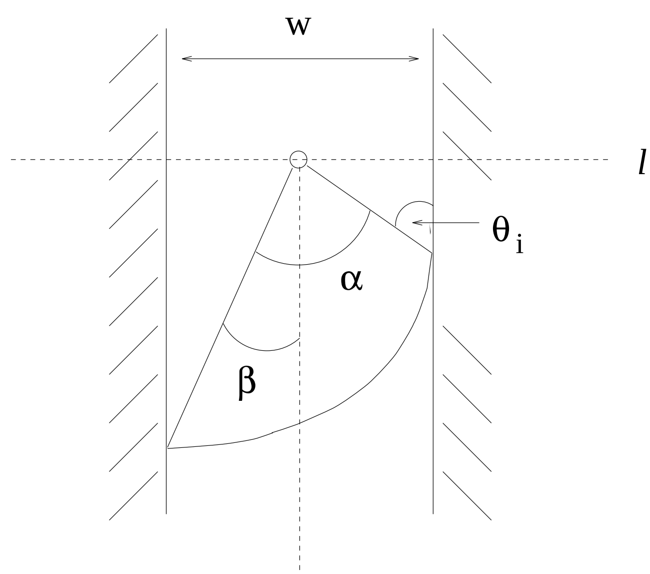

FIG. 4.

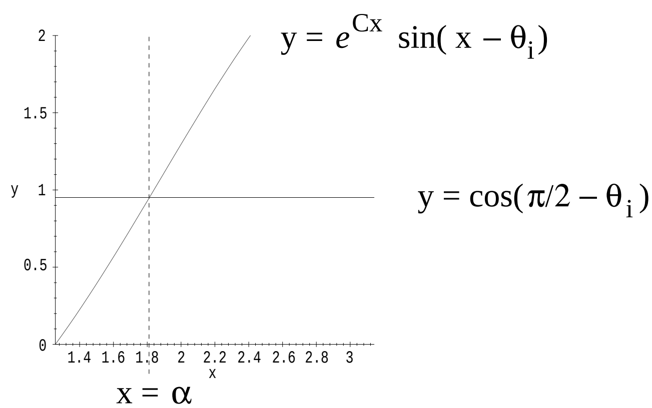

FIG. 5.

Substituting in the expression for $r(\theta)$, equating the expressions equal to $w/2$, and simplifying, we have

$$\cos(\pi/2 - \theta_i) = e^{C\alpha} \sin(\alpha - \theta_i). \tag{2.4}$$

This equation uniquely determines a value for α with $\theta_i < \alpha < \pi$, as can be seen, for example, from the graph of $y = e^{Cx} \sin(x - \theta_i)$ for $\theta_i = 2\pi/5$; see Figure 5.

Commercial cams actually sweep out an angle *bigger* than α so that a portion of the small radius side of the cam is never actually used.

3. Conclusion. Our investigation of the mathematics of friends began while the first two authors were enrolled in an introductory differential equations course taught by the last author. This probably explains our efforts to reduce the problem to a linear system of differential equations.

It is interesting to note that the inventor of friends, Ray Jardine, arrived at the same curve equation by an altogether different line of analysis. He derived a differential equation for the curve in polar coordinates by considering an infinitesimal angular displacement $d\theta$ and determining an equation for the resulting radial displacement dr. This method is perhaps in keeping with the style of proof most commonly encountered in calculus courses.

Whatever method strikes the reader as sufficient and/or appropriate, it cannot be denied that the logarithmic spiral has probably never before enjoyed as much exposure as it does today on countless cliffs and boulders around the world.

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REFERENCE

[1] P. Blanchard, R. Devaney, and G.R. Hall, *Differential Equations*, Brooks/Cole Publishing Co., Boston, MA, 1996.